Quantum Field Theory (4/5)

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Interacting QFT

QED Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi - e\bar{\psi}\gamma^{\mu}\psi A_{\mu} - \frac{1}{2\xi}(\partial_{\mu}A^{\mu})^2$$

$\xi = 0$ ⟶ Landau gauge

$\xi = 1$ ⟶ Feynman gauge

We will work on Feynman gauge.

Scattering amplitude

Let $|i\rangle$ be the initial (multi-particle) state $(t \to -\infty)$.

Let $|f\rangle$ be some final (multi-particle) state $(t \to \infty)$.

After a long time, the initial state involves into $S\,|i\rangle$.

The amplitude for this to be $|f\rangle$ is the S-matrix element:

$$S_{fi} = \langle f|\,S\,|i\rangle$$

The probability is proportional to $|\langle f|\,S\,|i\rangle|^2$.

The perturbative expansion of S implies that

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^4(p_f - P - i)\mathbf{T}_{fi}$$

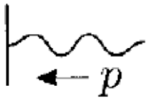
where

$$\mathbf{T}_{fi} = \langle i| T |i\rangle$$

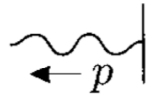
measures the genuine scattering amplitude for distinct $|i\rangle$ and $|f\rangle$.

QED Feynman rule

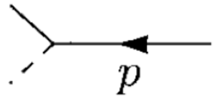
- draw all possible diagrams with allowed vertices and momentum conserved at each vertex
- external line

$\overbrace{A_\mu \ |\mathbf{p}\rangle}$ = 

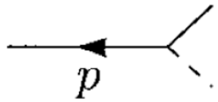
$= \epsilon_\mu(p)$

$\overbrace{\langle\mathbf{p}| \ A_\mu}$ = 

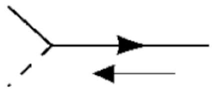
$= \epsilon^*_\mu(p)$

$\overbrace{\psi \underbrace{|\mathbf{p}, s\rangle}_{\text{fermion}}}$ = 

$= u^s(p)$

$\overbrace{\underbrace{\langle\mathbf{p}, s|}_{\text{fermion}} \bar{\psi}}$ = 

$= \bar{u}^s(p)$

$\overbrace{\bar{\psi} \ \underbrace{|\mathbf{k}, s\rangle}_{\text{antifermion}}}$ = 

$= \bar{v}^s(k)$

$\overbrace{\underbrace{\langle\mathbf{k}, s|}_{\text{antifermion}} \ \psi}$ = 

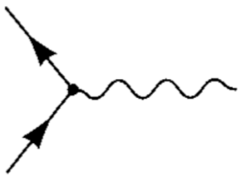
$= v^s(k)$

- propagators

$$= \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

$$= \frac{-ig_{\mu\nu}}{q^2 + i\epsilon}$$

- vertices



$$= -ie\gamma^\mu$$

QED Feynman rules (cont.)

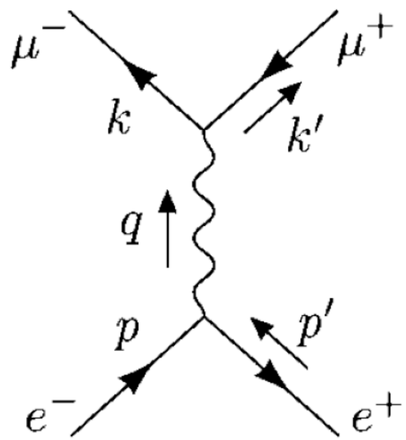
• relative minus sign between graphs with two identical fermions i.e. those that differ by exchange of the two fermion

• in loop diagram, have unconstrained integral over internal momentum

• “-” sign for closed fermion loop

•Devide by symmetry factor for loop diagram – to account for identical contributions

Example: electron-positron scattering



$$i\mathcal{M}_{(e^-e^+\to\mu^-\mu^+)} = \bar{v}^{s'}(p')(-ie\gamma^\mu)u^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}^r(k)(-ie\gamma^\nu)v^{r'}(k')$$

$$= \frac{ie^2}{q^2}\left(\bar{v}^{s'}(p')\gamma^\mu u^s(p)\right)\left(\bar{u}^r(k)\gamma_\mu v^{r'}(k')\right)$$

To compute the differential cross section, we need an expression for $|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^*$

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\left(\bar{v}^{s'}(p')\gamma^\mu u^s(p)\bar{u}^s(p)\gamma^\nu v^{s'}(p')\right)\left(\bar{u}^r(k)\gamma_\mu v^{r'}(k')\bar{v}^{r'}(k')\gamma_\nu u^r(k)\right)$$

Note that we use $(\bar{v}\gamma^\mu u)^* = \bar{u}\gamma^\mu v.$

In most experiment electron and positron are unpolarized, so we average over their spins. Muon detectors are normally blind to polarization, so we sum over muon spins

$$\frac{1}{2}\sum_s \frac{1}{2}\sum_{s'}\sum_r\sum_{r'}\left|\mathcal{M}_{(e^-e^+\to\mu^-\mu^+)}\right|^2$$

The spin sums can be performed using the completeness relations

$$\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \not{p} + m\,,$$

$$\sum_{s=1,2} v^s(p)\bar{v}^s(p) = \not{p} - m\,,$$

For example, we get

$$\sum_{s,s'} \bar{v}_a^{s'}(p')\gamma^{\mu}_{ab}u_b^s(p)\bar{u}_c^s(p)\gamma^{\nu}_{cd}v_d^{s'}(p') = (\not{p}' - m)_{da}\gamma^{\mu}_{ab}(\not{p} + m)_{bc}\gamma^{\nu}_{cd}$$

$$= \mathrm{trace}\big[(\not{p}' - m)\gamma^{\mu}(\not{p} + m)\gamma^{\nu}\big].$$

After some calculations, we arrive with

$$\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 = \frac{e^4}{4q^4} tr\,[(\not{p}' - m_e)\gamma^{\mu}(\not{p} + m_e)\gamma^{\nu}]\, tr\,[(\not{k} + m_m u)\gamma_m u(\not{k}' - m_{\mu})\gamma_{\nu}]\,.$$

This can simplify further by using trace theorem for the gamma-matrices:

$$
\begin{aligned}
\mathrm{tr}(\mathbf{1}) &= 4 \\
\mathrm{tr}(\text{any odd; \#of}\gamma'\text{s}) &= 0 \\
\mathrm{tr}(\gamma^\mu \gamma^\nu) &= 4g^{\mu\nu} \\
\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) &= 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \\
\mathrm{tr}(\gamma^5) &= 0 \\
\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^5) &= 0 \\
\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5) &= -4i\epsilon^{\mu\nu\rho\sigma}.
\end{aligned}
$$

Let us return to the square matrix elements. The electron part will give

$$\mathrm{tr}\left[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right] = 4\left[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p.p' + m_e^2)\right].$$

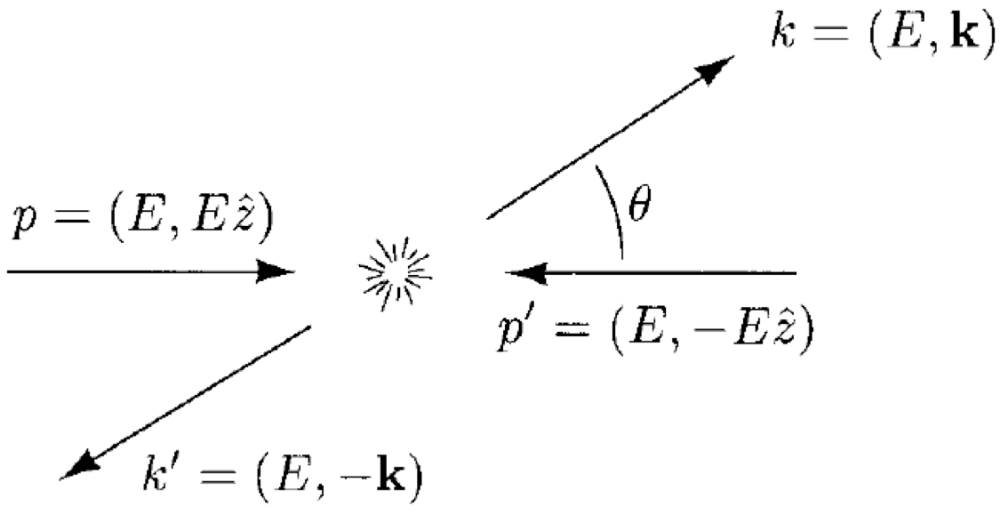
Similarly, the muon part will give

$$\mathrm{tr}\left[(\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_e)\gamma_\nu\right] = 4\left[k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(k.k' + m_\mu^2)\right].$$

We get the simple result

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{q^4}\left[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2(p\cdot p')\right].$$

$$\frac{m_e}{m_\mu} \sim \frac{1}{200} \longrightarrow m_e = 0$$



$$|\mathbf{k}| = \sqrt{E^2 - m_\mu^2}$$

$$\mathbf{k} \cdot \hat{z} = |\mathbf{k}| \cos\theta$$

We can now rewrite

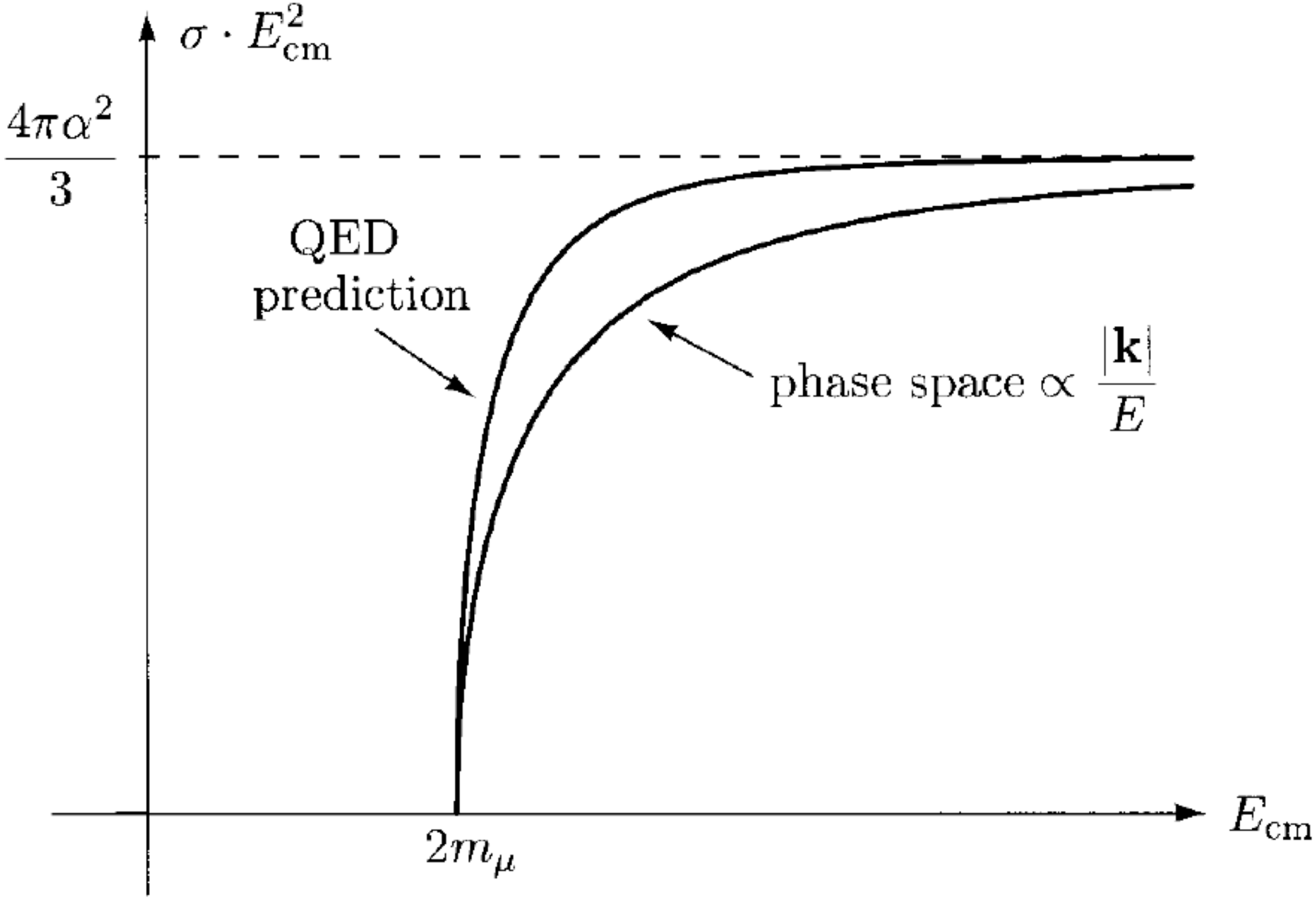
$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = e^4 \left[(1 + \frac{m_\mu^2}{E^2}) + (1 - \frac{m_\mu^2}{E^2}) cos^2\theta \right]$$

The scattering cross section can be written as

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_{cm}} \frac{|\vec{k}|}{16\pi^2 E_{cm}} \cdot \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2$$
$$= \frac{\alpha^2}{4E_{cm}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[(1 + \frac{m_\mu^2}{E^2}) + (1 - \frac{m_\mu^2}{E^2}) cos^2\theta \right]$$

Integrating over solid angle, we find the total cross section:

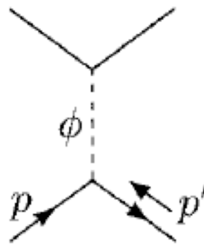
$$\sigma_{total} = \frac{4\pi\alpha^2}{3E_{cm}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left(1 + \frac{1}{2} \frac{m_\mu^2}{E^2} \right).$$



$\sigma \cdot E_{\rm cm}^2$
$\frac{4\pi\alpha^2}{3}$
QED prediction
phase space $\propto \frac{|\mathbf{k}|}{E}$
$2m_\mu$
$E_{\rm cm}$

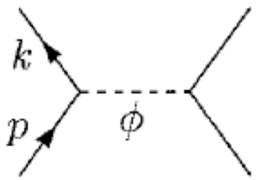
Mandelstam variable and channel

s-channel:



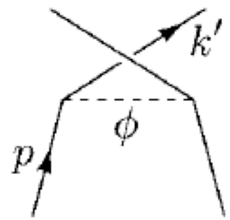
$$\mathcal{M} \propto \frac{1}{s - m_\phi^2}$$

t-channel:



$$\mathcal{M} \propto \frac{1}{t - m_\phi^2}$$

u-channel:



$$\mathcal{M} \propto \frac{1}{u - m_\phi^2}$$