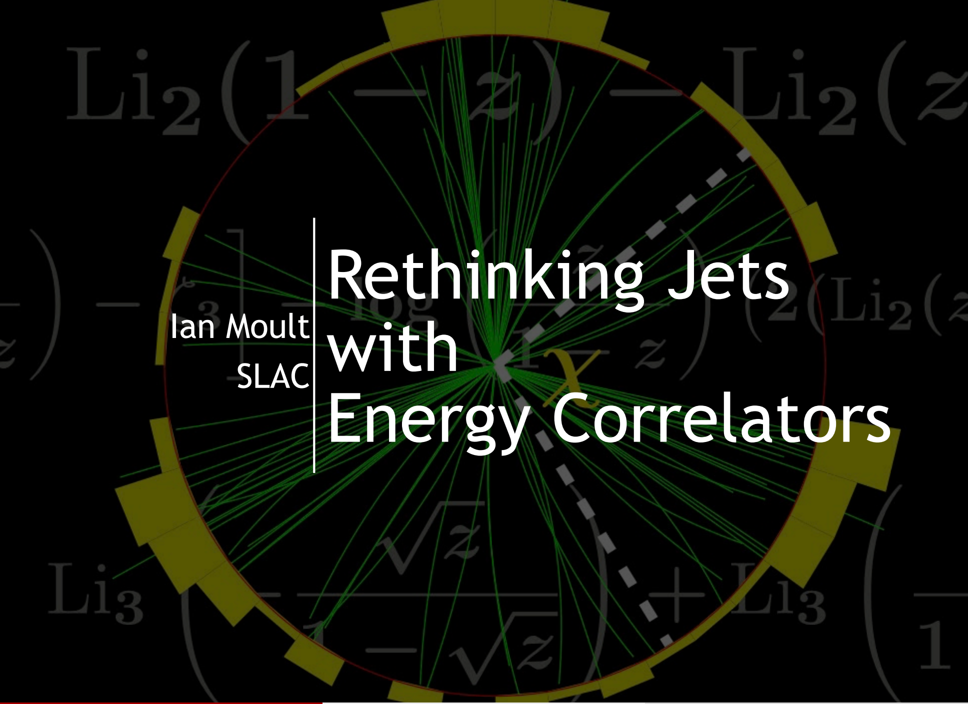
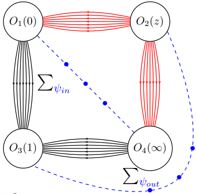


Rethinking Jets with Energy Correlators

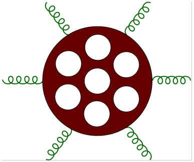
Ian Moult
SLAC

- In the last few years there has been significant progress in understanding correlation functions and scattering amplitudes, often driven by insights from $\mathcal{N} = 4$ SYM.

See e.g. "What can we learn about QCD and Collider Physics from $\mathcal{N} = 4$" by Henn

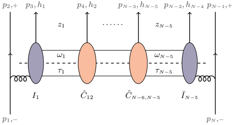


[Coronado, Korchemsky, Belitsky, ...]



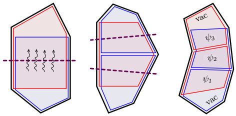
[Dixon et al.]

Regge:



[Del Duca, Duhr et al.]

Collinear:

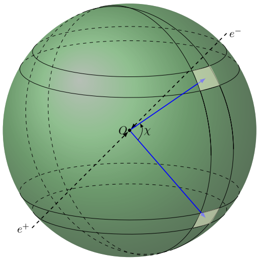


[Basso, Sever, Vieira]

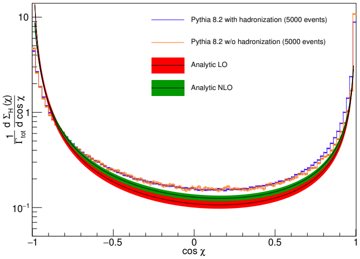
- Exploits enhanced symmetries, behavior in kinematic limits and functional/analytic properties of these objects.

Motivation

- Another class of field theoretic quantities that has received much less attention from the formal community are cross section level observables that measure the flow of energy (i.e. event shapes).



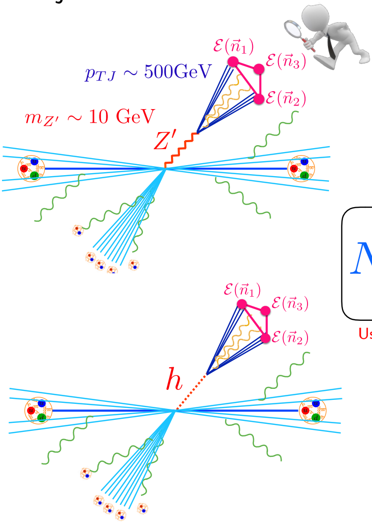
From [Luo, Shtabovenko, Yang, Zhu]

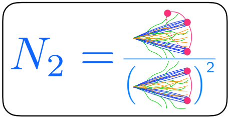


- Such observables have a long history for studying QCD in e^+e^- colliders, but have had a massive renewal of interest due to the invention of jet substructure at the LHC.

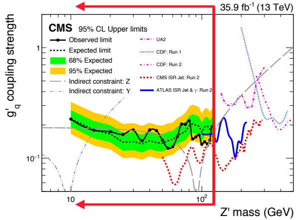
Jet Substructure: Searches

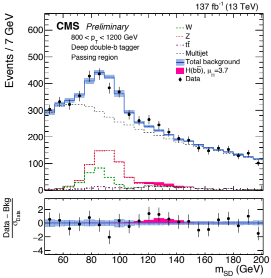
- Observables that probe complicated energy flows play a central role in jet substructure searches.





Using [IM, Necib, Thaler]

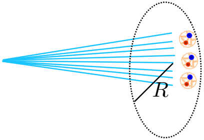


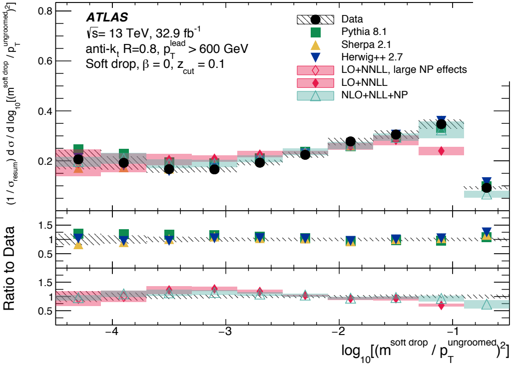


Jet Substructure: Precision

- Precision calculations of energy flow within jets offer new opportunities to measure fundamental constants with jet substructure.

Soft Drop Mass





Using [Frye, Larkoski, Schwartz, Yan]

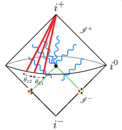
Rethinking Jet Substructure

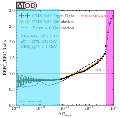
- Unfortunately, there has been limited interaction between formal developments and jet substructure.
- This is often an issue of observables: observables of practical use are often not theoretically nice (e.g. complicated algorithms), and observables that are theoretically nice are often practically useless.

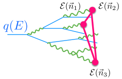
- As jet substructure transitions to a precision era, it is important to ask the following two questions:
    - Practical Question: Can we formulate jet substructure in a manner that facilitates more precise/ more differential calculations?
    - Formal Question: Can we formulate jet substructure in a manner that facilitates connections with more formal developements?

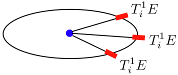
Outline

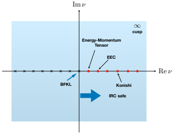
- Energy Correlators and Weighted Cross Sections
- Projected Energy Correlators and Scaling
- Beyond Scaling: Shape Dependence of the Three Point Correlator
- Making Tracks Tractable
- Bonus: The Analytic Continuation of Jet Substructure







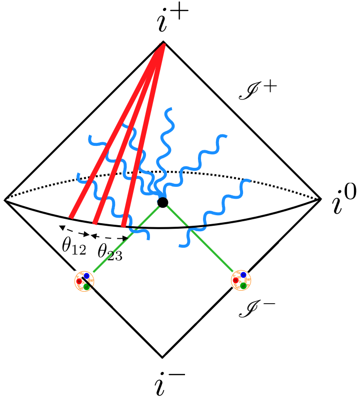




References

- Based on:
  - Dixon, Moult, Zhu, arXiv:1905.01310
  - Chen, Luo, Moult, Yang, Zhang, Zhu, arXiv:1912.11050
  - Chen, Moult, Zhang, Zhu, arXiv:2004.11381
  - Dixon, Komiske, Moult, Thaler, Zhu, Forthcoming

Energy Correlators and Weighted Cross Sections

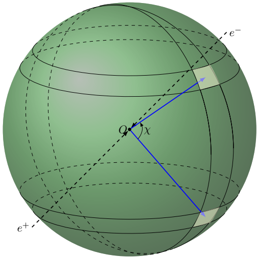


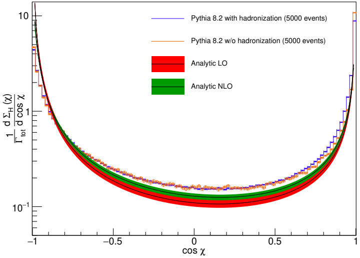
Energy-Energy Correlators

- To understand the structure of energy flow observables, one should start with those that are most closely tied to simple field theoretic objects.
- Arguably the simplest is the two-point correlator, which is called the Energy-Energy Correlator.

[Basham, Brown, Ellis, Love]

$$\boxed{\frac{d\sigma}{dz} = \sum_{i,j} \int d\sigma \; \frac{E_i E_j}{Q^2} \delta\left(z - \frac{1-\cos\chi_{ij}}{2}\right)}$$



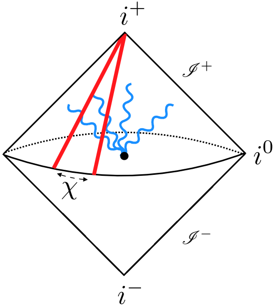


From [Luo, Shtabovenko, Yang, Zhu]

Energy-Energy Correlators

- The EEC admits an alternative formulation as a four point function of light ray (ANEC) operators

$$\mathcal{E}(\vec{n}) = \int_0^\infty dt \lim_{r\to\infty} r^2 n^i T_{0i}(t, r\vec{n})$$



$$\frac{1}{\sigma_{\text{tot}}}\frac{d\sigma}{dz} = \frac{\int d^4x\, e^{iq\cdot x}\langle \mathcal{O}(x)\mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\mathcal{O}^\dagger(0)\rangle}{\int d^4x\, e^{iq\cdot x}\langle \mathcal{O}(x)\mathcal{O}^\dagger(0)\rangle}$$

[Korchemsky; Maldacena, Hofman]

- Simplest extension of a standard four point correlator of local operators $\Longrightarrow$ has led to significant recent progress.

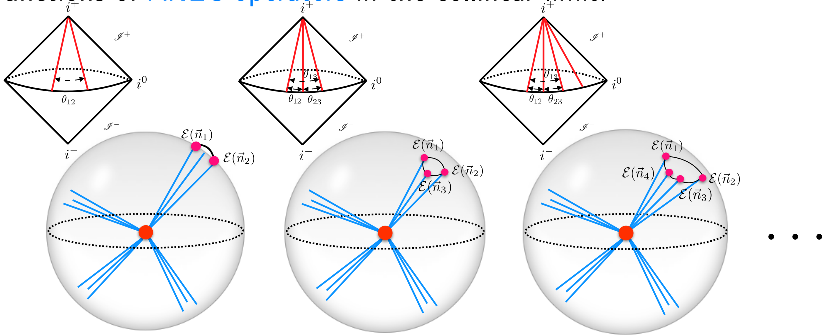
[Chicherin, Henn, Sokatchev, Yan,Simmons Duffin, Kologlu, Kravchuk, Zhiboedov,Korchemsky, Moult, Dixon, Zhu,...]

- This has a natural generalization to higher point correlation functions:

$$\langle \mathcal{O}(x)\mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\mathcal{E}(\vec{n}_3)\mathcal{O}^\dagger(0)\rangle\,, \qquad \langle \mathcal{O}(x)\mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\mathcal{E}(\vec{n}_3)\mathcal{E}(\vec{n}_4)\mathcal{O}^\dagger(0)\rangle\,, \qquad \cdots$$

Energy Flow Operators

- From this perspective, Jet Substructure is the study of correlation functions of ANEC operators in the collinear limit.

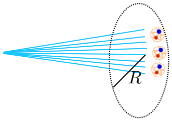


$\langle \hat{O}\mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\hat{O}^\dagger \rangle$ $\langle \hat{O}\mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\mathcal{E}(\vec{n}_3)\hat{O}^\dagger \rangle$ $\langle \hat{O}\mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\mathcal{E}(\vec{n}_3)\mathcal{E}(\vec{n}_4)\hat{O}^\dagger \rangle$

- Much recent progress in understanding the small angle limit of ANECs in (Non-)Conformal Field Theories.
- Are these different/related to "Standard Jet Substructure Observables"?

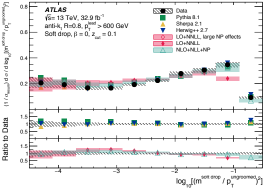
Two Ways to Make a Distribution

- If I want to make a differential distribution, there are two approaches:
- Approach 1: "Standard Observable"
  For each jet (or event), observable returns a number, make a distribution of the values.
- e.g. Mass, energy correlation functions, all substructure observables...



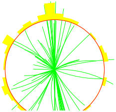
$$e_2^{(\beta)} = \frac{1}{p_{TJ}^2} \sum_{i<j\in J} p_{Ti} p_{Tj} (R_{ij})^\beta \longrightarrow \#$$

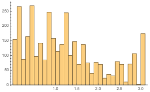
Distribution



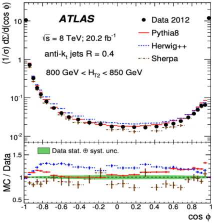
Two Ways to Make a Distribution

- Approach 2: "Weighted Cross Section"
  For each jet (or event), observable returns a distribution, then average the distributions
- e.g. Transverse energy energy correlator





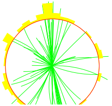
$$\frac{d\Sigma}{d\cos\chi} = \sum_{i,j}\int \frac{E_i E_j}{Q^2}\delta(\vec{n}_i\cdot\vec{n}_j - \cos\chi)d\sigma$$

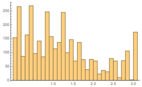


Energy Flow Operators and Weighted Cross Sections

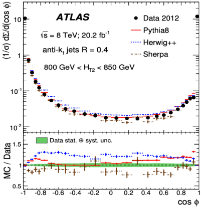
- How does this connect to weighted cross sections?

  $\Longrightarrow$ Energy correlation functions are exactly weighted cross sections!





$$\frac{d\Sigma}{d\cos\chi} = \sum_{i,j} \int \frac{E_i E_j}{Q^2} \delta(\vec{n}_i \cdot \vec{n}_j - \cos\chi) d\sigma$$



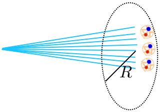
Field Theory Definition

$$\boxed{\frac{d\Sigma}{d\cos\chi} \sim \langle 0|\mathcal{O}\mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\mathcal{O}^\dagger|0\rangle}$$

- Weighted cross sections can be expressed as a correlation function of energy flow operators! Manifest symmetries, can use fancy techniques, etc.

Complexities of Standard Observables

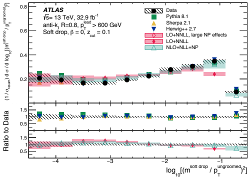
- Why are "standard observables" more complicated?



$$e_2^{(\beta)} = \frac{1}{p_{TJ}^2} \sum_{i<j\in J} p_{Ti} p_{Tj} (R_{ij})^\beta$$

#

Distribution



Field Theory Definition

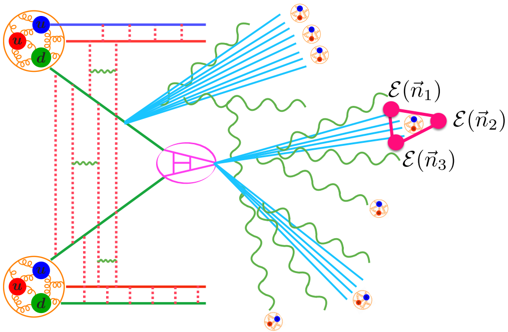
$$\boxed{\langle 0|\mathcal{O}\delta(e_2 - f(\mathcal{E}(\vec{n}_1), \mathcal{E}(\vec{n}_2))\mathcal{O}^\dagger|0\rangle}$$

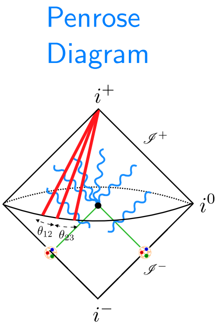
$$\boxed{\delta(e_2 - f(\mathcal{E}(\vec{n}_1), \mathcal{E}(\vec{n}_2)) = \delta(e_2) + f(\mathcal{E}(\vec{n}_1), \mathcal{E}(\vec{n}_2))\delta^{(1)}(e) + \cdots + \frac{[f(\mathcal{E}(\vec{n}_1), \mathcal{E}(\vec{n}_2))]^n}{n!}\delta^{(n)}(e) + \cdots}$$

- "Standard observables" require an infinite number of correlators.
- Their moments are weighted cross sections and hence simple.
- This complexity will come back to bite you even more when you try and incorporate non-perturbative information such as tracks.

Energy Correlators in the Collinear Limit

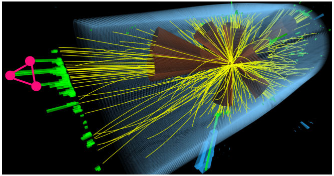
- Given this theoretical simplicity of the energy correlators, lets explore what they can give phenomenologically.
- The fact that they probe correlations of energy flow in the collinear limit is a good start!





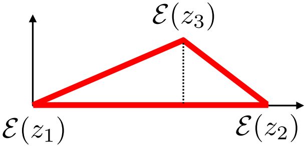
The Basic Structure

- In a CFT, energy correlators take a simple form in the small angle limit:



$$\boxed{\frac{d\sigma}{dx_L\, d\mathrm{Shape}} = C_{\mathrm{Shape}}(x_L = 1, \alpha_s)\, x_L^{\gamma_{N+1}(\alpha_s)-1}}$$

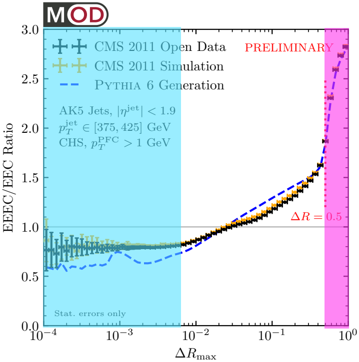
e.g.



$$= \underbrace{f_4\left(\frac{|z_3-z_1|^2}{|z_2-z_1|^2}, \frac{|z_3-z_2|^2}{|z_2-z_1|^2}\right)}_{\text{Shape}} \cdot \underbrace{\left(\frac{1}{|z_2-z_1|^2}\right)}_{\text{Scaling}}{}^{1-\gamma_4(\alpha_s)}$$

- Will explore both shape and scaling of multi-point correlators.

Projected Energy Correlators and Scaling



Scaling

- Most basic property of their correlators is scaling with size.
- Begin with the two-point correlator to gain intuition.
- In a conformal theory, Maldacena and Hofman showed:

$$\frac{d\sigma}{dz} = C(\alpha_s)\, z^{\gamma_3(\alpha_s)-1}$$

- γ_N is the twist-2 spin-N spacelike anomalous dimension.
- Power law scaling corresponds to a "single logarithmic" (collinear) observable. (As compared with Sudakov observables).

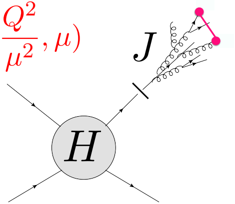
- Would like to generalize this to a non-conformal theory such as QCD.

Energy Correlators in QCD

[Dixon, Moult, Zhu]

- We can derive a timelike factorization formula for the 2-point correlator in a non-CFT (e.g. QCD):

$$\Sigma(z, \ln\frac{Q^2}{\mu^2}, \mu) = \int_0^1 dx\, x^2 \vec{J}(\ln\frac{zx^2Q^2}{\mu^2}, \mu) \cdot \vec{H}(x, \frac{Q^2}{\mu^2}, \mu)$$



- The jet function satisfies the renormalization group equation:

$$\frac{d\vec{J}(\ln\frac{zQ^2}{\mu^2}, \mu)}{d\ln\mu^2} = \int_0^1 dy\, y^2 \vec{J}(\ln\frac{zy^2Q^2}{\mu^2}, \mu) \cdot \widehat{P}_T(y, \mu)$$

- At LL, have correspondence with CFT result (up to running coupling):

$$\vec{J}^T_{LL} = (J_q, J_g) \exp\left(\frac{\widehat{\gamma}(3)}{2\beta_0} \ln\frac{\alpha_s(z^{1/2}Q)}{\alpha_s(Q)}\right)$$

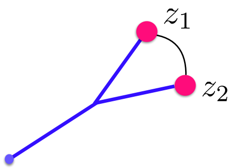
- In a non-CFT, beyond LL, derivatives $\gamma'(N+1)$, $\gamma''(N+1)$, ... also enter.

Basso-Korchesmky Reciprocity

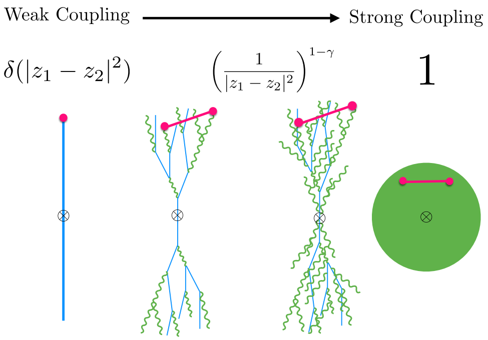
- Equivalence of spacelike and timelike formulations can be proven in a CFT using Basso-Korchemsky Reciprocity.
- Consider for concreteness $\mathcal{N}=4$ where SUSY reduces the evolution equations to scalar equations.
- In a CFT we can make a power law ansatz for the jet function:
$$J(zQ^2,\mu) = C_J(\alpha_s)\left(\frac{zQ^2}{\mu^2}\right)^{\gamma_J^{\mathcal{N}=4}(\alpha_s)}$$
- Substituting this into the evolution equation, we find
$$\begin{aligned} 2\gamma_J^{\mathcal{N}=4}(\alpha_s) &= -2\int_0^1 dy\, y^{2+2\gamma_J^{\mathcal{N}=4}(\alpha_s)} P_{T,\text{uni.}}(y,\alpha_s) \\ &= 2\gamma_T^{\mathcal{N}=4}(1+2\gamma_J^{\mathcal{N}=4},\alpha_s) \end{aligned}$$
- Basso-Korchemsky reciprocity provides the following relation between spacelike and timelike twist 2 anomalous dimensions
$$2\gamma_S^{\mathcal{N}=4}(N,\alpha_s) = 2\gamma_T^{\mathcal{N}=4}(N+2\gamma_S^{\mathcal{N}=4},\alpha_s)$$
- We then find $\gamma_J^{\mathcal{N}=4}(\alpha_s)=\gamma_S^{\mathcal{N}=4}(1,\alpha_s)$ as required. Interesting relation between spacelike and timelike dynamics.

Partonic Interpretation

- Scaling has a simple interpretation from parton splitting:



$$\frac{1}{p^2} \sim \frac{1}{E_1 E_2 \theta_{12}^2} \sim \frac{1}{E_1 E_2 |z_1 - z_2|^2}$$



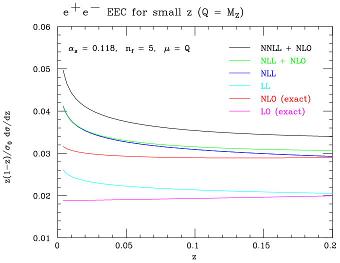
- Small angle enhancement of the correlation function

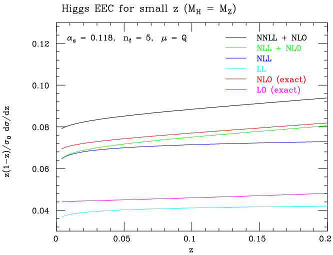
$\Longrightarrow$ reason for jets at weak coupling.

NNLL+NLO Results

- Resumed results at NNLL+NLO:

Quark Jets (From e^+e^-) Gluon Jets (From Higgs)



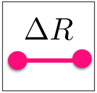


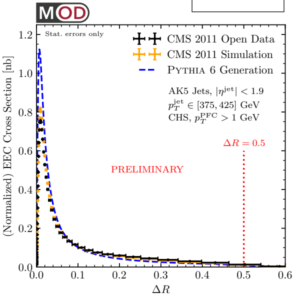
- Distribution depends very sensitively on quark vs gluon!
- In a unitary CFT, $\gamma_N > 0$. In QCD there is an interplay between the β-function, and the $\gamma_N > 0$: for gluons $\gamma_N > 0$ wins and they behave quite like in a CFT, for quarks the β function wins.

Test of 2-point Correlator with Open Data

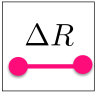
- Scaling of two-point correlator:

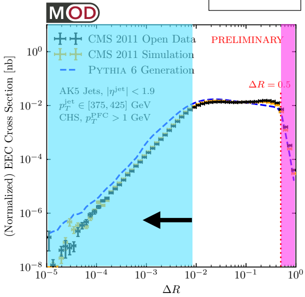
$$\frac{d\sigma}{d\Delta R}$$





$$\Delta R \frac{d\sigma}{d\Delta R}$$





Non-Perturbative

- Perturbative, single log scaling over wide range (like SD mass).
- Note: no grooming was required to make it single log!

Projected Energy Correlators

- How can we generalize this to obtain a family of "scaling observables"?
- We can reduce higher point correlators by integrating out shape information, keeping only the longest side x_L. This is a proxy for its size.

$$\frac{d\sigma^{[N]}}{dx_L} = \sum_n \sum_{1\le i_1,\ldots,i_N\le n} \int d\sigma_{e^+e^-\to X_n} \frac{\prod_{a=1}^N E_{i_a}}{Q^N} \cdot \delta(x_L - \max\{R_{i_1 i_2}, R_{i_1 i_3}, \ldots, R_{i_{N-1} i_N}\})$$

- This directly generalizes the two point correlator, and we will see it inherits its nice properties, in particular, the scaling with twist-2 spin-j operators.

Projected Energy Correlators

- In analogy with the two point correlator one can derive a timelike factorization formula for the ν-point projected correlator

$$\boxed{\Sigma^{[\nu]}(x_L, \ln\frac{Q^2}{\mu^2}) = \int_0^1 dx\, x^\nu \vec{J}^{[\nu]}(\ln\frac{x_L x^2 Q^2}{\mu^2}) \cdot \vec{H}(x, \frac{Q^2}{\mu^2})}$$

- The hard and jet functions satisfy the RGs:

$$\frac{d\vec{H}(x, \ln\frac{Q^2}{\mu^2})}{d\ln\mu^2} = -\int_x^1 \frac{dy}{y}\widehat{P}(y)\cdot\vec{H}\left(\frac{x}{y}, \ln\frac{Q^2}{\mu^2}\right)$$

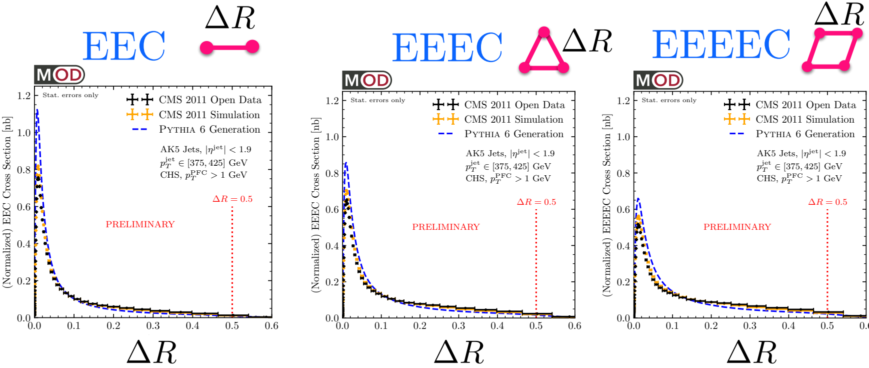
$$\frac{d\vec{J}^{[\nu]}(\ln\frac{x_L Q^2}{\mu^2})}{d\ln\mu^2} = \int_0^1 dy\, y^\nu \vec{J}^{[\nu]}(\ln\frac{x_L y^2 Q^2}{\mu^2})\cdot\widehat{P}(y)$$

- In a CFT, the projected ν point correlator exhibits a powerlaw scaling with exponent the twist-2 spin-ν anomalous dimension:

$$\boxed{\frac{d\sigma^{[\nu]}}{dx_L} = C^{[\nu]}(\alpha_s)\gamma_{J[\nu]}^{\mathcal{N}=4}(\alpha_s)\,\frac{x_L^{\gamma_{J[\nu]}^{\mathcal{N}=4}(\alpha_s)}}{x_L}}$$

Projected Energy Correlators

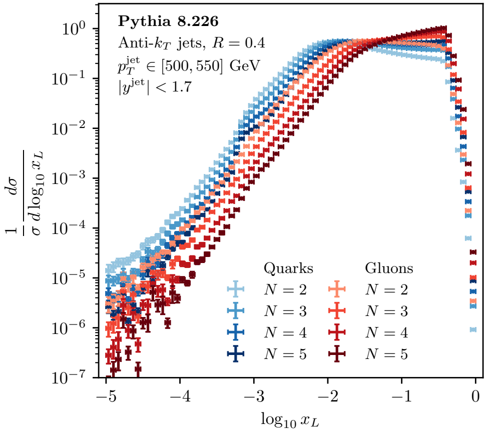
- We can probe these scalings in open data:



- First theoretically understood probes of higher point correlations!

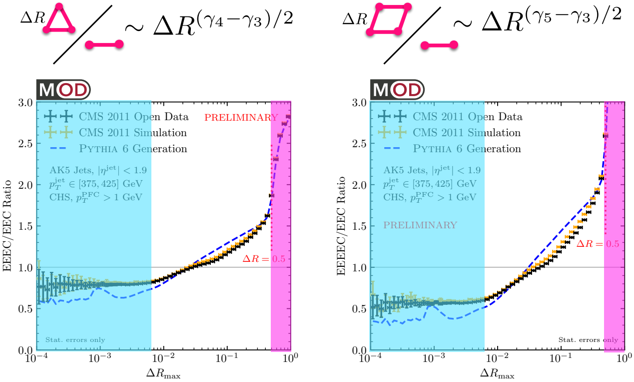
Behavior of Projected Correlators

- Generalizes the two point correlator to an infinite family of single logarithmic (groomed mass like) observables.



- Multiple observables of same family $\Longrightarrow$ can take ratios!
- Ratios of correlators offer a particularly robust observable.

$\boxed{\text{Scaling Behavior } \checkmark}$



- Slope is directly proportional to α_s.

NLL Calculation

- NLL calculation requires the 2 loop anomalous dimensions, and the one loop jet function constants.
- It is well known that the twist-2 spin-j anomalous dimensions are analytic functions of j (harmonic sums).
- Remarkably, we find that the jet function constants are an analytic function of $\nu = N$.
- In $\mathcal{N} = 4$, we find an extremely simple result:

$$2^\nu J_1^{\mathcal{N}=4,[\nu]} = -8N_c(\Psi(\nu)+\gamma_E)\left(\frac{1}{\epsilon} - \ln\frac{x_L Q^2}{\mu^2}\right) - 4N_c[\pi^2 + 2(\Psi(\nu)+\gamma_E)^2 - 6\Psi'(\nu)]$$

- Close connection to field theoretic quantities leads to remarkable simplicity.
- Enables calculation to NLL for all N!

NLL Calculation

- Result in QCD has more complicated rational dependence on ν

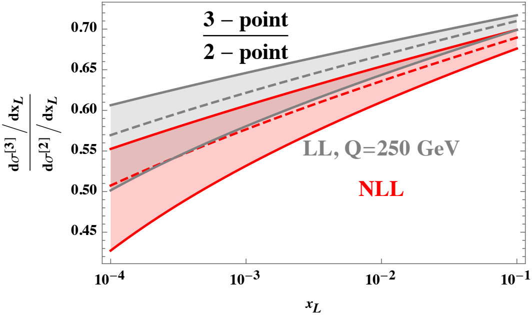
$$
\begin{aligned}
J_1^{q,[\nu]} &= C_F \left[\frac{3(\nu-1) - 4(\nu+1)(\Psi(\nu)+\gamma_E)}{\nu+1} \left(\frac{1}{\epsilon} - \ln \frac{x_L Q^2}{\mu^2} \right) \right. \\
&\quad \left. + \frac{13\nu^3 + 24\nu^2 - 25\nu - 12}{\nu(\nu+1)^2} - 4(\Psi(\nu)+\gamma_E)^2 - \frac{12(\Psi(\nu)+\gamma_E)}{\nu+1} + 12\Psi'(\nu) - 2\pi^2 \right],
\end{aligned}
$$

$$
\begin{aligned}
J_1^{g,[\nu]} &= \left[C_A \left(\frac{(\nu-1)\left(11\nu^2+53\nu+66\right)}{3(\nu+1)(\nu+2)(\nu+3)} - 4(\Psi(\nu)+\gamma_E) \right) - \frac{2(\nu-1)\left(\nu^2+4\nu+6\right) n_f}{3(\nu+1)(\nu+2)(\nu+3)} \right] \left(\frac{1}{\epsilon} - \ln \frac{x_L Q^2}{\mu^2} \right) \\
&\quad + C_A \left[\frac{2\left(67\nu^7 + 804\nu^6 + 3634\nu^5 + 7380\nu^4 + 4723\nu^3 - 5520\nu^2 - 8712\nu - 2376\right)}{9\nu(\nu+1)^2(\nu+2)^2(\nu+3)^2} - 4(\Psi(\nu)+\gamma_E)^2 \right. \\
&\quad \left. - \frac{8\left(2\nu^2+9\nu+11\right)(\Psi(\nu)+\gamma_E)}{(\nu+1)(\nu+2)(\nu+3)} + 12\Psi'(\nu) - 2\pi^2 \right] \\
&+ n_f \left[\frac{-23\nu^7 - 276\nu^6 - 1190\nu^5 - 2376\nu^4 - 1703\nu^3 + 1644\nu^2 + 3060\nu + 864}{9\nu(\nu+1)^2(\nu+2)^2(\nu+3)^2} + \frac{4\left(\nu^2+3\nu+4\right)(\Psi(\nu)+\gamma_E)}{(\nu+1)(\nu+2)(\nu+3)} \right]
\end{aligned}
$$

- Principle of maximal transcendentality is obeyed.
- Very interesting to calculate these constants to two loops.

$3/2$ Ratio at NLL

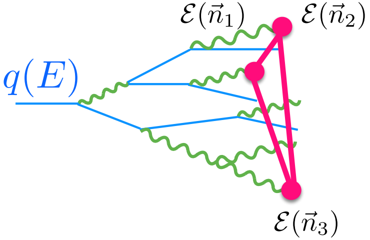
- Example: $3/2$ point ratio for quark jets.



(scale variation is by a factor of 5 instead of the standard 2)

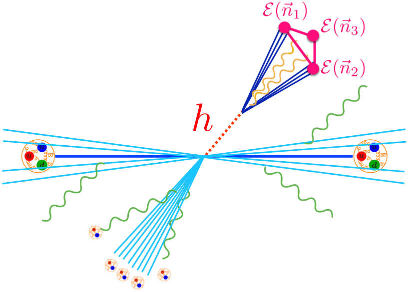
- Hope to extend to NNLL (single log) very shortly. We are missing one number, preliminary tests show significant further reduction in scale variation.
- Promising for precision extraction of α_s.

Beyond Scaling: Shape Dependence of the Three Point Correlator



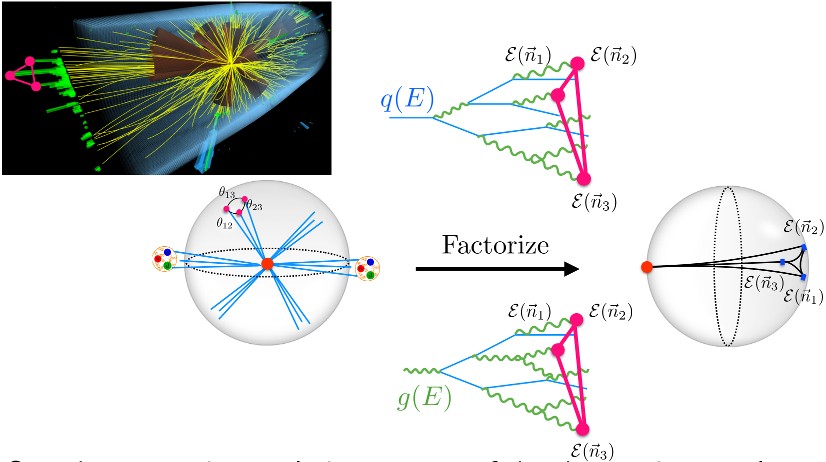
The Celestial Sphere and Multi-Point Correlators

- Full shape dependence of higher point correlators probes detailed aspects of theory. (analogy 2 point vs 3 point correlators for CMB.)
- Interesting for:
  - Probing $1 \to 3$ splitting. e.g. Monte Carlo tuning?
  - Probing detailed structure of quark and gluon jets.
- Multi-point correlations are central in jet substructure.
- Unfortunately no previous analytic calculations.



The Underlying Field Theoretic Problem

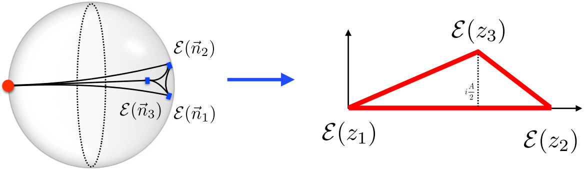
- Shape dependence of multi-point correlators described by universal jet functions.



- Start by computing analytic structure of the three point correlator.

The Celestial Sphere and $\mathcal{N} = 4$ SYM

- $\mathcal{N} = 4$ Super Yang-Mills is a field theory similar to QCD, but it exhibits scale (conformal) symmetry.
- We can use this theory as a guide for understanding QCD, where scale symmetry is weakly broken by the β function.
- To manifest these symmetries, it is convenient to exchange vectors with complex coordinates z_i on the celestial sphere:



Parametrizing a Unit Triangle

- Since we understand scaling, can focus on a unit triangle.
- Parametrize unit triangle using a complex variable z:

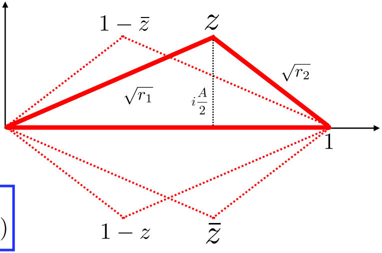
$S_3 \times \mathbb{Z}_2$ Symmetry:

$$z \to 1 - z \qquad z \to 1 - \frac{1}{z}$$

$$z \to \frac{1}{z} \qquad z \to \frac{z}{1-z}$$

$$z \to \frac{1}{1-z} \qquad z \to \bar{z}$$

$$\boxed{\begin{aligned} r_1 &= z\bar{z} \\ r_2 &= (1-z)(1-\bar{z}) \end{aligned}}$$



- Correlator is a single valued function of $z, \bar{z}$.

Result in $\mathcal{N} = 4$

- Result in $\mathcal{N} = 4$ takes quite a simple form

$$
\begin{aligned}
G(z) = {} & \frac{(1+|z|^2+|1-z|^2)}{2|z|^2|1-z|^2}(1+\zeta_2) + \frac{(-1+|z|^2+|z|^4-|z|^6-|1-z|^4-|z|^2|1-z|^4+2|1-z|^6)}{2|z|^2|1-z|^2(z-\bar{z})^2}\log|1-z|^2 \\
& + \frac{(-1-|z|^4+2|z|^6+|1-z|^2-|z|^4|1-z|^2+|1-z|^4-|1-z|^6)}{2|z|^2|1-z|^2(z-\bar{z})^2}\log|z|^2 \\
& + \frac{|z|^4-1}{2|z|^2|1-z|^4}D_2^+(z) + \frac{|1-z|^4-1}{2|z|^4|1-z|^2}D_2^+(1-z) + \frac{(|z|^2-|1-z|^2)(|z|^2+|1-z|^2)}{2|z|^2|1-z|^2}D_2^+\left(\frac{z}{z-1}\right) \\
& + \frac{2iD_2^-(z)}{2|1-z|^4|z|^4(z-\bar{z})^3}p_3(|z|^2,|1-z|^2)
\end{aligned}
$$

- Expressed in terms of rational prefactors and the following weight 2 functions

$$
\begin{aligned}
2iD_2^-(z) &= \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\left(\log(1-z) - \log(1-\bar{z})\right)\log(z\bar{z}) \\
D_2^+(z) &= \left(\mathrm{Li}_2\left(1-|z|^2\right) + \frac{1}{2}\log\left(|1-z|^2\right)\log\left(|z|^2\right)\right)
\end{aligned}
$$

A Surprising Duality

- Interestingly, all rational prefactors can be removed by writing the result in terms of dual Feynman integrals.
- The integrals appearing are over the energy fractions of splitting functions, with angles fixed:

$$\frac{1}{\sigma_{\rm tot}}\frac{d^3\Sigma}{dx_1dx_2dx_3}=\mathcal{N}\int d\omega_1 d\omega_2 d\omega_3\delta(1-\omega_1-\omega_2-\omega_3)\frac{(\omega_1\omega_2\omega_3)^2}{16}\times P_{1\to3}$$

- Write all Mandelstam's in terms of celestial coordinates: $s_{ij}=Q^2\omega_i\omega_j|z_i-z_j|^2$.
- Consider for simplicity a particular term in the splitting function:

$$P_{1\to3}\supset\frac{1}{\omega_1\omega_3 s_{12}s_{123}}\sim\frac{1}{\omega_1^2\omega_2\omega_3|z_{12}|^2s_{123}}$$

writing $\quad s_{123}=Q^2(\omega_1\omega_2z_{12}^2+\omega_1\omega_3z_{13}^2+\omega_2\omega_3z_{23}^2)\,,$

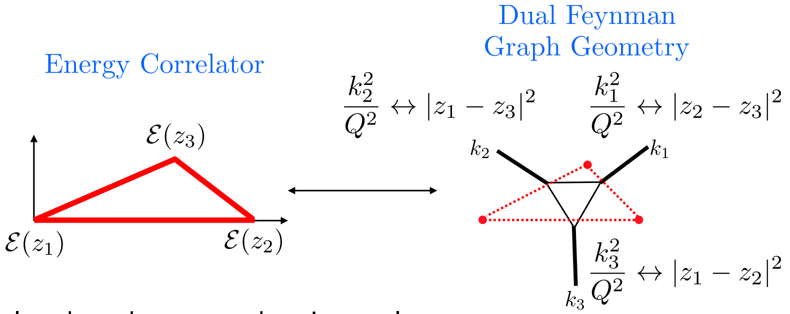
$$\to\mathcal{N}\frac{1}{2|z_{12}|^2}\times\int d\omega_1 d\omega_2 d\omega_3\delta(1-\omega_1-\omega_2-\omega_3)\frac{\omega_2\omega_3}{\omega_1\omega_2z_{12}^2+\omega_1\omega_3z_{13}^2+\omega_2\omega_3z_{23}^2}$$

A Surprising Duality

- This is recognized as a dual Feynman parameter integral, where the $|z_{ij}|^2$ are the dual coordinates.

$$x_i^\mu - x_{i+1}^\mu = p_i^\mu \,, x_{ij}^2 = (x_i - x_j)^2 = (p_i + \cdots p_{j-1})^2$$

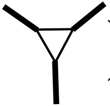
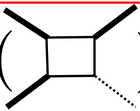
$$x_{ij}^2 \leftrightarrow |z_{ij}|^2$$



- Related to three mass box integral.

Result in $\mathcal{N} = 4$ Super Yang Mills

- Obtain one line result for three point correlator in $\mathcal{N} = 4$:
- Schematically:

$f_4 = \mathrm{Dual}\Big($  $\Big) + \mathrm{Dual}\Big($  $\Big)$

- Explicitly:

$$\boxed{\frac{d^2\sigma}{dz\,d\bar{z}} \propto \left(\mathcal{J}^{(d=8)}(2,2,1) + \mathcal{J}^{(d=10)}(2,2,2,\widetilde{1}) + \frac{\zeta_2 - 1}{2x_L(1-z)(1-\bar{z})} \right)}$$

- All rational prefactors of transcendental functions eliminated.
- Why?
- Does this persist to higher loop orders, higher points?

Shape Dependence in QCD

- Shape dependence in QCD involves same transcendental functions

Quarks:

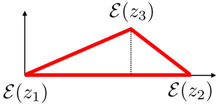
Gluons:

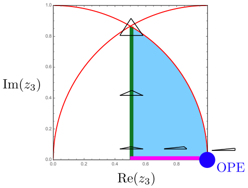
- ...but many more rational prefactors...

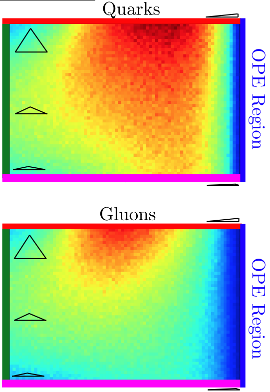
Shape Dependence

- A remarkably detailed probe of QCD in jets!

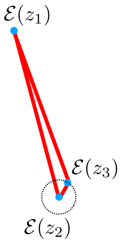
Shape ✓







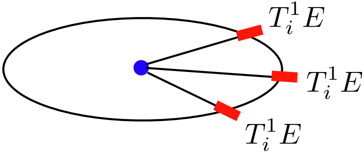
$$\langle q(E)\mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\mathcal{E}(\vec{n}_3)\rangle$$



$$\langle g(E)\mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\mathcal{E}(\vec{n}_3)\rangle$$

- Directly probe celestial correlators.
- Useful for probing $1 \to 3$ splitting, Parton Shower tuning, ...

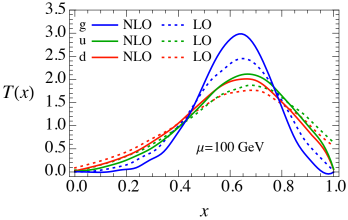
Making Tracks Tractable



Track Functions

- Tracks offer many experimental advantages.
- There is an elegant formalism for incorporating tracks (Chang, Procura, Waalewijn, Thaler 2013) using Track Functions, $T_i(x)$.
- Track functions are a non-perturbative function describing energy fraction of a parton going into tracks, $\bar{p}_i^\mu = x p_i^\mu + \mathcal{O}(\Lambda_{\mathsf{QCD}})$. (Analogous to a fragmentation function).

$$\int_0^1 dx\, T_i(x,\mu) = 1$$



- It obeys a non-linear RG:

$$\mu \frac{d}{d\mu} T_i(x,\mu) = \frac{1}{2} \sum_{j,k} \int dz dx_j dx_k \frac{\alpha_s(\mu)}{\pi} P_{i\to jk}(z)$$
$$\cdot T_j(x_j,\mu) T_k(x_k,\mu) \delta[x - z x_j - (1-z) x_k]$$

Track Functions

- Why hasn't it been put to use for "standard Jet Substructure Observables"?
  - Calculations are very complicated.
  - Calculations involve full shape of non-perturbative $T(x)$.
- Consider e.g. Track Thrust at LO [Chang, Procura, Thaler, Waalewijn]

$$\frac{d\sigma}{d\bar{\tau}} = \int_0^1 dy_1 dy_2 \frac{d\bar{\sigma}(\mu)}{dy_1 dy_2} \int_0^1 dx_1 dx_2 dx_3 T_q(x_1) T_q(x_2) T_g(x_3) \delta\left[\bar{\tau} - \bar{\tau}(y_1, y_2, x_1, x_2, x_3)\right]$$

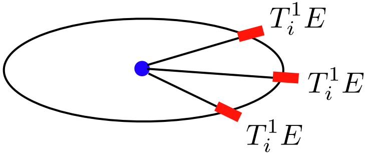
$$\frac{d\bar{\sigma}(\mu)}{dy_1 dy_2} = \sigma_0 \frac{\alpha_s(\mu) C_F}{2\pi} \frac{\theta(y_1 + y_2 - 1)(y_1^2 + y_2^2)}{(1-y_1)(1-y_2)}$$

where $y_1 = 2E_q/Q$, $y_2 = 2E_{\bar{q}}/Q$ are the normalized parton energy, and the measurement function for track thrust is

$$\begin{aligned}\bar{\tau} = {} & \theta[x_1 x_3(1-y_2) - x_1 x_2(1-y_3)] \cdot \theta[x_2 x_3(1-y_1) - x_1 x_2(1-y_3)] x_1 x_2 (1-y_3) \\ & + \theta[x_2 x_3(1-y_1) - x_1 x_3(1-y_2)] \theta[x_1 x_2(1-y_3) - x_1 x_3(1-y_2)] x_1 x_3(1-y_2) \\ & + \theta[x_1 x_3(1-y_2) - x_2 x_3(1-y_1)] \cdot \theta[x_1 x_2(1-y_3) - x_2 x_3(1-y_1)] x_2 x_3(1-y_1)\end{aligned}$$

Tracks and Energy Correlators

- Energy correlators are weighted by energy flow through detector cells as a function of angle.
- How to go from full calorimeter to tracks? simply multiply by "average energy deposited into tracks".



$$E_i \to \int dx_i \, x_i T_i(x_i) E_i = T_i^{(1)} E_i$$

- Upshot: Any perturbative calculation of energy correlators that can be done, can also be done on tracks just by weighting pieces of calculation by $T_i^{(1)}$! (higher moments only appear as contact terms)

Two Point Correlator on Tracks

- As an example, consider LO calculation of EEC on tracks. Just weight qg correlation by $T_q^{(1)}T_g^{(1)}$ and $q\bar{q}$ correlation by $T_q^{(1)})^2$:

$$\mathsf{EEC}^{\mathsf{tr}}(z) = \sigma_0 \frac{\alpha_s}{2\pi} C_F \left((T_q^{(1)})^2 I_1(z) + 2T_q^{(1)} T_g^{(1)} I_2(z) \right)$$

$$I_1 = \left(\frac{1}{6z^2} + \frac{1}{z^3} - \frac{4}{z^4} \right) \frac{1}{1-z} + \left(\frac{3}{z^4} - \frac{4}{z^5} \right) \frac{\ln(1-z)}{1-z},$$

$$I_2 = \left(\frac{53}{12z^2} - \frac{41}{4z^3} + \frac{13}{2z^4} \right) \frac{1}{1-z} + \left(\frac{13}{2z^5} - \frac{7}{z^4} + \frac{2}{z^3} \right) \ln(1-z)$$

- Or the calculation of jet functions in the collinear limit with/without tracks

$$j_g(z) = \delta(z) + \frac{\alpha_s}{4\pi} \left(\frac{14}{5} C_A + \frac{1}{5} n_f \right) \left[\frac{1}{z} \right]_+ + \delta(z) \frac{\alpha_s}{4\pi} \left(-\frac{898}{75} C_A - \frac{14}{25} n_f \right)$$

$$j_g^{\mathsf{tr}}(z) = \delta(z) T_g^{(2)} + \frac{\alpha_s}{4\pi} \left(\frac{14}{5} C_A (T_g^{(1)})^2 + \frac{1}{5} n_f (T_q^{(1)})^2 \right) \left[\frac{1}{z} \right]_+$$

$$+ \delta(z) \frac{\alpha_s}{4\pi} \left(-\frac{898}{75} C_A (T_g^{(1)})^2 - \frac{14}{25} n_f (T_q^{(1)})^2 \right)$$

The Underlying Reason

- This is directly due to the fact that weighted observables are defined in terms of a finite number of energy correlators, while "standard observables" involve an infinite number (hence all moments)

$$\boxed{\langle 0|\mathcal{O}\mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\mathcal{O}^\dagger|0\rangle} \implies \text{Easy for Tracks!}$$

$$\boxed{\langle 0|\mathcal{O}\delta(e_2 - f(\mathcal{E}(\vec{n}_1),\mathcal{E}(\vec{n}_2))\mathcal{O}^\dagger|0\rangle} \implies \text{Hard for Tracks!}$$

- Clear manifestation of difference in complexity. Can't just work harder to overcome.

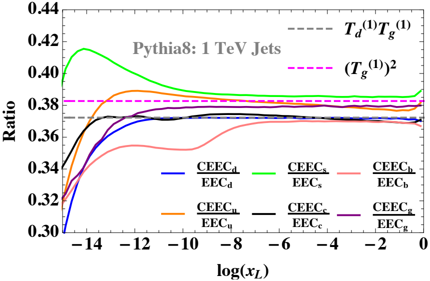
Tracks and Resummation

- Interfaces nicely with resummation. e.g.Two point correlator at LL for pure gluons:

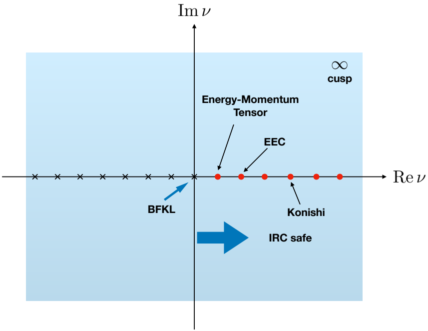
$$\Sigma^{[2]}(x_L) = \frac{1}{2}\left(\frac{\alpha_s(\sqrt{x_L}Q)}{\alpha_s(Q)}\right)^{-\frac{\gamma^{(0)}(3)}{\beta_0}}$$

$$\Sigma^{[2]}_{\text{tr}}(x_L) = \frac{1}{2}[T_g^{(1)}(Q)]^2\left(\frac{\alpha_s(\sqrt{x_L}Q)}{\alpha_s(Q)}\right)^{-\frac{\gamma^{(0)}(3)}{\beta_0}}$$

- With both quarks and gluons there is a matrix, but still straightforward...



Bonus: The Analytic Continuation of Jet Substructure



The Analytic Continuation of Jet Substructure

- Many jet substructure observables have been proposed.
- Probe diverse physics. e.g. Jet mass to multiplicitiy.
- How can we organize them?
- How can we understand what physics we can probe with jets? and extend what we can probe.
- Ultimately want to make this precise and link it to the underlying field theory.

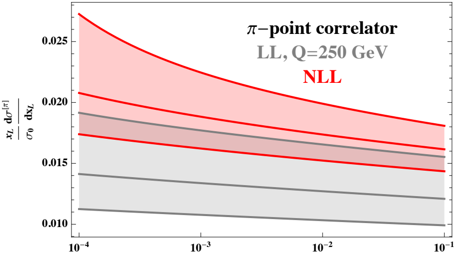
Analytic Continuation

- Results for the ν point correlators are analytic (more precisely meromorphic) functions of ν:

$$J_1^{q,[\nu]} = C_F \left[\frac{3(\nu-1) - 4(\nu+1)(\Psi(\nu)+\gamma_E)}{\nu+1} \left(\frac{1}{\epsilon} - \ln \frac{x_L Q^2}{\mu^2} \right) + \frac{13\nu^3 + 24\nu^2 - 25\nu - 12}{\nu(\nu+1)^2} - 4(\Psi(\nu)+\gamma_E)^2 - \frac{12(\Psi(\nu)+\gamma_E)}{\nu+1} + 12\Psi'(\nu) - 2\pi^2 \right],$$

- What is the meaning of this? Is it a mathematical curiosity, or can it be measured?

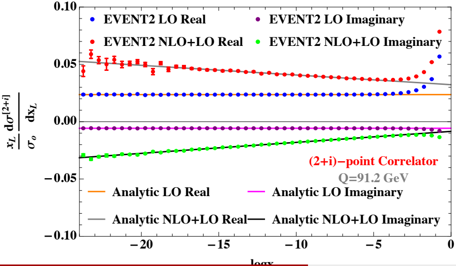
π-Point Correlator



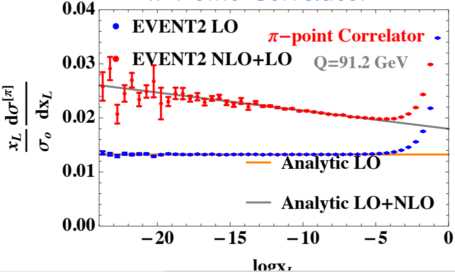
- It turns out that we can define a ν-point correlator that can be measured on actual jets. It correlates infinite combinations of particles (up to the fact that there are a finite number in a jet). The precise definition is given in the paper.

- Test by applying this algorithm in Monte Carlo and comparing with our analytic calculation

$2+i$-Point Correlator



π-Point Correlator

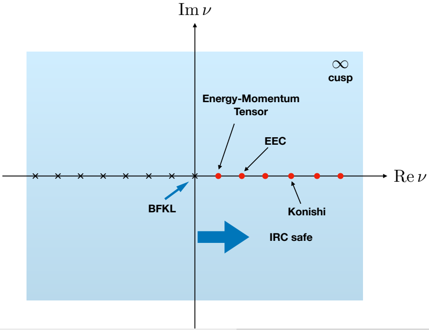


Wandering in the Complex Plane

- Interesting structure in the complex plane.
- Pole at $\nu \to 0$ corresponds to multiplicity with infrared resolution x_L, but approach governed by BFKL. Can we probe BFKL physics in timelike jets?

$$J_1^{q,[\nu]} = C_F \left[\frac{3(\nu-1) - 4(\nu+1)(\Psi(\nu) + \gamma_E)}{\nu+1} \left(\frac{1}{\epsilon} - \ln \frac{x_L Q^2}{\mu^2} \right) + \frac{13\nu^3 + 24\nu^2 - 25\nu - 12}{\nu(\nu+1)^2} - 4(\Psi(\nu)+\gamma_E)^2 - \frac{12(\Psi(\nu)+\gamma_E)}{\nu+1} + 12\Psi'(\nu) - 2\pi^2 \right],$$

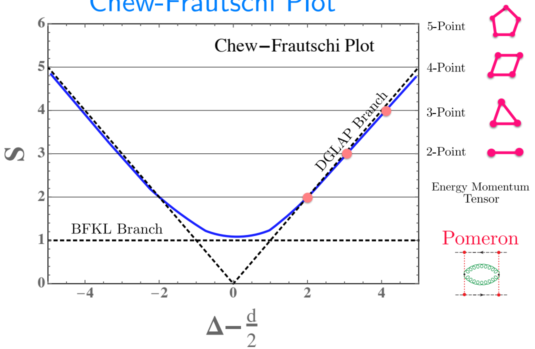
- Places jet substructure observables into an analytic family.
- What other physics can we probe?



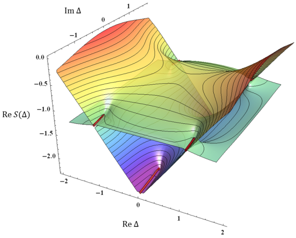
Chew-Frautschi and Jet Substructure

- Measuring Energy-Energy Correlators allows direct reconstruction of the spectrum of operators in the theory.

Chew-Frautschi Plot



Twist Two Spectral Surface

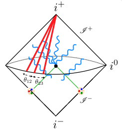


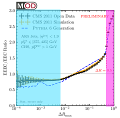
[Gromov, Levkovich-Maslyuk, Sizov]

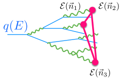
- Measurement would be a remarkable probe of field theory!
- Underlying reason for simplicity of the EECs: this surface exists, and is smooth (describable by an analytic function)!

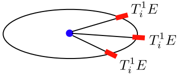
Summary

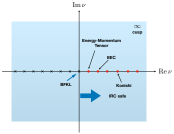
- Weighted cross sections and energy correlators offer many advantages.
- Projected correlators probe scaling behavior.
- Full shape dependence of higher point correlators analytically computed.
- Track information can be incorporated in high order perturbative calculations.
- Non-integer point correlators can be experimentally measured, and probe interesting physics.

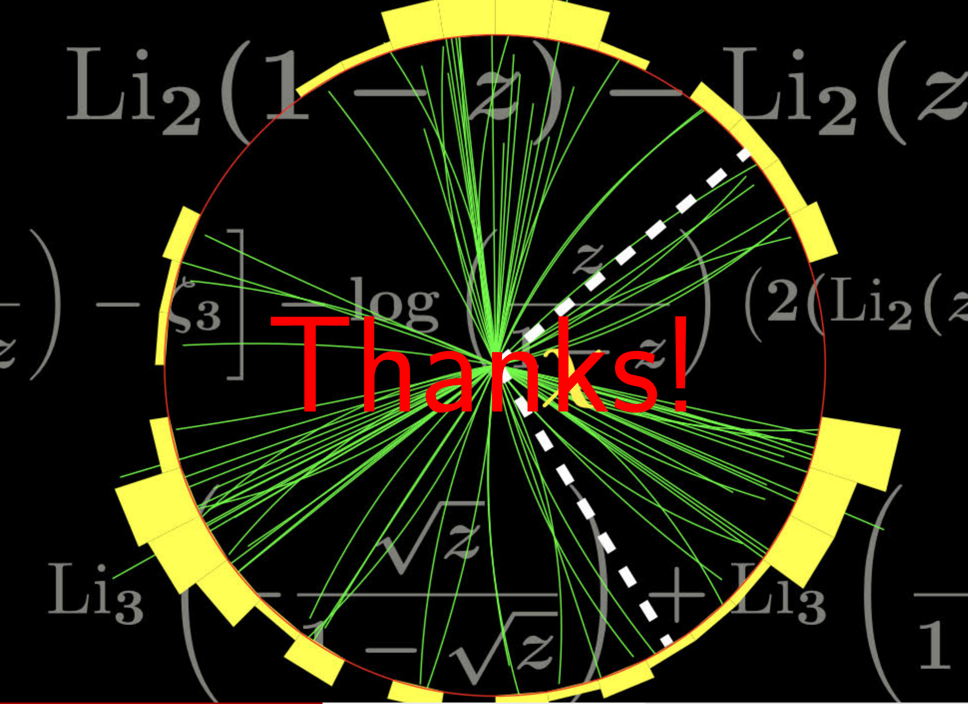












Thanks!