THEORY / PHENOMENOLOGY OF MPI

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- Incoherence and MPI

- The Probabilistic Picture of MPI: A Functional Approach

- Cancellation of Unitarity Corrections

- Inclusive and "Exclusive" Cross Sections, Sum Rules

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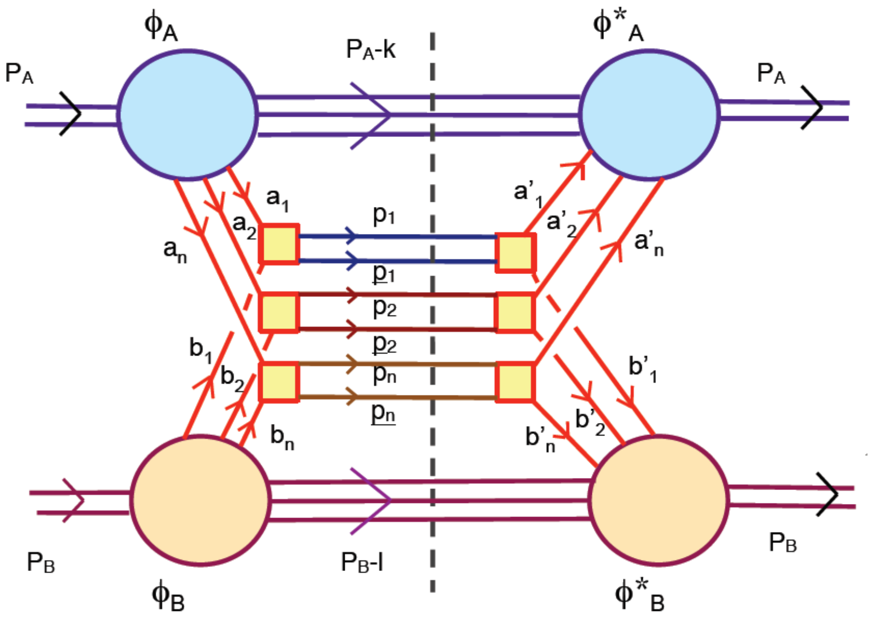
1. Incoherence and MPI

A given final state may be generated by various competing processes, characterized by different numbers of partonic collisions, which however populate the final state phase space in a different way. One hence has ***diagonal*** and ***off-diagonal*** contributions to the cross section.

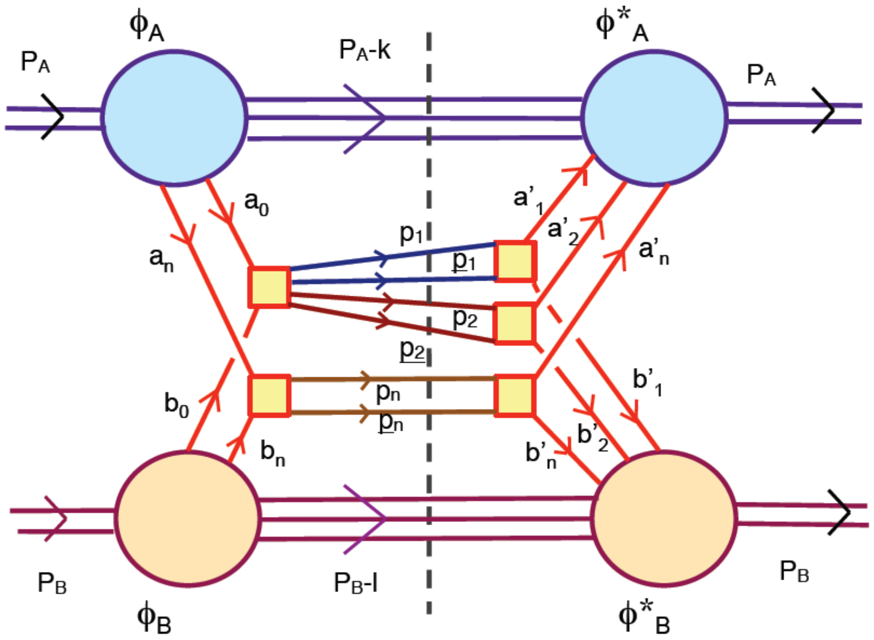


diagonal contribution

A ***diagonal contribution***, corresponding to a term with n-partons in the initial state, is given by the ***incoherent super-positions of n disconnected parton interactions, localized in n different points in transverse space***

MPI are generated by the increasingly large flux of partons at small fractional momenta. Once the final state is fixed, the ***parton flux is maximized*** in the channel where the ***hard component of the interaction is maximally disconnected.***

The flux factor is hence smaller in off-diagonal terms. Diagonal and off-diagonal terms populate however the final state phase space in a different way and, in the kinematical regions where the contributions to the cross section are similar, important interference effects should be expected.

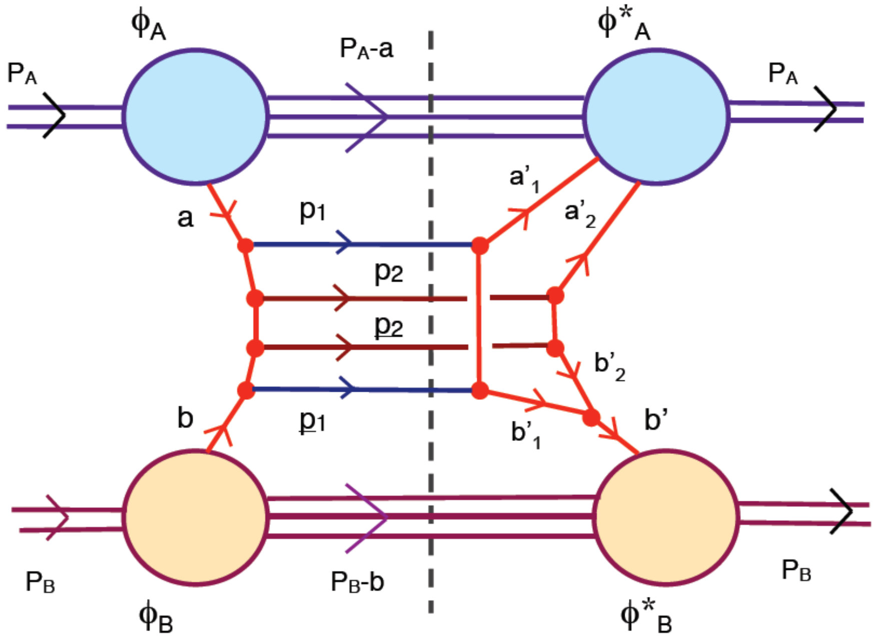


off diagonal contribution

The hard component of the interaction corresponding to ***off diagonal contributions*** is disconnected and ***localized in no more than n-1 points in transverse space***.

One may hence argue that ***interference terms*** do not represent corrections to the n-partons scattering inclusive cross section. They rather ***correct the (n-1)-partons*** (or less) ***scattering inclusive cross section***.

Partons are in fact localized in the hadron by the momenta exchanged in the interaction. When partons within a given hadron are localized inside non overlapping regions, much smaller as compared to the hadron size, they are only connected one with another through soft exchanges and the picture of independent parallel collisions is meaningful. If, on the contrary, partons are localized by the interaction within overlapping regions, much smaller as compared to the hadron size, they may interact by exchanging momenta compatible with their virtuality.



Because of the localization of the hard component of the interaction, the problem of ***interference*** is hence strictly linked to the problem to evaluate the ***scattering amplitude, at higher orders*** in the coupling constant and including higher twists in the hadron structure.

Given the two very different scales, the hadron size and of the large momenta exchanged, the hard component of the interaction may be disconnected, with the different hard parts linked only through soft exchanges and localized in different regions in transverse space. The ***different MPI terms*** are thus conveniently understood as the contributions to the final state due to the ***different disconnected parts of the hard component of the interaction***. In the simplest case, in each different region the interaction may be evaluated at the lowest order in the coupling constant.

In each single partonic collision all transverse momenta balance and a ***MPI process*** contributes to the cross section generating ***different groups of final state partons where the large transverse momenta are compensated separately***.

When MPI are understood in the topological sense described above, different MPI terms, corresponding to different localizations in transverse space, ***do not interfere*** and the final cross section is obtained by the simple superposition of the cross sections, due to the contributions of the different topological configurations of the hard component of the interaction.

MPI hence ADD INCOHERENTLY. A remarkable consequence is that MPI allow a PROBABILISTIC DESCRIPTION

2. The Probabilistic Picture of MPI: A Functional Approach

As Multiple Parton Interactions add ***incoherently***, the problem may be discussed within a ***probabilistic framework***. A functional approach is most general. One may start by introducing the ***exclusive*** n-body parton ***distributions***

$$W_n(u_1 \dots u_n) \quad , \quad u_i \equiv (\mathbf{b}_i, x_i)$$

which represen the probabilities to find the hadron in configurations with n partons with coordinated $u_1 \dots u_n$, where $\mathbf{b}_i$ are the transverse parton coordinates and x_i the fractional momenta.
The ***exclusive*** n-body parton ***distributions*** allow to introduce the multi-parton generating functional $\mathcal{Z}$:

$$\mathcal{Z}[J] = \sum_n \frac{1}{n!} \int J(u_1) \dots J(u_n) W_n(u_1 \dots u_n) du_1 \dots du_n,$$

Probability conservation imposes the normalization condition $\mathcal{Z}[1] = 1.$

The ***exclusive*** n-body parton ***distribution*** are thus the coefficients of the expansion of $\mathcal{Z}$ around 0, the many-body densities, i.e. the ***inclusive distributions*** $D_n(u_1 \dots u_n)$ are the coefficients of the expansion of $\mathcal{Z}$ around 1:

$$D_1(u) = \frac{\partial \mathcal{Z}}{\partial J(u)}\bigg|_{J=1},$$

$$D_2(u_1, u_2) = \frac{\partial^2 \mathcal{Z}}{\partial J(u_1) \partial J(u_2)}\bigg|_{J=1},$$

$$\dots$$

The many-body ***parton correlations*** are introduced by expanding the logarithm of the generating functional $\mathcal{F}[J] = \ln(\mathcal{Z}[J])$ in the vicinity of *J*=1:

$$\mathcal{F}[J] = \int D(u)[J(u) - 1]du + \sum_{n=2}^{\infty} \frac{1}{n!} \int C_n(u_1 \dots u_n)\big[J(u_1) - 1\big] \dots \\ \dots \big[J(u_n) - 1\big] du_1 \dots du_n$$

To sum all disconnected collisions one may start from the following expression of the hard cross section

$$\sigma_H = \int d^2\beta \sigma_H(\beta) \quad \text{where} \quad \sigma_{inel} = \sigma_H + \sigma_{soft}$$

Prob. to find the hadron *B* in a configuration with *m* partons

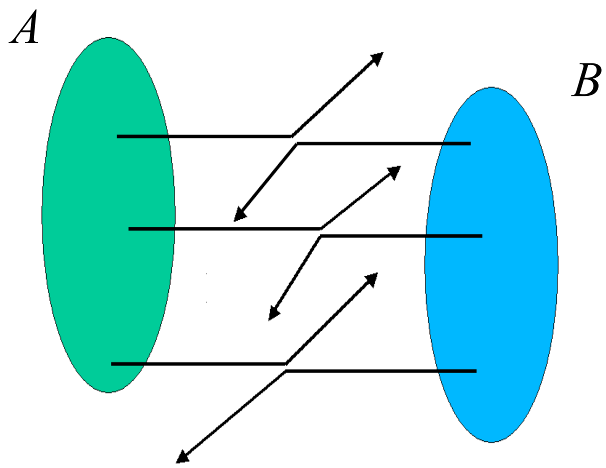
$$\sigma_H = \int d\beta \int \sum_n \frac{1}{n!} \frac{\partial}{\partial J(u_1)} \cdots \frac{\partial}{\partial J(u_n)} \mathcal{Z}_A[J] \times \sum_m \frac{1}{m!} \frac{\partial}{\partial J'(u'_1 - \beta)} \cdots \frac{\partial}{\partial J'(u'_m - \beta)} \mathcal{Z}_B[J'] \\ \times \left\{1 - \prod_{i=1}^{n} \prod_{j=1}^{m} [1 - \hat{\sigma}_{i,j}(u, u')]\right\} \prod du du' \Big|_{J=J'=0}$$

Prob. to find the hadron *A* in a configuration with *n* partons

Prob. to have at least one interaction between the two configurations

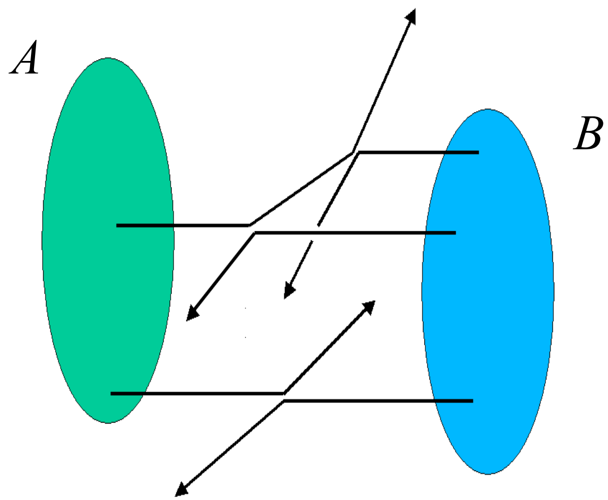
This expression for σ_H includes all possible interaction with *on-shell intermediate states*, between any configuration with n partons of hadron A and any configuration with m partons of hadron B.

Interestingly one may show that in the above expression of σ_H all unitarity corrections cancel when one evaluates the average number of collisions: $\langle N \rangle \, \sigma_H = \sigma_S$



Both disconnected collisions

and rescatterings



are included in $\boldsymbol{\sigma_H}$

To obtain only the ***disconnected collisions***, we remove all addenda with repeated indices in the interaction probability

$$\left\{1-\prod_{i,j}^{n,m}[1-\hat{\sigma}_{ij}]\right\} \Rightarrow \sum_{ij}\hat{\sigma}_{ij}-\frac{1}{2!}\sum_{ij}\sum_{k\neq i,l\neq j}\hat{\sigma}_{ij}\hat{\sigma}_{kl}+\ldots$$

Because of the symmetry of the derivative operators, the expression may be replaced by

$$nm\hat{\sigma}_{11}-\frac{1}{2!}n(n-1)m(m-1)\hat{\sigma}_{11}\hat{\sigma}_{22}+\ldots$$

The sums can be performed and the cross section is thus expressed in a ***closed analytic form***

$$\sigma_H(\beta)=\Big[1-\exp\big(-\partial\cdot\hat{\sigma}\cdot\partial'\big)\Big]\mathcal{Z}_A[J]\mathcal{Z}_B[J']\Big|_{J=J'=1}$$

Notice that ***all MPI are included in the expression above, while multi-parton correlations are kept into account at all orders***.

3. Cancellation of Unitarity Corrections

The hard cross section may be expressed as a sum of MPI

$$
\begin{aligned}
\sigma_H(\beta) &= \Big[1 - \exp\big(-\partial \cdot \hat{\sigma} \cdot \partial'\big)\Big] \mathcal{Z}_A[J]\mathcal{Z}_B[J'] \Big|_{J=J'=1} \\
&= \sum_{N=1}^{\infty} \frac{\big(\partial \cdot \hat{\sigma} \cdot \partial'\big)^N}{N!} e^{-\partial \cdot \hat{\sigma} \cdot \partial'} \mathcal{Z}_A[J]\mathcal{Z}_B[J'] \Big|_{J=J'=1}
\end{aligned}
$$

Rather generally one may now show that the ***average number of collisions*** is given by the ***single scattering inclusive cross section***:

$$
\begin{aligned}
\langle N \rangle \sigma_H(\beta) &= \sum_{N=1}^{\infty} \frac{N\big(\partial \cdot \hat{\sigma} \cdot \partial'\big)^N}{N!} e^{-\partial \cdot \hat{\sigma} \cdot \partial'} \mathcal{Z}_A[J]\mathcal{Z}_B[J'] \Big|_{J=J'=1} \\
&= \partial_{J_1} \cdot \hat{\sigma} \cdot \partial_{J'_1} \sum_{N=0}^{\infty} \frac{\big(\partial \cdot \hat{\sigma} \cdot \partial'\big)^N}{N!} e^{-\partial \cdot \hat{\sigma} \cdot \partial'} \mathcal{Z}_A[J]\mathcal{Z}_B[J'] \Big|_{J=J'=1} \\
&= \big(\partial_{J_1} \cdot \hat{\sigma} \cdot \partial_{J'_1}\big) \mathcal{Z}_A[J]\mathcal{Z}_B[J'] \Big|_{J=J'=1} \\
&= \int D_A(x_1; b_1)\hat{\sigma}(x_1 x'_1) D_B(x'_1; b_1 - \beta) dx_1 dx'_1 d^2 b_1 \equiv \sigma_S(\beta)
\end{aligned}
$$

One may analogously show the validity similar relations for higher moments of the distribution in the Multiplicity of collisions:

Single scatt. inclusive cross section

$$\langle N\rangle\sigma_H=\sigma_S \quad \text{and} \quad \frac{1}{2}\langle N(N-1)\rangle\sigma_H=\sigma_D$$

Double scatt. inclusive cross section

$$\frac{1}{K!}\langle N(N-1)\dots(N-K+1)\rangle\sigma_H=\sigma_K$$

K^{th} scatt. inclusive cross section

Notice that all effects of multi-parton correlations are taken into account in the derivation of the relations above, including the correlations induced by energy conservation

In the case of disconnected collisions all ***unitarity corrections cancel in the inclusive cross section***, which is given by the expression of the single scattering term (AGK cancellation).
The ***inclusive cross section*** is in fact given by the ***average number*** of partonic collision.
The ***K-partons inclusive cross section*** is analogously given by the ***K^{th} moment of the distribution*** in the number of partonic collisions.

When ONLY DISCONNECTED COLLISIONS are included in the interaction, the INCLUSIVE CROSS SECTIONS give the MOMENTS OF THE DISTRIBUTION in the number of hard collisions and are not affected by the unitarity corrections induced by multiple parton interactions with larger numbers of partonic collisions. The statement holds for ANY CORRELATION within the hadron structure

Correlations and σ_H

While keeping into account disconnected multi-parton scatterings only and when ***limiting correlations to the two-body case***, one may express the cross section with a gaussian functional integral, which ***allows obtaining a closed expression for σ_H***

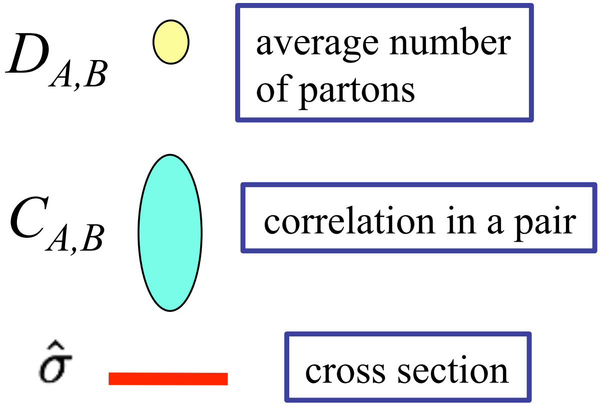
$$\mathcal{F}_{A,B}[J+1]=\int D_{A,B}(u)J(u)du+\frac{1}{2}\int C_{A,B}(u,v)J(u)J(v)dudv$$

D(*u*) is the average number of partons and *C*(*u,v*) the two-body correlation. The final expression obtained for σ_H is :

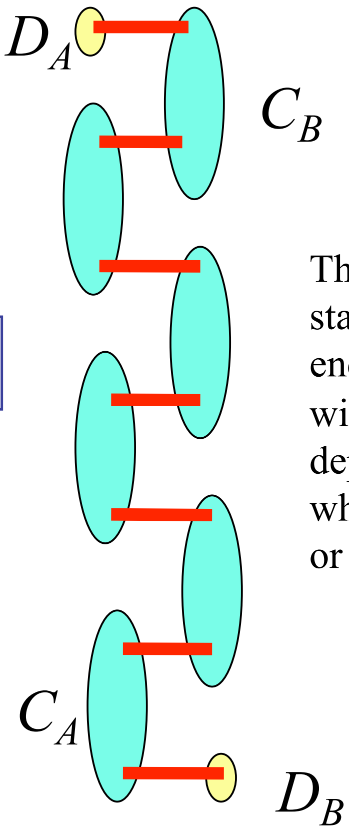
$$\sigma_H(\beta)=1-\exp\left[-\tfrac{1}{2}\sum_n a_n-\tfrac{1}{2}\sum_n b_n/n\right]$$

$$a_n=(-1)^{n+1}\int \quad D_A(u_1)\hat{\sigma}(u_1,u_1')C_B(u_1',u_2')\hat{\sigma}(u_2',u_2)C_A(u_2,u_3)\dots$$
$$\cdots\hat{\sigma}(u_n,u_n')D_B(u_n')\prod_{i=1}^{n}du_idu_i'$$
$$b_n=(-1)^{n+1}\int \quad C_A(u_n,u_1)\hat{\sigma}(u_1,u_1')C_B(u_1',u_2')\dots$$
$$\cdots C_B(u_{n-1}',u_n')\hat{\sigma}(u_n',u_n)\prod_{i=1}^{n}du_idu_i'$$

The structures a_n and b_n allow a graphic representation

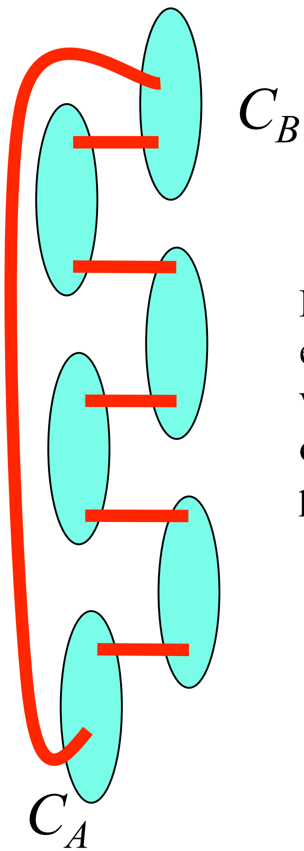


a_n



The chain which starts with A may end either with A or with B, depending whether n is even or odd

b_n



Here only even values of n are possible

4. Inclusive and "Exclusive" Cross Sections, Sum Rules

In proton-proton collisions, the ***inclusive cross sections*** are the ***moments of the distribution in the number of MPI***.

The most basic information on the distribution in the number of collisions, the average number, is hence given by the single scattering inclusive cross section of the QCD parton model. Analogously the K^{th} scattering inclusive cross section gives the K^{th} moment of the distribution in the number of collisions and is related directly to the K-partons distribution of the hadron structure.

A way alternative to the set of moments, to provide the whole information on the distribution, is represented by the set of the different terms of the probability distribution of multiple collisions. Correspondingly, in addition to the set of the inclusive cross sections, one may consider the set of the ***"exclusive" cross sections***, where one selects ***events where only a given number of collisions are present***. The cross sections called now "exclusive" are in fact partially inclusive, since one sums over all large p_t partons outside a given phase space interval of interest and on all soft fragments.

A brief recall about *inclusive* and *exclusive* cross sections:

As a simplest example consider pion production.

***Inclusive* cross section**:

To measure the inclusive cross section ***one has to count with the same weight each produced pion which enters the detector*** (if the case, at different bins of rapidity, transverse momenta etc.).

If n pions are produced in a single event, ***the event contributes n times to the inclusive cross section***. The inclusive cross section is hence an average quantity with respect to the distribution in multiplicity of the produced pions.

***Exclusive* cross section**:

To measure the exclusive cross section one has to look at the pions produced in each single event and ***each event is counted either with weight 1 or with weight 0***, depending on the exclusive cross section one is interested in.

If one is looking at the exclusive cross section of n pions production, an event where n pions are produced contributes with weigh 1 to the cross section, while all other events contribute with weight 0.

<u>*Inclusive*</u> and <u>*exclusive*</u> cross sections are thus results of <u>*independent measurements.*</u>
Notice that, being an average, the *inclusive* cross section is a much simpler quantity as compared to the *exclusive* cross section

Interestingly, in its study of double parton collisions, ***the CDF experiment did not measure the double parton scattering inclusive cross section***. The events selected where in fact only those which contained just double parton collisions, while all events with triple scatterings (about 17% of the sample of all events with double parton scatterings) where removed.

The resulting quantity measured by CDF is hence different with respect to the inclusive cross sections usually discussed in large momentum transfer physics. In fact ***the quantity measured by CDF is precisely the double parton scattering "exclusive" cross section.***

Notice that ***the "exclusive" cross sections are NOT given by the usual pQDC-parton model expression for the large p_t processes***. While the inclusive cross sections are in fact linked directly to the multi-parton structure of the hadron, the link of the "exclusive“ cross sections with the hadron structure is more elaborate.

The requirement of having only events with a given number of hard collisions implies in fact that the corresponding cross section (being proportional to the probability of not having any further hard interaction) ***depends***, at least in principle, ***on the whole series of multiple hard collisions.***

One has:

$$\sigma_H \equiv \sum_{N=1}^{\infty} \tilde{\sigma}_N, \qquad \sigma_K \equiv \sum_{N=K}^{\infty} \frac{N(N-1)\dots(N-K+1)}{K!}\tilde{\sigma}_N$$

$\tilde{\sigma}_N$ N^th scattering “exclusive” cross sections (where one selects the events where only N partonic collisions are present)

σ_K K^th scattering inclusive cross section

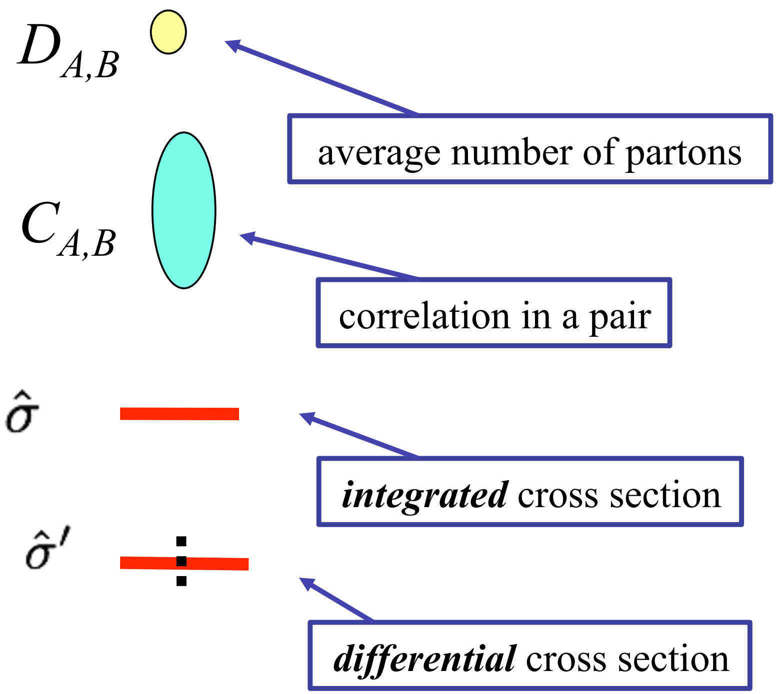
Notice that the relations above represents also a set of ***sum rules connecting the inclusive and the "exclusive" cross sections***

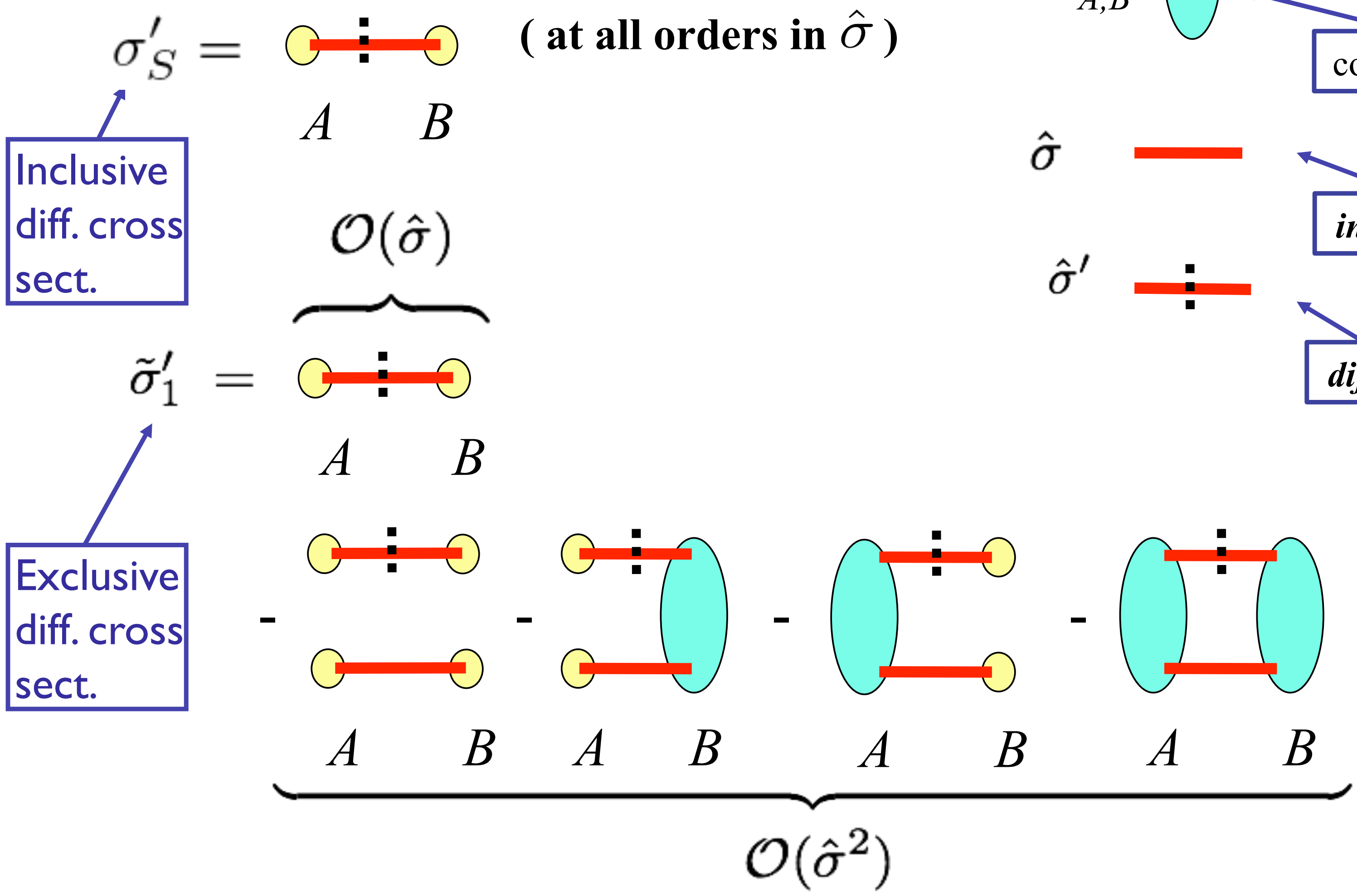
The number of hard partonic collisions which can be observed directly is limited, the ***"exclusive" cross sections*** may hence be expressed in terms of known quantities after ***expanding*** the expressions above ***in the number of elementary interactions.***

At order $\hat{\sigma}^3$ one obtains :

$$\begin{aligned} \tilde{\sigma}_1 &= \sigma_S - 2\sigma_D + 3\sigma_T \\ \tilde{\sigma}_2 &= \sigma_D - 3\sigma_T \\ \tilde{\sigma}_3 &= \sigma_T \end{aligned}$$

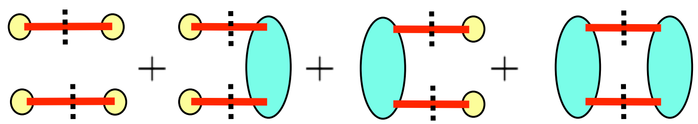
Single scattering differential "exclusive" cross sections at order $\hat{\sigma}^2$ including the two-body parton correlations:





Double parton scattering differential "exclusive" cross section at order $\hat{\sigma}^3$ taking into account the two-body parton correlations

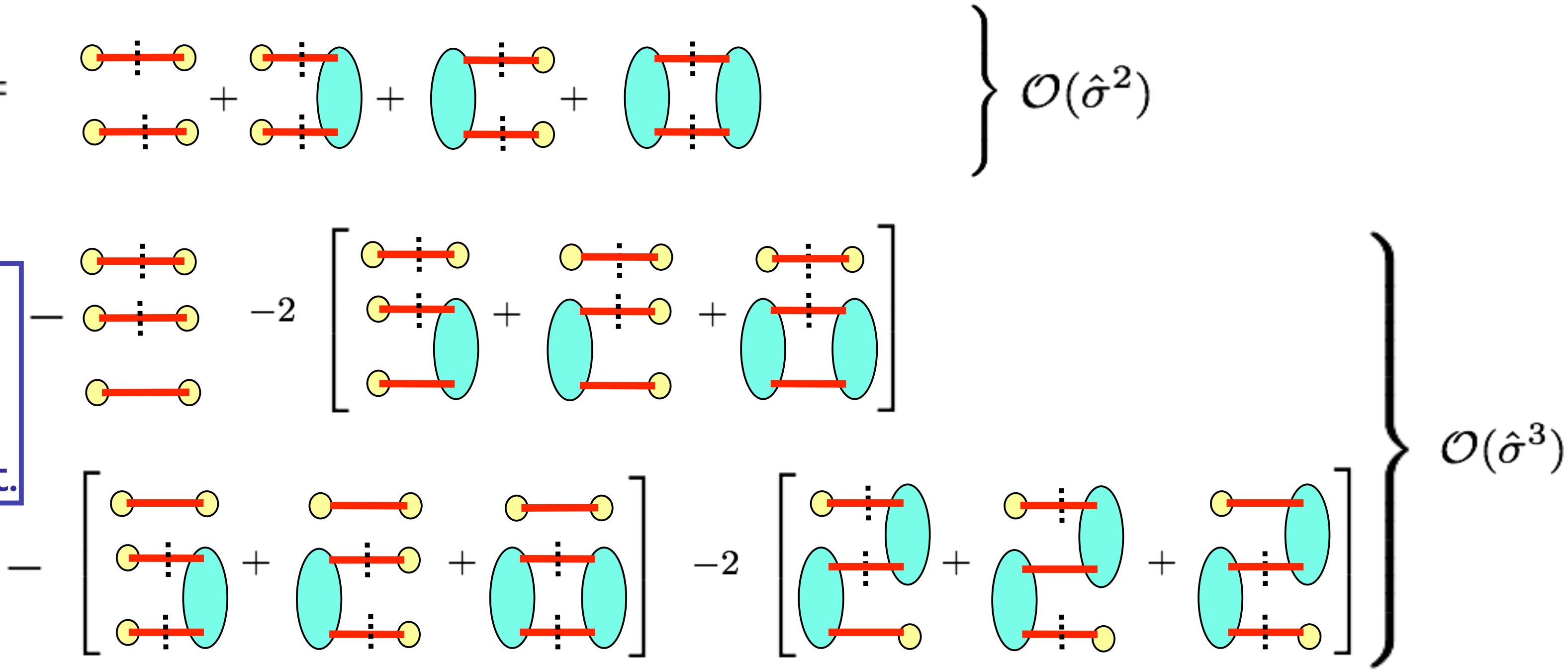
$2\sigma''_D =$



(at all orders in $\hat{\sigma}$)

Inclusive double diff. cross sect.

$2\tilde{\sigma}''_2 =$



Exclusive double diff. cross sect.

Notice that MPI add incoherently in the final cross section ***in each different phase space interval***, leading to a different ***probabilistic picture*** of the process in each different phase space interval. One may hence associate a ***different distribution in multiplicity of MPI to each different phase space choice*** to observe the final state.

Any final state phase space window identifies an interval in momentum transfer and in fractional momenta, which identifies the domain of definition of the probability distribution of multiple collisions.

One should point out that the same interval in momentum transfer and fractional momenta represents also the integration domain of the integrated terms, which appear in the "exclusive" differential cross sections. Integrated terms appear in the "exclusive” differential cross sections because of the normalization of the probability distribution. ***Normalization*** hence ***fixes unambiguously the integration limits of the virtual terms***, which must coincide with the kinematical limits imposed to the real terms by the choice adopted to select the final state.

Inclusive diff. cross sect.

$$d\sigma_S(u,u') = D_A(u)d\hat{\sigma}(u,u')D_B(u')$$

$$d\tilde{\sigma}_1(u,u') = D_A(u)d\hat{\sigma}(u,u')D_B(u')\Big[1-\int D_A(u_1)\hat{\sigma}(u_1,u_1')D_B(u_1')du_1du_1'\Big]$$
$$-\Big[\int D_A(u)d\hat{\sigma}(u,u')C_B(u',u_1')\hat{\sigma}(u_1',u_1)D_A(u_1)du_1du_1' + A\leftrightarrow B\Big]$$
$$-\int C_A(u_1,u)d\hat{\sigma}(u,u')C_B(u',u_1')\hat{\sigma}(u_1',u_1)du_1du_1'$$

Exclusive diff. cross sect.

By restricting the phase space interval of the observed final state, $\tilde{\sigma}_1$ is well expressed by the term linear in $\hat{\sigma}$. In that limit the probability of interaction is well approximated by the average of the distribution, while $\tilde{\sigma}_2$ is negligibly small and ***the single parton scattering "exclusive" cross section is well represented by the single scattering expression of the simple QCD parton model***.

When the phase space volume is increased, the single parton scattering ***"exclusive" cross section becomes increasingly different*** from the prediction of the single scattering expression of the simple QCD parton model and the difference allows a direct measure of the importance of correlations.

Although the ***distribution in multiplicity of MPI is different in each different phase space interval,*** one can show that ***the components of the hadron structure***, namely the terms D and C, ***are well defined quantities, as they do not mix when the kinematical limits adopted to selelect the final state are changed***.
The effect, to modify the kinematical limits adopted to selelect the final state, is in fact only to change the integration domain of each term, namely to probe the multi-parton structure of the hadron in different domains in x and Q^2.

By summing the expressions of $\tilde{\sigma}_1$, $\tilde{\sigma}_2$ etc. at a given order in $\hat{\sigma}$ with the proper multiplicity factors one obtains the inclusive single parton scattering cross section, which is correctly given by the QCD parton model expression. ***By comparing the measured inclusive cross section with the sum of the measured "exclusive" cross sections***, taken with the proper multiplicity factors, up to a given order in the number of collisions, ***one hence has a direct indication of the importance of higher order multiparton processes*** in a given phase space interval.

Inclusive and "exclusive" cross sections result from independent measurements and, if in a given phase space interval only single and double collisions give sizable contributions, the following relations between the measured integrated cross sections hold:

$$\begin{aligned} \sigma_S &= \tilde{\sigma}_1 + 2\tilde{\sigma}_2 \\ \sigma_D &= \tilde{\sigma}_2 \end{aligned} \qquad \text{at } \mathcal{O}(\hat{\sigma}^2)$$

By increasing the phase space interval triple collisions may become important and, in such a case, the relations become:

$$\begin{aligned} \sigma_S &= \tilde{\sigma}_1 + 2\tilde{\sigma}_2 + 3\tilde{\sigma}_3 \\ \sigma_D &= \tilde{\sigma}_2 + 3\tilde{\sigma}_3 \\ \sigma_T &= \tilde{\sigma}_3 \end{aligned} \qquad \text{at } \mathcal{O}(\hat{\sigma}^3)$$

By checking the validity of the sum rules above, one hence controls the effects of the presence of higher order multiparton processes in a given phase space interval. Once, as an example only single and double collisions give sizable contributions, in a given phase space interval, one may obtain ***information on the multi-parton correlations*** by looking at the difference between the single scattering inclusive and "exclusive" differential cross sections:

$$\frac{d\sigma_S}{dyd\mathbf{p}_t} - \frac{d\tilde{\sigma}_1}{dyd\mathbf{p}_t} = \frac{d\sigma_S}{dyd\mathbf{p}_t}\frac{\sigma_S}{\sigma_{eff}}$$

The value of the effective cross section is thus obtained by measuring the single scattering inclusive and "exclusive" cross sections. By looking at the difference between the rigth and left hand side of the equation above, as a function of fractional momenta and rapidity, one obtains information on the dependence of the effective cross section on y and p_t

Concluding Summary

In each single parton collision all transverse momenta balance and a ***MPI process*** contributes to the cross section by generating ***different groups of final state partons where the large transverse momenta are compensated separately***.

different MPI terms, corresponding to different localizations in transverse space, ***do not interfere*** and the final cross section is obtained by the simple superposition of the cross sections, due to the contributions of the different topologies of the hard component of the interaction.

The physical picture of MPI is hence probabilistic and different final state phase space intervals are characterized by different probability distributions of MPI.

One may thus construct (and measure) ***two different sets of cross sections***, the ***inclusive*** cross sections, which are ***related to the moments*** of the probability distribution of MPI, and the ***"exclusive"*** cross sections, which are ***related to the different terms*** of the probability distribution.

All different terms of the distribution in multiplicity of partonic interactions contribute to the inclusive cross sections (each with a different multiplicity factor). The inclusive cross sections give a direct measurement of the multiparton distributions.

Each single term of the distribution of MPI gives an "exclusive" cross section.

Inclusive and "exclusive" cross section are measured independently.

The average number of MPI depends on the final state phase space interval under consideration, while the number of MPI may be controlled by adjusting the final state phase space interval.

Inclusive and "exclusive" cross sections are linked by sum rules, which are saturated by a different number of terms in each different final state phase space interval. ***The number of terms needed to saturate the sum rules provide a quantitative measure of the importance of higher order multiparton processes*** in the phase space interval under consideration.

For a given number of hard interactions ***the non perturbative input to the "exclusive" cross sections is given explicitly*** in terms of well defined non perturbative properties of the hadron structure.