Analytic methods in QCD
Light and heavy-quark symmetries

Sébastien Descotes-Genon

Laboratoire de Physique Théorique
CNRS & Université Paris-Sud 11, 91405 Orsay, France

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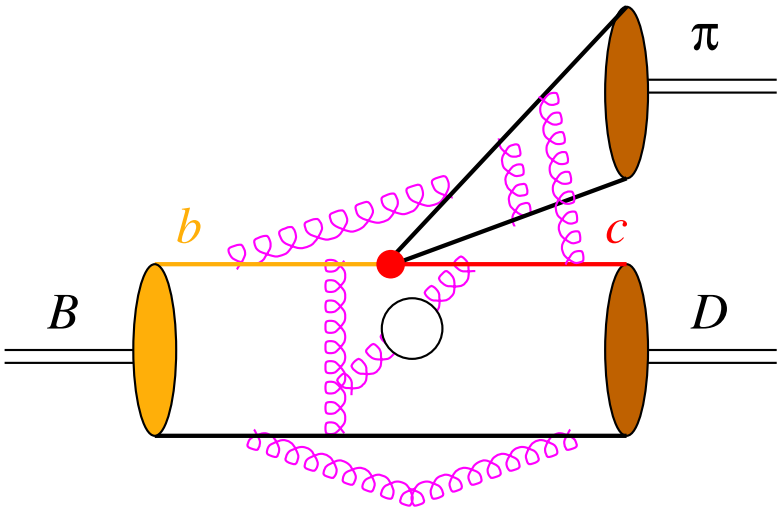
Contents

Hadronic matrix elements

Effective Hamiltonian yields $A(B \to H) = \sum \lambda_i \, C_i(\mu) \langle H|\mathcal{O}_i|B\rangle(\mu)$

- above m_b, perturbative Wilson coefficients $C_i(\mu)$
- below m_b, operators yielding matrix elements $\langle H|\mathcal{O}_i|B\rangle(\mu)$

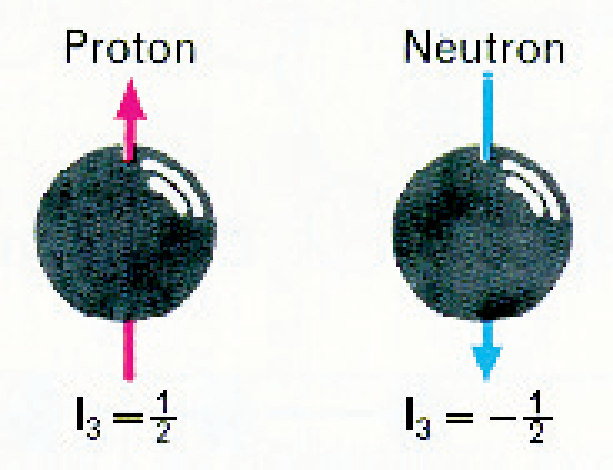


Strong interaction
in nonperturbative regime

How to compute $\langle H|\mathcal{O}_i|B\rangle$?

- Model building
- Lattice simulations
- Sum rules
- Light flavour symmetries (isospin, SU(3)...)
- Heavy flavour symmetries (HQET...)

Isospin symmetry



Isospin symmetry

QCD interactions are flavour blind : increase symmetry when $m_u = m_d$
(very good approximation, since $m_u, m_d \ll \Lambda$)

$\Longrightarrow$Notion of isospin $\quad I = 1/2 : \begin{array}{c} I_z = 1/2 \\ I_z = -1/2 \end{array} \begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}$

$I = 0 \; D_s \qquad I = 1/2 \; (B_d, B^+) \qquad I = 1 \; (\pi^+, \pi^0, \pi^-)$

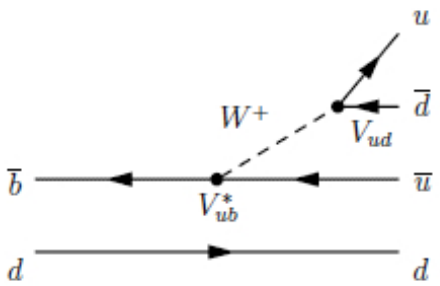
Useful due to Wigner-Eckart theorem, to reduce matrix elements
matrix element = spin projections $\times$ reduced matrix element

$$\langle jm | O^k_q | j'm' \rangle = \langle jm | j'm'; kq \rangle \times \langle j || O^k || j' \rangle$$

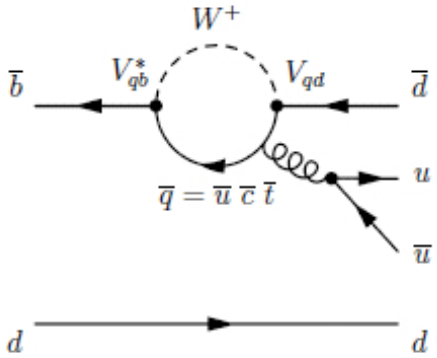
- k is rank of operator O (scalar, vector, tensor...), q its component
- $\langle jm | j'm'; kq \rangle$ Clebsch-Gordan coefficient
- $\langle j || T^k || j' \rangle$ reduced matrix element (independent of m's)

$\Longrightarrow$possible to analyse all processes with isospin amplitude

Penguin pollution in $B \to \pi^+\pi^-$



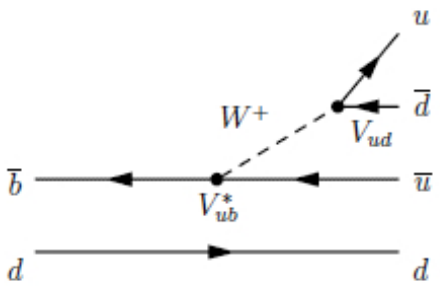
Tree



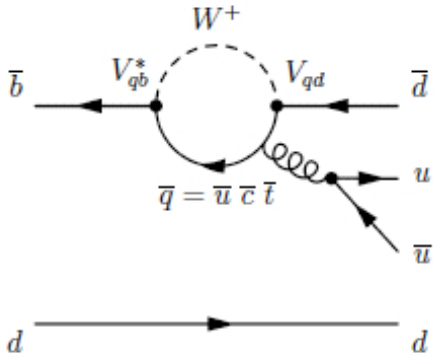
Penguin

$$A(B^0 \to \pi^+\pi^-) = V_{ud}V^*_{ub}t + \sum_{q=u,c,t} V_{qd}V^*_{qb}p_q$$

Penguin pollution in $B \to \pi^+\pi^-$



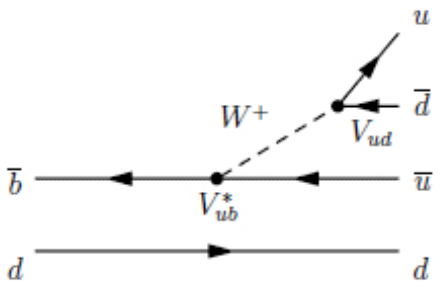
Tree



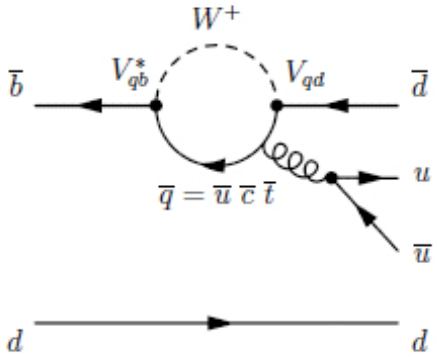
Penguin

$$A(B^0 \to \pi^+\pi^-) = V_{ud}V^*_{ub}(t + p_u) + (-V_{ud}V^*_{ub} - V_{td}V^*_{tb})p_c + V_{td}V^*_{tb}p_t$$

Penguin pollution in $B \to \pi^+\pi^-$



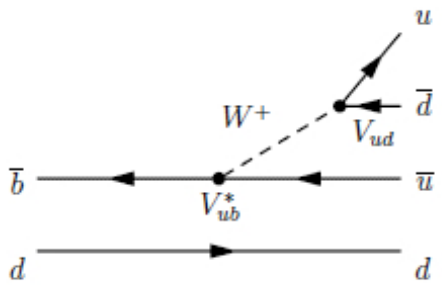
Tree



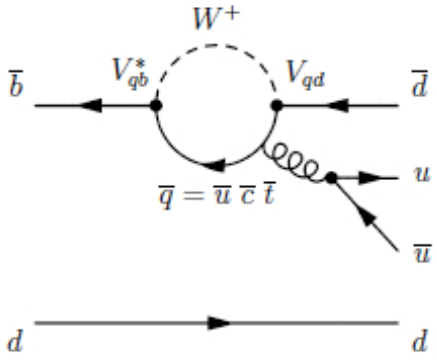
Penguin

$$A(B^0 \to \pi^+\pi^-) = V_{ud}V^*_{ub}t^{+-} + V_{td}V^*_{tb}p^{+-}$$

Penguin pollution in $B \to \pi^+\pi^-$



Tree



Penguin

$$A(B^0 \to \pi^+\pi^-) = V_{ud}V^*_{ub}t^{+-} + V_{td}V^*_{tb}p^{+-}$$

Time-dependent asymmetry

$$\begin{aligned} A(t) &= S_{\pi^+\pi^-}\sin(\Delta m t) - C_{\pi+\pi-}\cos(\Delta m t) \\ &= \sqrt{1 - C^2_{\pi^+\pi^-}}\sin 2\alpha_{eff}\sin(\Delta m t) - C_{\pi+\pi-}\cos(\Delta m t) \end{aligned}$$

Combining CKM for t^{+-} and B-$\bar{B}$ mixing: $S_{\pi+\pi-} = \sin(2\alpha) + O(\frac{p^{+-}}{t^{+-}})$

$\Longrightarrow$Penguin pollution: handle on p^{+-} and t^{+-} to extract $\sin(2\alpha)$?

Isospin analysis for $B \to \pi\pi$

In terms of isospin quantities $Q_{I_z}^{(I)}$

- Two operators for $\bar{b} \to \bar{u}u\bar{d}$: $O_{1/2}^{(3/2)}$ and $O_{1/2}^{(1/2)}$ (adding 3 $I = 1/2$)
- Two inital states: $|B^+\rangle = |B_{1/2}^{(1/2)}\rangle$ and $|B^0\rangle = |B_{-1/2}^{(1/2)}\rangle$
- Three final states: [$I = 1$ forbidden by Bose symmetry]

$$\langle\pi^+\pi^0| = \langle\pi\pi_1^{(2)}| \qquad \begin{aligned} \langle\pi^+\pi^-| &= \sqrt{\frac{1}{3}}\langle\pi\pi_0^{(2)}| + \sqrt{\frac{2}{3}}\langle\pi\pi_0^{(0)}| \\ \langle\pi^0\pi^0| &= \sqrt{\frac{2}{3}}\langle\pi\pi_0^{(2)}| - \sqrt{\frac{1}{3}}\langle\pi\pi_0^{(0)}| \end{aligned}$$

Isospin analysis for $B \to \pi\pi$

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$$\langle\pi^+\pi^0| = \langle\pi\pi^{(2)}_1| \qquad \langle\pi^+\pi^-| = \sqrt{\frac{1}{3}}\langle\pi\pi^{(2)}_0| + \sqrt{\frac{2}{3}}\langle\pi\pi^{(0)}_0|$$

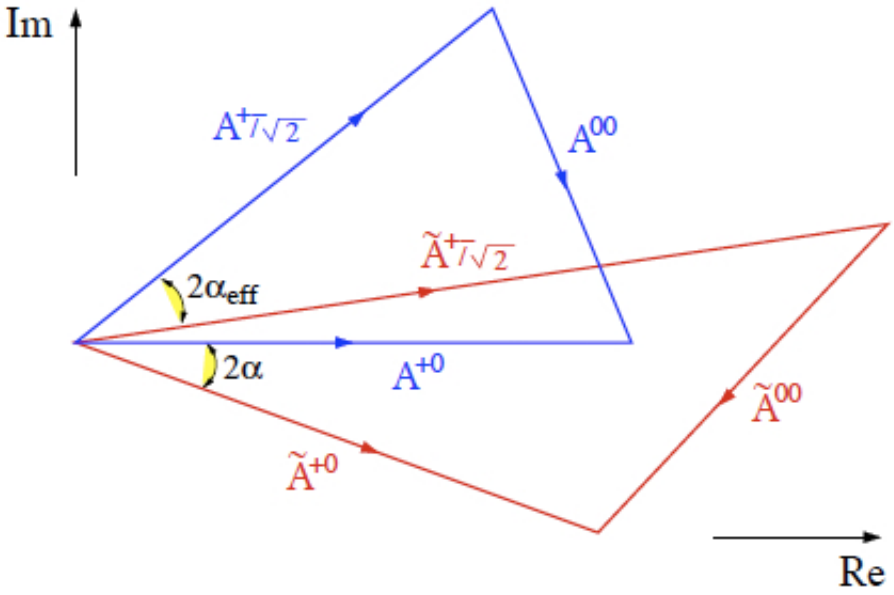
$$\langle\pi^0\pi^0| = \sqrt{\frac{2}{3}}\langle\pi\pi^{(2)}_0| - \sqrt{\frac{1}{3}}\langle\pi\pi^{(0)}_0|$$

From $B^{(1/2)}$, $O^{(3/2)}$ can only yield $I = 2$ final states,
and $O^{(1/2)}$ only $I = 0$, so two reduced amplitudes

$$A_2 = \frac{1}{2\sqrt{3}}\langle\pi\pi^{(2)}||O^{(3/2)}||B^{(1/2)}\rangle \qquad A_0 = -\frac{1}{\sqrt{6}}\langle\pi\pi^{(0)}||O^{(1/2)}||B^{(1/2)}\rangle$$

Trapping the penguin in $B^0 \to \pi^+\pi^-$

$$B^+, B^0 \quad : \quad A^{+0} = 3A_2 \qquad A^{+-} = \sqrt{2}(A_2 - A_0) \qquad A^{00} = 2A_2 + A_0$$
$$B^-, B^0 \quad : \quad \bar{A}^{+0} = 3\bar{A}_2 \qquad \bar{A}^{+-} = \sqrt{2}(\bar{A}_2 - \bar{A}_0) \qquad \bar{A}^{00} = 2\bar{A}_2 + \bar{A}_0$$



Two triangular relations

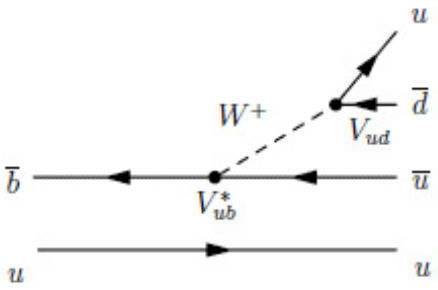
$$A^{+-} + \sqrt{2}A^{00} = \sqrt{2}A^{+0}$$
$$\bar{A}^{+-} + \sqrt{2}\bar{A}^{00} = \sqrt{2}\bar{A}^{+0}$$

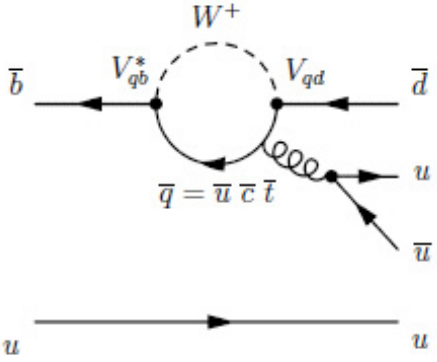
allowing one to build two triangles from Br and CP-asymmetries

Trapping the penguin in $B^0 \to \pi^+\pi^-$

A^{+0} is $I = 2$ final state, only from tree and $I = 3/2$ penguins

- Electroweak penguins $I = 1/2, 3/2$
  (Z, γ emit $I = 0, 1$ pairs $\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d$, $\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d$)
  $\Longrightarrow$only $I = 3/2$ contributes to A^{+0} penguin
- Gluon penguins $I = 1/2$ (gluon emits $I = 0$ pairs $\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d$)
  $\Longrightarrow$do not contribute to A^{+0} penguin

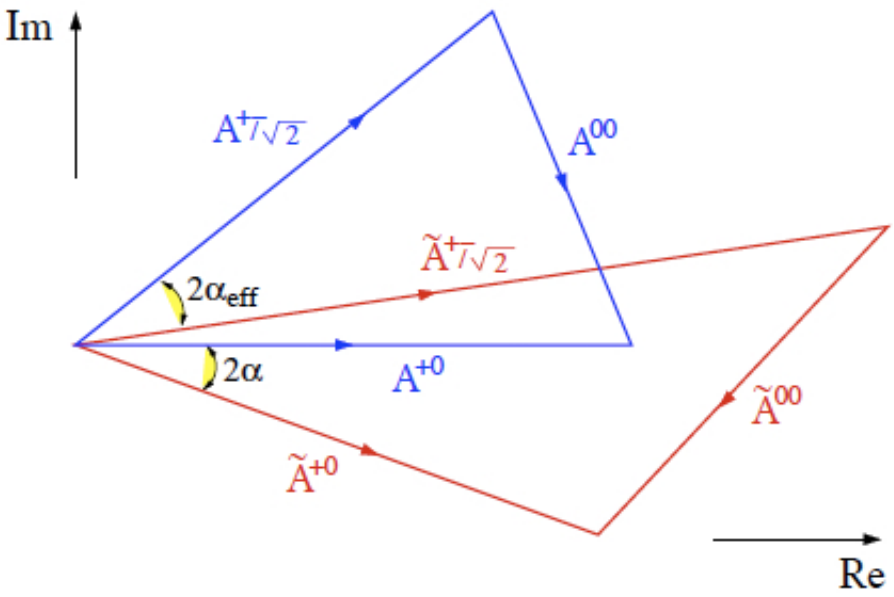




Neglecting $I = 3/2$ electroweak penguin, A^{+0} is pure tree

α from $B \to \pi\pi$

Introducing P and T including CKM moduli



$$
\begin{aligned}
A^{+0} &= e^{i\gamma}(T^{00} + T^{+-}) \\
\sqrt{2}A^{+-} &= e^{i\gamma}T^{+-} - Pe^{-i\beta} \\
\sqrt{2}A^{00} &= e^{i\gamma}T^{00} + Pe^{-i\beta}
\end{aligned}
$$

$$
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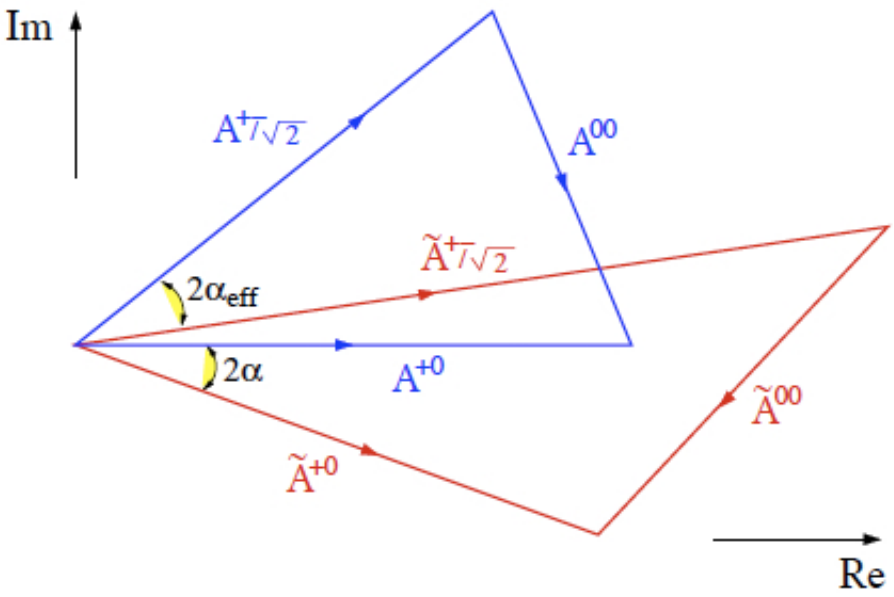
α from $B \to \pi\pi$

Introducing P and T including CKM moduli



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\end{aligned}
$$

$$
\bar{A}^{+0} = e^{-i\gamma}(T^{00} + T^{+-})
$$

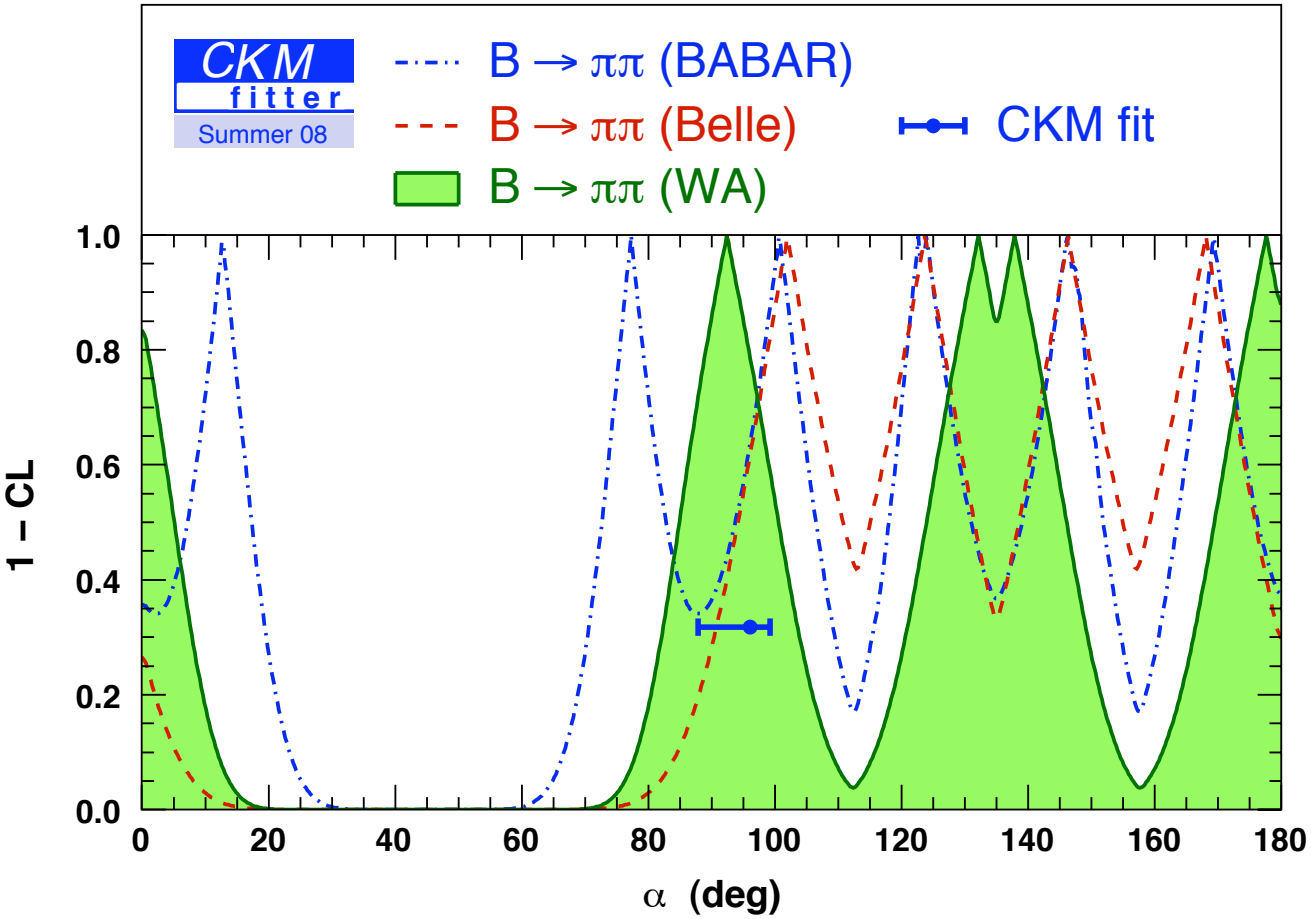
Introducing $\tilde{A}^{ij} = exp(-2i\beta)\bar{A}^{ij}$,

2α between A^{+0} and $\tilde{A}^{+0}$, $2\alpha_{eff}$ between A^{+-} and $\tilde{A}^{+-}$

- Reconstruct 2 independent triangles from BR and direct asym
- Measure mixed CP-asymmetry in $\pi^+\pi^-$ as $\sin(2\alpha_{eff})$
- Up to discrete ambiguity, determine $\sin(2\alpha)$

Gronau, London

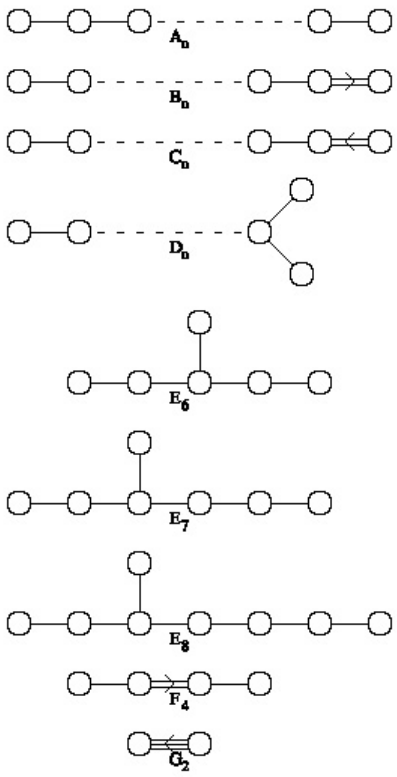
Current application of the method



$$\alpha = (89.0^{+4.4}_{-4.2})^\circ \qquad (\pi\pi, \rho\pi, \rho\rho)$$

So accurate that isospin-breaking effects should be taken into account

$SU(3)$ symmetry



$SU(3)$

$$m_u = m_d = m_s = m : \qquad \mathcal{L}_q = \Psi i \partial \Psi - m \bar{\Psi} \Psi \qquad \Psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

$\mathcal{L}_q$ inv. under $SU(3)$ flavour rotations $\Psi \to V \Psi \quad V^\dagger V = 1 \quad \det V = 1$

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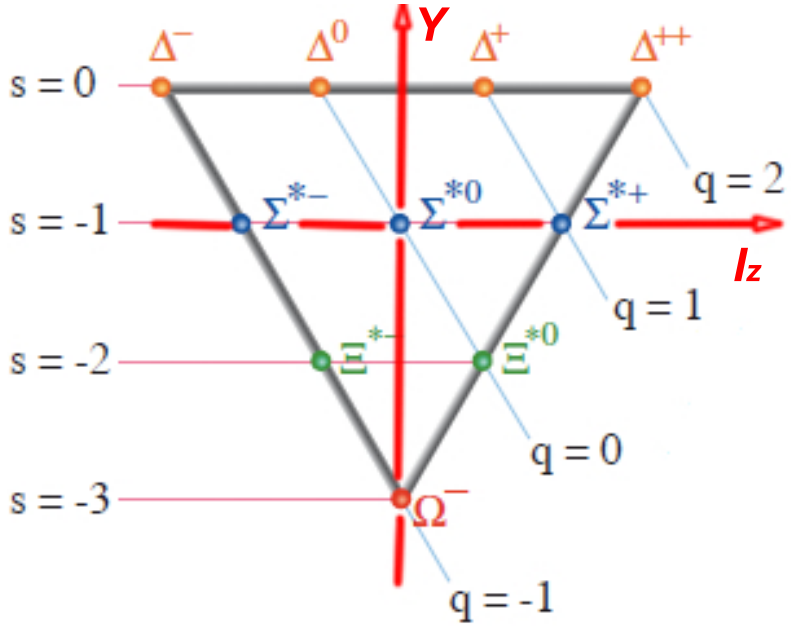
$\mathcal{L}_q$ inv. under $SU(3)$ flavour rotations $\Psi \to V\Psi \quad V^\dagger V = 1 \quad \det V = 1$

Two axes of quantisation: isospin I_z and hypercharge $Y = 2(Q - I_z)$



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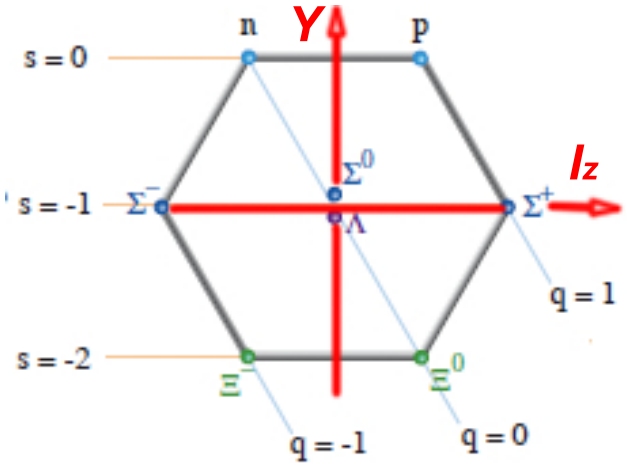


octet **8**

$SU(3)$ decomposition

- Representations are not $I = 0, 1/2, 1 \ldots$ but $\mu = \mathbf{1}, \mathbf{3}, \mathbf{8}, \mathbf{10} \ldots$

$$\mathbf{3} : (B^+, B_d, B_s), (D^0, D^-, D_s^-) \qquad \mathbf{8} + \mathbf{1} : (\pi, K, \eta, \eta'), (\rho, K^*, \phi, \omega)$$

- A particular composant is not only given by I_z, but by $\nu = Y, I, I_z$
- Clesbch-Gordan are replaced by composition symbols, actually

Clebsch-Gordan $\times$ isoscalar symbol

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix} = \langle I_1 I_{1z} I_2 I_{2z}; I_3 I_{3z} \rangle \times \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ I_1 Y_1 & I_2 Y_2 & I_3 Y_3 \end{pmatrix}$$

Wigner-Eckart theorem can be used again to express a given process with (tabulated) composition symbols & (unknown) reduced amplitudes

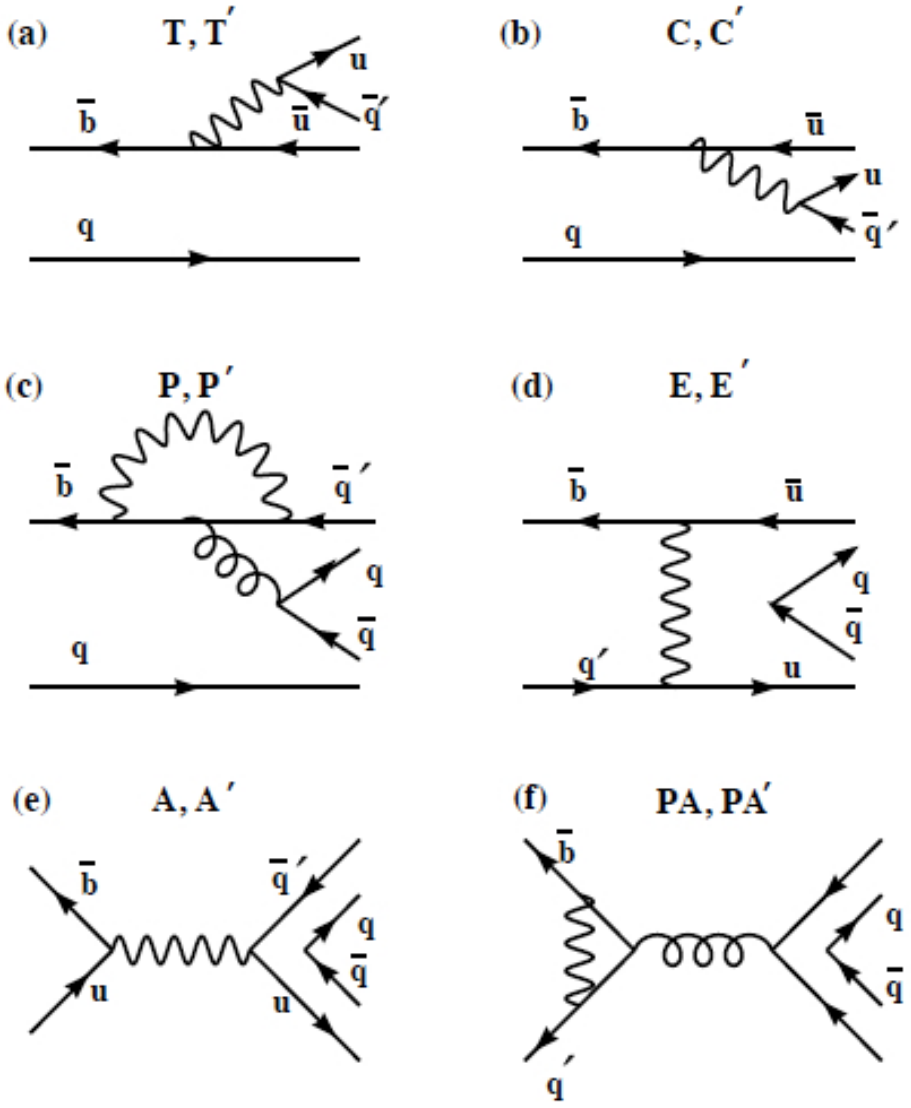
$$\langle (M_1 M_2)^{(\mu)} || O^{(\mu')} || B^{(3)} \rangle$$

Diagrammatic representation

Exactly like for isospin, possible to use a diagrammatic representation (but overcomplete)

- T : (colour-favoured) tree
- C : colour-suppressed tree
- P : penguin
- E : exchange
- A : annihilation
- PA : penguin annihilation

(penguin with $V_{tD} V_{tb}^*$, tree with $V_{uD} V_{ub}^*$)



$SU(3)$ symmetry illustrated

In addition to isospin symmetry (embedded by definition)

- among $\Delta S = 1$ processes:

$$\begin{aligned} A(B^0 \to \pi^- K^+) &= A(B_s \to K^+ K^-) - A(B_s \to \pi^+ \pi^-) \\ -(T' - P' e^{-i\gamma}) &= -(T' - P' e^{-i\gamma} + E' + PA') + (E' + PA') \end{aligned}$$

- between $\Delta S = 0$ and $\Delta S = 1$ processes

$$\begin{aligned} \sqrt{2} A(B^+ \to \pi^0 K^+) + A(B^+ \to \pi^+ K^0) &= \frac{V_{us}}{V_{ud}} \sqrt{2} A(B^+ \to \pi^+ \pi^0) \\ -(T' + C' - P' e^{-i\gamma} + A') + (-P' e^{-i\gamma} + A') &= \frac{V_{us}}{V_{ud}} (-C - T) \end{aligned}$$

(relation between isospin 2 $\pi\pi$ and isospin 3/2 πK)

Gronau, Hernandez, London, Rosner

$SU(3)$ illustrated for α

$$A(B^0 \to \pi^+\pi^-) = Te^{i\gamma} - Pe^{-i\beta} \qquad A(B^+ \to K^0\pi^+) = P'$$

- Estimate of α by bounding modulus of P from $B^+ \to \pi^+ K^0$?
- Not competitive for PP modes because $r = P/T$ large
- But can be used for longitudinally polarised $\rho^+\rho^-$ and $K^{*0}\rho^+$

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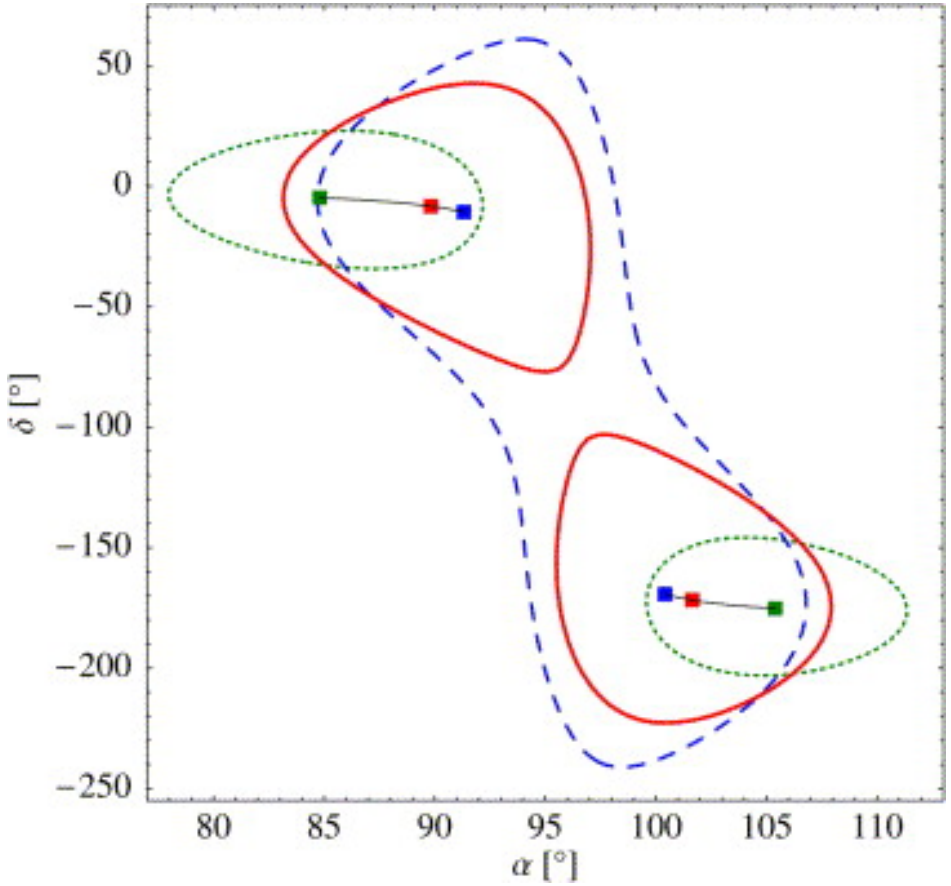
$$A_L(B^0 \to \rho^+\rho^-) = T_L e^{i\gamma} - P_L e^{-i\beta} \qquad A_L(B^+ \to K^{*0}\rho^+) = P'_L$$

$$|P_L| = \xi' \frac{|V_{td}|}{|V_{ts}|} \frac{f_\rho}{f_{K^*}} |A_L(B^+ \to K^{*0}\rho^+)|$$

- ξ' estimate of $SU(3)$ breaking (to be determined !)
- $|V_{td}|/|V_{ts}|$ moduli of CKM matrix elements included in P and P'
- f_ρ/f_{K^*} takes into account the production of different mesons (factorisable corrections)

α from $\rho\rho$ and $K^*\rho$



Using older values of $B(K^{*0}\rho^+)$ and $f_L(K^{*0}\rho^+)$

- ξ' large range (central value inspired by factorisation)
- δ strong phase between P_L and T_L
- Contours for different values of $SU(3)$ symmetry breaking

Using current values, $\alpha = [91.2^{+9.1}_{-6.6}{}^{+1.2}_{-3.9}]^\circ$

Beneke, Gronau, Rohrer, Spranger

$SU(3)$ illustrated for B_s decays

A subgroup of $SU(3)$ is U-spin : $d \leftrightarrow s$ (rather than $u \leftrightarrow d$)
$\Longrightarrow$Very interesting in connection with B_s decays

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$\Longrightarrow$Very interesting in connection with B_s decays

$$\begin{aligned} A(B^+ \to K^0\pi^+) &= P \\ A(B^0 \to K^+\pi^-) &= Te^{i(\delta_d+\gamma)} + P \\ \xi A(B_s \to K^-\pi^+) &= \frac{1}{\tilde{\lambda}} Te^{i(\delta_s+\gamma)} - \tilde{\lambda} P \end{aligned}$$

- T and P are real (and include CKM moduli),
- $\delta^{d,s}$ strong phases between T and P
- $\tilde{\lambda} = |V_{us}|/|V_{ud}|$ from $d \leftrightarrow s$
- ξ estimate of $SU(3)$ breaking (inspired by factorisation)

$$\xi = \frac{f_K}{f_\pi} \frac{F_{B^0\pi}(m_K^2)}{F_{B^sK}(m_\pi^2)} \frac{m_{B^0}^2 - m_\pi^2}{m_{B^s}^2 - m_K^2}$$

$\Longrightarrow$CDF/D0 data on Br and asymmetry for $B_s \to K^-\pi^+$

γ from $B_{d,s} \to K\pi$

One can build four (measured) quantities for $K^\pm\pi^\mp$

$$\frac{\Gamma(B^0 \to K^+\pi^-) + \Gamma(\bar{B}^0 \to K^-\pi^+)}{\Gamma(B^+ \to K^0\pi^+) + \Gamma(\bar{B}^- \to \bar{K}^0\pi^-)} = 1 + r^2 + 2r\cos\gamma\cos\delta_d$$

$$\xi^2 \frac{\Gamma(B_s \to K^-\pi^+) + \Gamma(\bar{B}^s \to K^+\pi^-)}{\Gamma(B^+ \to K^0\pi^+) + \Gamma(\bar{B}^- \to \bar{K}^0\pi^-)} = \tilde{\lambda}^2 + (r/\tilde{\lambda})^2 - 2r\cos\gamma\cos\delta_s$$

$$\frac{\Gamma(B^0 \to K^+\pi^-) - \Gamma(\bar{B}^0 \to K^-\pi^+)}{\Gamma(B^+ \to K^0\pi^+) + \Gamma(\bar{B}^- \to \bar{K}^0\pi^-)} = 2r\sin\gamma\sin\delta_d$$

$$\xi^2 \frac{\Gamma(B_s \to K^-\pi^+) - \Gamma(\bar{B}^s \to K^+\pi^-)}{\Gamma(B^+ \to K^0\pi^+) + \Gamma(\bar{B}^- \to \bar{K}^0\pi^-)} = -2r\sin\gamma\sin\delta_s$$

γ from $B_{d,s} \to K\pi$

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$$\xi^2\frac{\Gamma(B_s \to K^-\pi^+) + \Gamma(\bar{B}^s \to K^+\pi^-)}{\Gamma(B^+ \to K^0\pi^+) + \Gamma(\bar{B}^- \to \bar{K}^0\pi^-)} = \tilde{\lambda}^2 + (r/\tilde{\lambda})^2 - 2r\cos\gamma\cos\delta_s$$

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Enough to fit 4 unknowns : $r = T/P$, δ_d, δ_s, γ, with current inputs

$$(r, \gamma, \delta_d, \delta_s) = (-0.128, 60^\circ, 23^\circ, 155^\circ)$$
$$(r, \gamma, \delta_d, \delta_s) = (-0.121, 25^\circ, 58^\circ, 111^\circ)$$

$\Longrightarrow$Large U-spin breaking ($\delta_{d,s}$, $\xi_T \neq \xi_P$?) *Chiang, Gronau, Rosner*

Limitations

Isospin, U-spin, $SU(3)$ flavour symmetry:

- Relate process by exploit underlying symmetries of QCD
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- difference of masses (quarks, and thus mesons)
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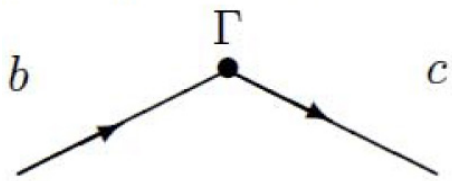
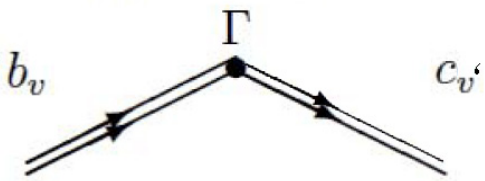
Difficulty to assess precisely the error for $SU(3)$ or U-spin

- Estimation from other methods
- Guesstimate (often 30%, but depends on the quantity considered)

Heavy-quark symmetry and HQET

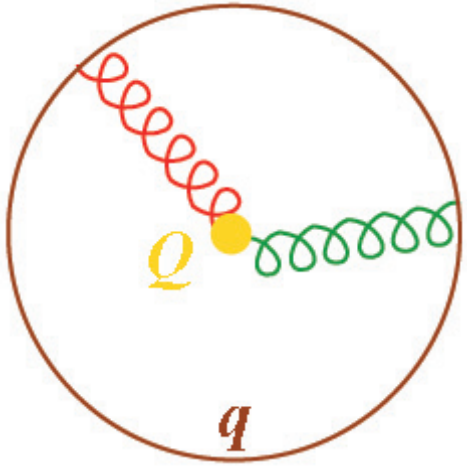




Heavy-quark symmetry

Hierachy of scale in heavy-light systems

- heavy quark of mass M_Q,
- light quark dynamics interacting through soft gluons
- dynamics with energy of order $\Lambda \ll M_Q$



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$\Longrightarrow$Heavy quark static source of gluons,
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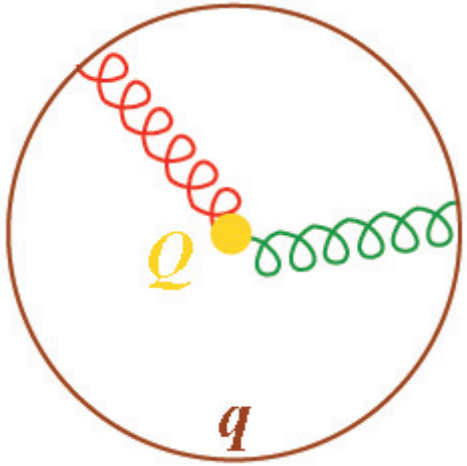

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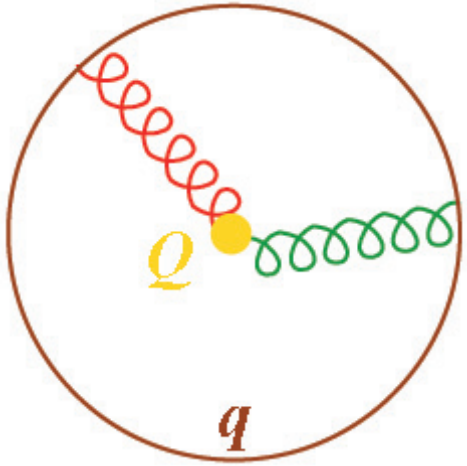

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When $M_Q \gg$ other scales in presence, properties of heavy hadrons independent of spin and mass of the heavy source of colour

Effective theory of an infinitely heavy quark

Heavy quark with momentum: $p^\mu = M_Q v^\mu + k^\mu$

where v^μ velocity of hadron ($p_B^\mu = m_B v^\mu$, $v^2 = 1$) and $k = O(\Lambda)$

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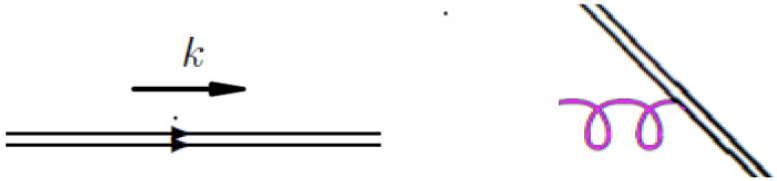

- Propagation of heavy quark

$$\frac{i}{\not p - M_Q} = \frac{i(\not p + M_Q)}{p^2 - M_Q^2} = \frac{i[M_Q(\not v + 1) + \not k]}{2(v\cdot k) + k^2} = \frac{i}{v\cdot k} P_+ + O(k/M_Q)$$

with projectors $P_\pm = \frac{1 \pm \not v}{2}$ $\qquad P_+^2 = P_+, P_-^2 = P_-, P_\pm P_\mp = 0$

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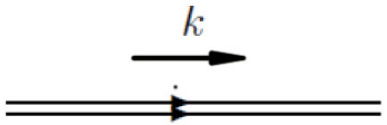

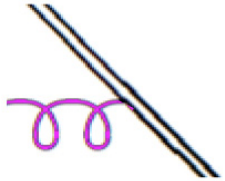

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$$i g_s P_+ \gamma_\mu T^a P_+ = i g_s P_+ (P_- \gamma_\mu + 2 v_\mu) T^a = i g_s v_\mu T^a$$

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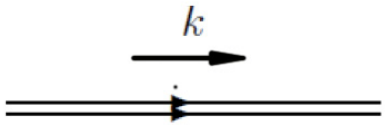

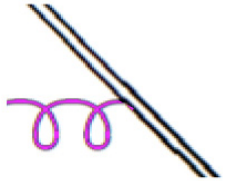

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Effective theory for the projection of heavy quark (only 1 spin d.o.f) ?

$$h_v(x) = \exp(i m_Q v\cdot x) P_+ Q(x)$$

Effective theory and symmetries

Infinitely heavy quark described by Lagrangian

$$\mathcal{L} = \bar{h}_v(iv^\mu \partial_\mu + gT^a v^\mu G^a_\mu)h_v = \bar{h}_v(iv^\mu D_\mu)h_v$$

can be extended to two heavy flavours (b and c) at the same velocity v

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- Flavour $SU(2)$ (as long as $m_b \gg \Lambda$ and $m_c \gg \Lambda$)

$$\begin{pmatrix} b_v \\ c_v \end{pmatrix} \to U \begin{pmatrix} b_v \\ c_v \end{pmatrix} \qquad UU^\dagger = U^\dagger U = 1 \quad \det U = 1$$

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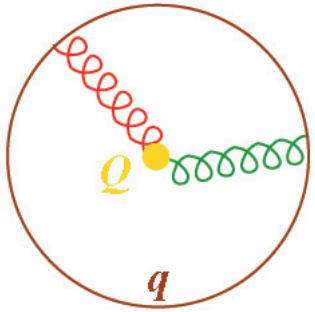
- No explicit Lorentz invariance (preferred direction: v_μ)

Spectrum

In the rest frame of the heavy meson: $J = L + S$

- L angular momentum of light d.o.f.
- S angular momentum for heavy quark



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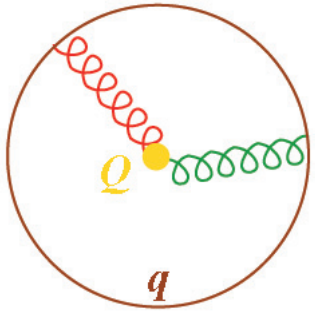

Mesons fully characterised by $(l, m_l; s, m_s)$

spectrum degenerate in m_s, organised in doublets

- $l = 0$ $\quad j = 1/2$ $\quad \Lambda_b(5620)$
- $l = 1/2$ $\quad j = 0, 1$ degenerate pseudoscalar and vector

  $B(5279), B^*(5325)$ $\quad B_s(5366), B_s^*(5412)$

  $D(1869), D^*(2010)$ $\quad D_s(1968), D_s^*(2112)$

- $l = 1$ $\quad j = 1/2, 3/2$ $\Sigma_b(5807), \Sigma_b^*(5829)$
- $l = 3/2$ $\quad j = 1, 2$ $\quad D_1(2420), D_2^*(2460)$ $D_{s1}(2536), D_{s2}(2573)$

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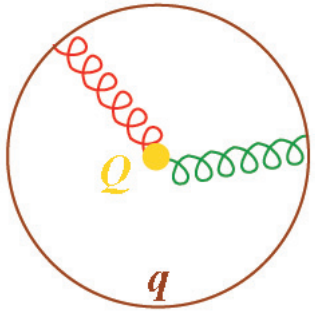

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Splitting is spin breaking $\propto \Lambda^2/m_Q$: $\frac{m_{B^*} - m_B}{m_{D^*} - m_D} = \frac{m_{B_s^*} - m_{B_s}}{m_{D_s^*} - m_{D_s}} = \frac{m_c}{m_b} = 1/3$

Description of (B, B^*)

How to describe spin degrees of freedom of (B, B^*) ?

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- In rest frame $v = (1, \vec{0})$, spin operator $S = \gamma_5 \gamma^0 \vec{\gamma}/2$

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- Spin 0: $$u^{\uparrow}_Q \bar{v}^{\downarrow}_q + u^{\downarrow}_Q \bar{v}^{\uparrow}_q = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} = \frac{1+\gamma_0}{2}\gamma_5$$

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  where $\epsilon^{(\pm,0)}_\mu$ polarisation vectors

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For an arbitrary velocity: $\tilde{M}(v) = \dfrac{1+\not{v}}{2}\gamma_5\,, \quad \tilde{M}^*(v,\epsilon) = \dfrac{1+\not{v}}{2}\not{\epsilon}$

with normalisation $|\tilde{X}\rangle = \lim_{M_Q \to \infty} \frac{1}{\sqrt{M_Q}} |X\rangle$

$B \to D^{(*)}\ell\nu$ form factors

$B \to D^{(*)}$ described by form factors, function of $q^2 = (p - p')^2$

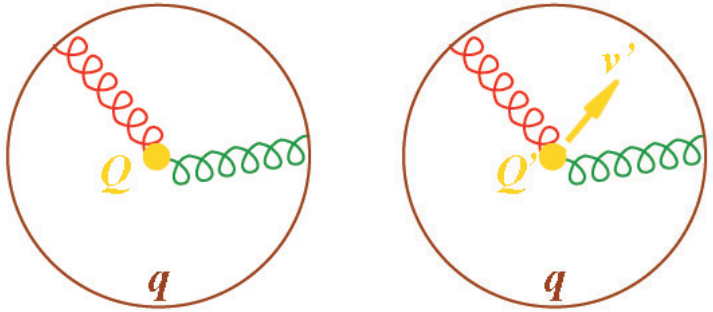
$$\langle D(p')|\bar{c}\gamma_\mu b|\bar{B}(p)\rangle = (p + p')_\mu f_+ + \frac{M_B^2 - M_D^2}{q^2} q_\mu [f_0 - f_+]$$

$$\langle D^*(p',\epsilon)|\bar{c}\gamma_\mu\gamma_5 b|\bar{B}(p)\rangle = [M_B + M_{D^*}]\epsilon^*_\mu A_1 + \frac{\epsilon^* \cdot q}{M_B + M_{D^*}}(p + p')_\mu A_2$$
$$+ \frac{\epsilon \cdot q}{q^2} q_\mu \left[(M_B + M_{D^*})A_1 - (M_B - M_{D^*})A_2 - 2M_{D^*}A_0\right]$$

$$\langle D^*(p',\epsilon)|\bar{c}\gamma_\mu b|\bar{B}(p)\rangle = \frac{-2i}{M_B + M_{D^*}} \epsilon_{\mu\nu\rho\sigma}\epsilon^{*\nu} p^\rho p'^\sigma V$$

- Meson velocities $v_\mu = p_{B\mu}/M_B$, $v'_\mu = p_{D\mu}/M_D$
- Recoil energy of D in B rest frame $E = m_D(v \cdot v' - 1)$
- $q^2 = m_B^2 + m_D^2 - 2m_B m_D(v \cdot v')$ up to $q^2_{\max} = (m_B - m_D)^2$
- $v \cdot v'$ varies between no-recoil limit $(v \cdot v' - 1)_{\min} = 0$
  and $(v \cdot v' - 1)_{\max} = \frac{(m_B - m_D)^2}{2m_B m_D} \simeq 0.6$

Physical picture



In the heavy quark limit, for $B \to D^{(*)}\ell\nu$

- Relations between D and D^* by heavy-quark symmetry on c spin
- In no-recoil limit $v = v'$, $b \to c$ unnoticed by light quark
- For $v \neq v'$, exchange of (soft) gluons to reorganise light cloud
- ...decreasing the overlap between initial B and final D

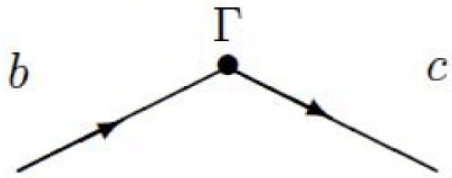
Form factors and Isgur-Wise function

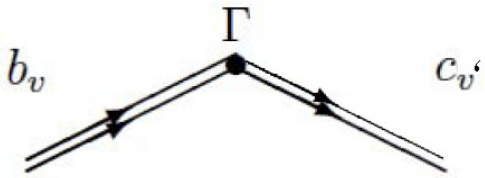
Another embodiment of Wigner-Eckart theorem

$$\langle D(v')|\bar{c}\Gamma b|B(v)\rangle \quad \to \quad -\xi(v\cdot v')\mathrm{Tr}[\bar{\tilde{D}}(v')\Gamma\tilde{B}(v)]$$
$$\langle D^*(v',\epsilon)|\bar{c}\Gamma b|B(v)\rangle \quad \to \quad -\xi(v\cdot v')\mathrm{Tr}[\bar{\tilde{D}}^*(v',\epsilon)\Gamma\tilde{B}(v)]$$

- $Tr(\ldots) \equiv$ Clebsch-Gordan (configuration of spin projections)
- $\xi \equiv$ reduced matrix element

  Isgur-Wise function, depending only on $v\cdot v'$ (since $v^2 = v'^2$=1)





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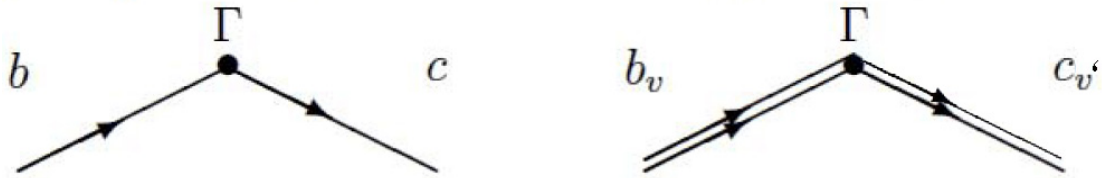
- $Tr(\ldots) \equiv$ Clebsch-Gordan (configuration of spin projections)
- $\xi \equiv$ reduced matrix element

  Isgur-Wise function, depending only on $v\cdot v'$ (since $v^2 = v'^2$=1)



In heavy-quark limit, form factors expressed in terms of ξ

$$\frac{M_B+M_D}{2\sqrt{M_BM_D}}\xi(v\cdot v') \quad = \quad f_+ = \left(1-\frac{q^2}{M_B+M_D)^2}\right)^{-1} f_0$$
$$\frac{M_{B^*}+M_D}{2\sqrt{M_{B^*}M_D}}\xi(v\cdot v') \quad = \quad V = A_0 = A_2 = \left(1-\frac{q^2}{M_{B^*}+M_D)^2}\right)^{-1} A_1$$

Isgur-Wise function ξ

- ξ also arises in

$$\langle B(v)|\bar{c}_v\gamma^0 b_v|\rangle B(v)\rangle = -\xi(v^2=1)\mathrm{Tr}[\bar{\tilde{B}}(v)\gamma^0\tilde{B}(v)] \Longrightarrow \xi(1)=1$$

  - Conservation of B-number
  - If $v\cdot v'=1$, $q^2_{\max}=(M_B-M_D)^2$ ($b\to c$ unnoticed by light quark)
    $f_+[(M_B-M_D)^2]=(M_D\pm M_B)/(2\sqrt{M_BM_D})$ useful normalisation point

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- Models to get away from $\omega = v\cdot v'=1$

$$\xi(\omega)=1-\rho^2(\omega-1)+O[(\omega-1)^2]$$

$$\ldots\left(\frac{2}{\omega+1}\right)^{2\rho^2}, e^{-\rho^2(\omega-1)}, \frac{2}{\omega+1}\exp\left[-(2\rho^2-1)\frac{\omega-1}{\omega+1}\right]\ldots$$

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- Determine them using non-perturbative methods (lattice, sum rules), or extract from one decay to get another

Heavy-quark expansion and HQET

Highers order in $1/M_Q$: Heavy-Quark Effective Theory

Grinstein, Wise, Neubert...

- $(i\not{D} - M_Q)Q = 0$ separated in "good" and "bad" components

$$Q = e^{-iM_Q v\cdot x}\left[\frac{1+\not{v}}{2} + \frac{1-\not{v}}{2}\right]Q = e^{-iM_Q v\cdot x}[h_v + H_v]$$

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- h_v has no mass term, whereas H_v is the "heavy" component

$$i(v\cdot D)h_v = -P_+ i\not{D}H_v \qquad i(v\cdot D)H_v + 2M_Q H_v = P_- i\not{D}h_v$$

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- $H_v = O(1), h_v = O(1/M_Q)$, solved recursively

$$\mathcal{L} = h_v(iv\cdot D)h_v + \frac{1}{2M_Q}h_v\left[D^2 - (v\cdot D)^2 + \frac{g_s}{2}\sigma^{\mu\nu}G_{\mu\nu}\right]h_v + O\left(\frac{1}{M_Q^2}\right)$$

  - Corrections to kinetic term (motion of heavy quark in meson)
  - Chromomagnetic moment (mass splitting in doublets)

Higher orders

- High-energy (above M_Q): corrections in α_s
  - Integrate out physics from M to μ between M and Λ
    (hard gluons, energetic light quarks)
  - Matching QCD/HQET to get same physics
- Low-energy (below M_Q): corrections in $1/M_Q$

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$$\mathcal{L} = \bar{h}_v (iv \cdot D) h_v + \frac{1}{2M_Q} \bar{h}_v \left[[D^2 - (v \cdot D)^2] + C_{mag}(m_Q, \mu) \frac{g_s}{2} \sigma^{\mu\nu} G_{\mu\nu} \right] h_v + O\left(\frac{1}{M_Q^2}\right)$$

$$\bar{c}\gamma^\mu b \to C_1(m_b, m_c, \omega, \mu) \bar{c}_{v'} \gamma^\mu b_v + C_2(m_b, m_c, \omega, \mu) \bar{c}_{v'} v^\mu b_v + \frac{1}{M_Q} \sum_i D_i(m_b, m_c, \omega, \mu)\, \bar{c}_{v'} \Gamma_i^\mu b_v + O\left(\frac{1}{M_Q^2}\right)$$

$\Longrightarrow$ Computable corrections $C, D \ldots$

but also new $1/m_{b,c}$-suppressed functions for $B \to D^{(*)}$

Illustration for $|V_{cb}|$ (1)

$$\frac{d\Gamma(B \to D^* \ell \nu)}{d\omega} = \frac{G_F^2}{48\pi^3} |V_{cb}|^2 m_{D^*}^3 \sqrt{\omega^2 - 1} P(\omega) |\mathcal{F}(\omega)|^2$$

$$\frac{d\Gamma(B \to D \ell \nu)}{d\omega} = \frac{G_F^2}{48\pi^3} |V_{cb}|^2 (m_B + m_D)^2 m_D^3 (\omega^2 - 1)^{3/2} |\mathcal{G}(\omega)|^2$$

- $\omega = v \cdot v'$, $P(\omega)$ phase space
- $\mathcal{F}$ and $\mathcal{G}$ form factors, related to ξ, include $1/M$-corrections

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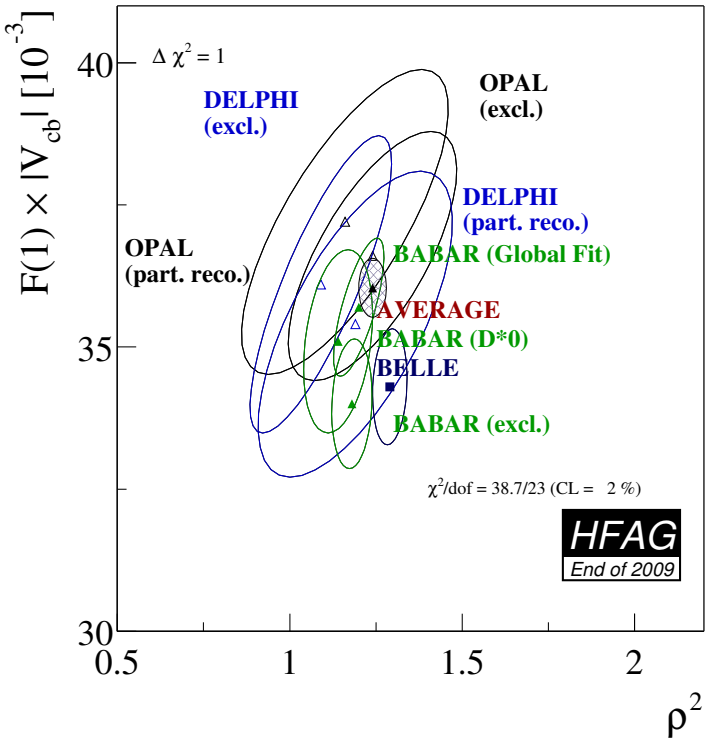
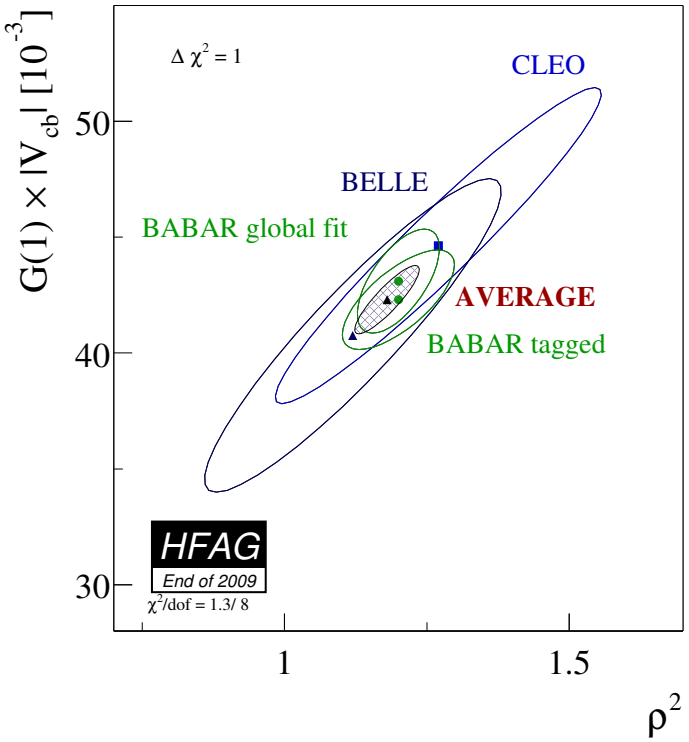
$$\mathcal{F}(\omega) = \eta_{QED} \eta_A \left[1 + O\left(\left(\frac{1}{m_b} - \frac{1}{m_c} \right)^2 \right) \right] + (\omega - 1)\rho^2 + O[(\omega - 1)^2]$$

$$\mathcal{G}(1) = \eta_{QED} \eta_V \left[1 + O\left(\frac{M_B - M_D}{M_B + M_D} \right) \right]$$

- η_{QED} and $\eta_{A,V}$ perturbative corrections from QED and HQET
- $\mathcal{F}(\omega)$ has no $1/m_Q$ corrections (Luke's theorem)

Illustration for $|V_{cb}|$ (2)



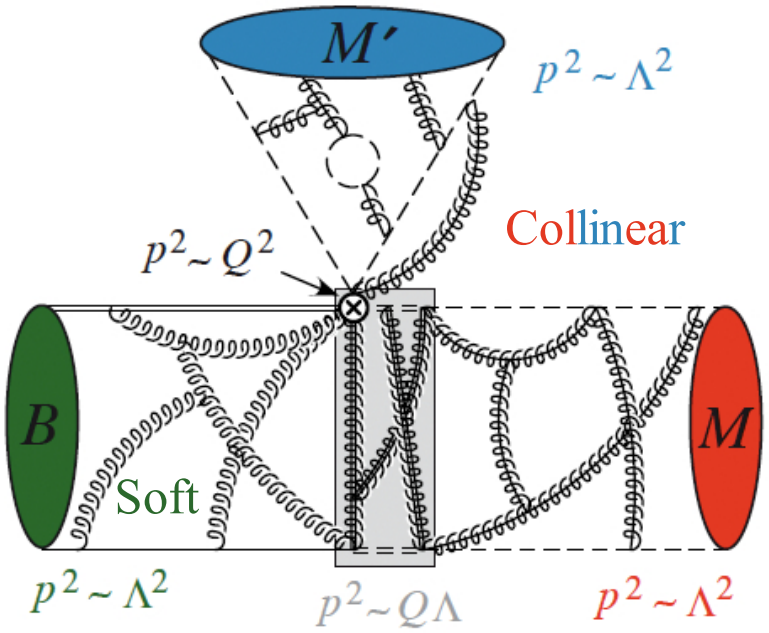


$\Longrightarrow$Need $\mathcal{F}(1)$ and $\mathcal{G}(1)$ from lattice, sum rules to determine $|V_{cb}|$

$|V_{cb}|$ also extracted from inclusive $B \to X_c \ell \nu$ by $1/m_b$ expansion

Stéphane's, Olivier's and Zoltan's lectures

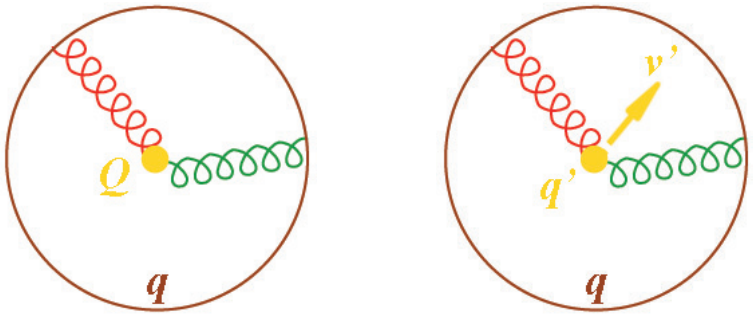
SCET, NRQCD, HMχPT...



HQET is not (always) enough

HQET: heavy-light system

where light degrees of freedom remain "soft" $E = O(\Lambda)$



What if in a process, the quarks, gluons become suddenly energetic ?

- $B \to \pi$: reorganisation of light quark through energetic gluons

  static b-quark and soft cloud $\to q\bar{q}$ pair colinear in π direction
- $B \to \pi\pi, B \to X_s\gamma$: repartition of momenta between light quarks even more complicated

Relevant degrees of freedom and scales

Several degrees of freedom

- hard gluons/quarks : $p_h = O(M, M, M, M)$
- soft gluons/quarks : $p_s = O(\Lambda, \Lambda, \Lambda, \Lambda)$
- collinear gluons/quarks : $p_c = (M, 0, 0, M) + O(\Lambda, \Lambda, \Lambda, \Lambda)$
  [energetic, but along one direction, with $p_c^2 = \Lambda^2$)]

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- $B \to \pi\pi$: $p_\pi^\mu = (E_\pi, 0, 0, \pm\sqrt{E_\pi^2 - m_\pi^2})$
  obtained through exchange between soft and collinear quarks
  with gluons of virtuality $(p_s + p_c)^2 = \sqrt{M\Lambda}$

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- $B \to X_s\gamma$: $p_\gamma^\mu = (E_\gamma, 0, 0, -E_\gamma)$, $p_X^\mu = (M_B - E_\gamma, 0, 0, E_\gamma)$
  cut on $E_\gamma = O(M_B/2) \Longrightarrow p_X^2 = M_B(M_B - 2E_\gamma) = O(M\Lambda)$

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$\Longrightarrow$Several scales: M, Λ, $\sqrt{M\Lambda}$

SCET

Soft-Collinear Effective Theory = Effective theory of QCD
keeping only the energy modes of quarks and gluons for factorisation
(others integrated out)

Stewart et al., Beneke et al.

- explains how soft and collinear quarks/gluons communicate with each other, and with hard interactions
- organizes the interactions in an expansion in Λ/M

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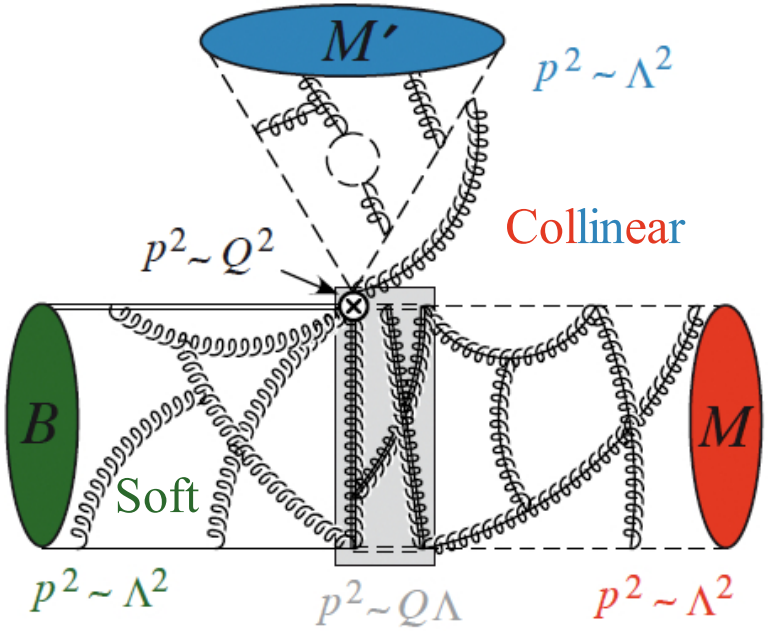
- organizes the interactions in an expansion in Λ/M

$\Longrightarrow$ SCET much more complicated Lagrangian than HQET

- large number of d.o.f. involved
- various interactions through soft/collinear gluons
- used for inclusive decays, nonleptonic decays...

Illustration 1: $B \to \pi\pi$



"Simple" language to separate scales $M, \sqrt{\Lambda M}, \Lambda$

- hard gluons integrated and separated
- ...from soft/collinear d.o.f., which define hadronic quantities

Illustration 2: $B \to$ light meson form factors

- Vector $V, A = q\gamma_\mu b, q\gamma_\mu\gamma_5 b$ (semileptonic $B \to \pi, \rho \ldots \ell\nu$)
- Tensor $T, T_5 = q[\gamma_\mu, \gamma_\nu]b, q[\gamma_\mu, \gamma_\nu]\gamma_5 b$ (radiative $B \to K^{(*)}\gamma$)

$$
\begin{aligned}
\langle P | V^\mu | B \rangle &= f_+ \left[p^\mu + p'^\mu - \frac{M^2 - m_P^2}{q^2} q^\mu \right] + f_0 \frac{M^2 - m_P^2}{q^2} q^\mu \,, \\
\langle P | T^{\mu\nu} q_\nu | B \rangle &= i \frac{f_T}{M + m_P} \left[q^2 (p^\mu + p'^\mu) - (M^2 - m_P^2) q^\mu \right] \,, \\
\langle V | V^\mu | B \rangle &= i \frac{2V}{M + m_V} \epsilon^{\mu\nu\rho\sigma} p^\nu p'^\rho \epsilon^{*\sigma} \,, \\
\langle V | A^\mu | B \rangle &= 2 m_V A_0 \frac{\epsilon^* \cdot q}{q^2} q^\mu + (M + m_V) A_1 \left[\epsilon^{*\mu} - \frac{\epsilon^* \cdot q}{q^2} q^\mu \right] \\
&\quad - A_2 \frac{\epsilon^* \cdot q}{M + m_V} \left[p^\mu + p'^\mu - \frac{M^2 - m_V^2}{q^2} q^\mu \right] \,, \\
\langle V | T^{\mu\nu} q_\nu | B \rangle &= -2 T_1 \epsilon^{\mu\nu\rho\sigma} p^\nu p'^\rho \epsilon^{*\sigma} \,, \\
\langle V | T_5^{\mu\nu} q_\nu | B \rangle &= -i T_2 \left[(M^2 - m_V^2) \epsilon^{*\mu} - (\epsilon^* \cdot q)(p^\mu + p'^\mu) \right] \\
&\quad - i T_3 (\epsilon^* \cdot q) \left[q^\mu - \frac{q^2}{M^2 - m_V^2} (p^\mu + p'^\mu) \right] \,.
\end{aligned}
$$

Relations between form factors

For energetic $E = O(M_B)$ light mesons, all form factors expressed in terms of three form factors $\zeta, \zeta_{//}, \zeta_\perp$ at leading order in α_s and E/M

Charles et al.

$$f_+(q^2) = \zeta(E_P)\,, \qquad f_0(q^2) = \left(1 - \frac{q^2}{M^2 - m_P^2}\right)\zeta(E_P)\,,$$

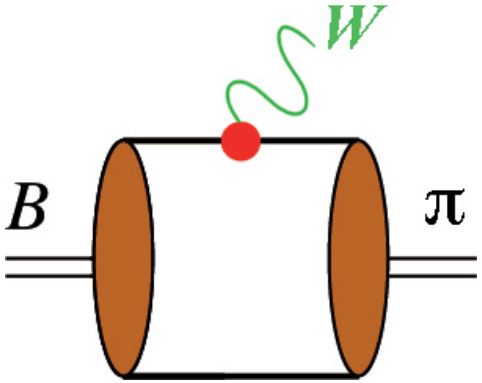
$$f_T(q^2) = \left(1 + \frac{m_P}{M}\right)\zeta(E_P)\,, \qquad A_0(q^2) = \left(1 - \frac{m_V^2}{ME_V}\right)\zeta_{//}(E_V) + \frac{m_V}{M}\zeta_\perp(E_V)\,,$$

$$A_1(q^2) = \frac{2E_V}{M + m_V}\zeta_\perp(E_V)\,, \qquad A_2(q^2) = \left(1 + \frac{m_V}{M}\right)\left[\zeta_\perp(E_V) - \frac{m_V}{E_V}\zeta_{//}(E_V)\right]\,,$$

$$V(q^2) = \left(1 + \frac{m_V}{M}\right)\zeta_\perp(E_V)\,, \qquad T_2(q^2) = \left(1 - \frac{q^2}{M^2 - m_V^2}\right)\zeta_\perp(E_V)\,,$$

$$T_1(q^2) = \zeta_\perp(E_V)\,, \qquad T_3(q^2) = \zeta_\perp(E_V) - \frac{m_V}{E}\left(1 - \frac{m_V^2}{M^2}\right)\zeta_{//}(E_V)\,.$$

Higher-order corrections



Corrections in α_s can be computed
Beneke, Feldmann...

$$f_i(q^2) = C_i(q^2)\xi_i(q^2) + \phi_B \otimes T_i \otimes \phi_\pi$$

- $\xi_i = \zeta, \zeta_{//}, \zeta_\perp$ are universal (soft) form factors
- C_i and T_i dominated by hard gluons (above $\sqrt{M\Lambda}$) and can be computed perturbatively: $C_i = 1 + O(\alpha_s)$, $T_i = O(\alpha_s)$
- ϕ_B and ϕ_π are light-cone distribution amplitudes

$$\langle 0|\bar{u}(z)\gamma_\mu\gamma_5 d(0)|\pi^+(p)\rangle = ip_\mu F_\pi \int_0^1 dx\ e^{ix(p\cdot z)}\phi(x) \qquad z^2 = 0$$

Hadronic quantity, corresponding to probability amplitude of finding in $\pi(p)$ a quark with long momentum xp

$\Longrightarrow$but still only leading-order in E/M

Illustration for $|V_{ub}|$

Differential decay rate as a function of leptonic q^2

$$\frac{d\Gamma(B \to \pi \ell \nu)}{dq^2} = \frac{G_F^2}{24\pi^3} p_\pi^3 |V_{ub}|^2 |f_+(q^2)|^2$$

where $p_\pi = \sqrt{\vec{p}_\pi^2}$ in B rest frame

- Soft-pion configurations (large-q^2) suppressed by phase space
- Still need a representation over all q^2 range

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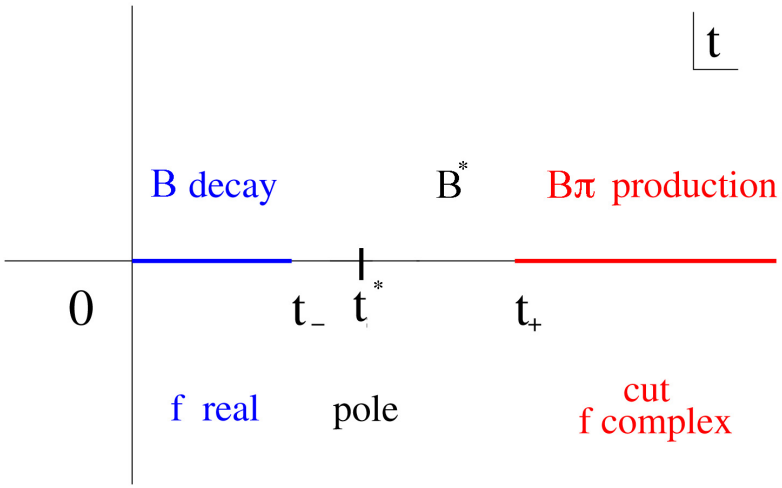
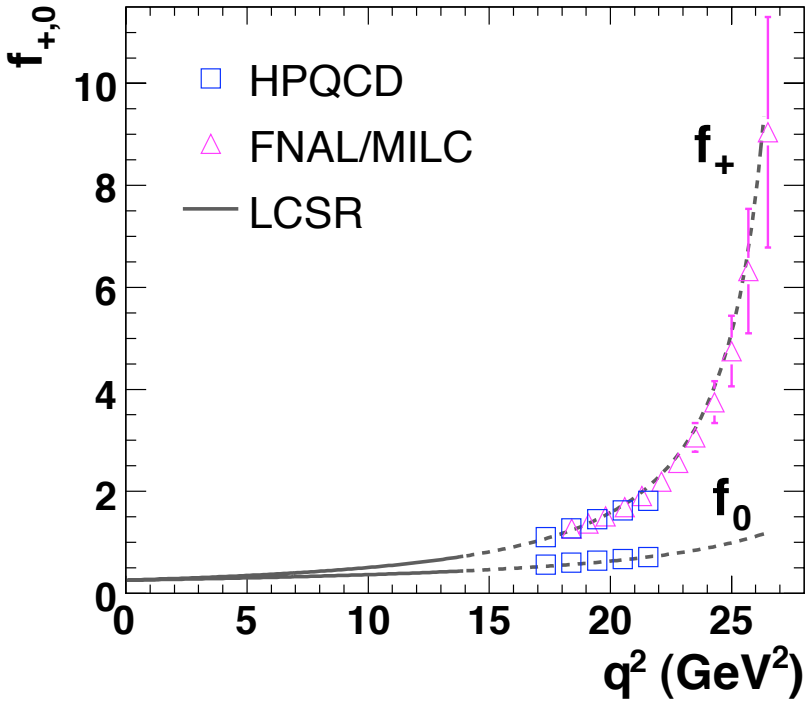


(Too) general representation

$$f_+(q^2) = \frac{g_{BB^*\pi}}{M_{B^*}^2 - q^2} + \frac{1}{\pi} \int_{(M_B+M_\pi)^2}^{\infty} dt \frac{\mathrm{Im}[f_+(t)]}{t - q^2}$$

Many parametrisation models (only poles, polynomial. . .)

More information on $B \to \pi$ form factors



Extract $|V_{ub}|$ from exclusive decays combining
Lattice QCD (high q^2) + Sum rule/SCET (low q^2) + parametrisation

$|V_{ub}|$ also extracted from inclusive $B \to X_u \ell \nu$ by $1/m_b$ expansion

Stéphane's, Olivier's and Zoltan's lectures

Other effective theories

Non-Relavistic QCD (NRQCD)

- Heavy-heavy systems (ψ, Υ, B_c)
- Hierarchy of scales
  - in quarkonium rest frame, heavy quark momenta $p = M_Q v + k$
  - slowly moving $k_0 \sim M_Q \mathrm{v}^2, \vec{k} \sim M_Q \mathrm{v}$, $\mathrm{v} \ll 1$
  - different from HQET where $k_0 \sim \vec{k} \sim \Lambda$
  - v and "softness" of scales depend on the system
  - several versions of the effective theory

Heavy-Meson Chiral Perturbation Theory (HMχPT)

- Treatment of soft pions and kaons interacting with heavy B, B^*
- Theory of mesons rather than quarks (no perturbative matching)
- Fundamental parameters
  - masses and decay constants of light and heavy-light mesons
  - $g_{BB^*\pi}$ coupling

Conclusions

Conclusions (1)

Hadronic quantities

- Defined in an unambiguous way (decay constants, form factors, matrix elements...)
- Parametrise our ignorance of low-energy QCD
- Symmetries powerful tool to simplify the problem (a bit)

Light-quark symmetries $m_q \to 0$

- Isospin, $SU(3)$, U-spin
- allows one to reduce the number of unknown parameters
- illustration with α and γ for nonleptonic decays
- generic, but with large corrections for $SU(3)$

Conclusions (1)

Heavy-quark symmetry $M_Q \to \infty$

- Useful if other scales comparable or lower than M_Q
- $1/M_Q$ expansion implemented in an effective theory (HQET)
- Spin of heavy quark unaffected by soft gluons
- Predictions on heavy-light spectrum, exclusive decay rates
- Interesting to determine $|V_{cb}|$
- Other effective theories when further relevant degrees of freedom (SCET, NRQCD...)

Not clever enough to determine hadronic quantities...
Need to work hand in hand with lattice QCD
to disentangle strong and weak interactions
in heavy-flavour dynamics