Quantum ChromoDynamics

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Outline

1. Basics
2. QCD Phenomenology at Tree Level
3. Higher Order corrections
4. Monte Carlo techniques

QCD Phenomenology Beyond Tree Level

- e^+e^- annihilation at one loop
  - Dimensional regularization
  - Phase space integrals
  - Loop integrals
  - Total cross section
  - α_s measurements
- Jet cross sections
  - Infrared safety
  - General methods
- Deep Inelastic Scattering revisited
  - Factorization
  - Evolution

e^+e^- Annihilation to Hadrons

- Partons produce hadrons with probability 1

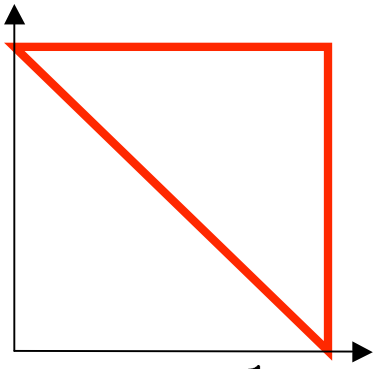
→ Need to calculate total cross section to produce any number/type of partons

- At $\mathcal{O}(\alpha_s)$, this means qqbar + qqbarg
- Recall total qqbarg cross section divergent

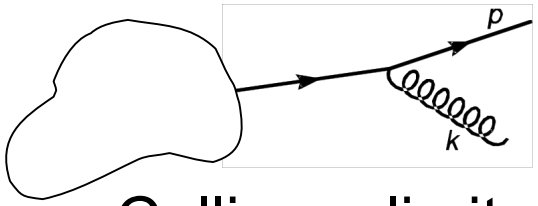
$$\sigma = \sigma_0 \, C_F \frac{\alpha_s}{2\pi} \int dx_1 \, dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

- Must be regularized in some way before proceeding

Origin of Divergences



- Propagator factors diverge:



$$\frac{1}{(p+k)^2} = \frac{1}{2p\cdot k} = \frac{1}{2E\omega(1-\cos\theta_{qg})} \approx \frac{1}{E\omega\theta_{qg}^2}$$

- Collinear limit, $\theta \to 0$:

$$|\mathcal{M}|^2 \sim \frac{1}{\theta^2} \qquad (\theta^2 \text{ cancelled by numerators})$$

- Soft limit, $\omega \to 0$:

$$|\mathcal{M}|^2 \sim \frac{p_1\cdot p_2}{p_1\cdot k\, p_2\cdot k} \sim \frac{1}{\omega^2}$$

- Phase space: $\dfrac{d^3k}{2\omega} = \frac{1}{2}\,\omega d\omega\, \sin\theta\, d\theta\, d\phi \sim \omega d\omega\, \theta d\theta$

Regularization

- Can regularize this cross section with cutoffs, eg on k_t
- But how to regularize all cross sections consistently?
- Need to modify theory to remove divergences
  such that QCD can be obtained as a smooth limit

$$QCD'(\varepsilon): \quad QCD'(0)=QCD$$

Gluon Mass Regularization

- Introduce a non-zero gluon mass,
  $m_g^2 = \varepsilon\ \mu^2$
- Minimum value of propagator > 0
  - = m_g^2 massless quarks
  - = $2m_q m_g$ massive quarks
- Total cross section

$$\sigma_{q\bar{q}g} = \sigma_0\, C_F \frac{\alpha_s}{2\pi}\left(\log^2\frac{1}{\epsilon} - 3\log\frac{1}{\epsilon} + 7 - \frac{\pi^2}{3} + \mathcal{O}(\epsilon)\right)$$

- Breaks gauge invariance → fails in general

Dimensional Regularization

- Only known method consistent with symmetries of QCD (Lorentz, gauge)
- Observe that divergences only arise in four dimensions
- Work in d>4 dimensions, $d=4-2\varepsilon$, $\varepsilon<0$!

- Feynman rules and Dirac algebra (fairly!) straightforward to generalize to d dimensions

→ d-dimensional matrix elements

with propagators $1/p^2$

and d-dependent numerators

- (cf dimensional reduction)

Dimensional Regularization Scale

- Gauge coupling is only dimensionless in 4 dimensions
- Convenient to extract its dimensional part…
- Define $g_s^{(d)} = g_s \, \mu^{\varepsilon}$,
  where g_s is the dimensionless coupling

→ $\alpha_s^{(d)} = \alpha_s \, \mu^{2\varepsilon}$

- μ is a completely arbitrary scale that will cancel in all finite cross sections
- should not be confused with renormalization scale μ_R (but often is)

Phase Space Integrals

- How to perform phase space integration in d dimensions?

$$d^d k\,\delta(k^2) = \frac{d^{d-1}k}{2\omega} = \tfrac{1}{2}\,\omega^{d-3} d\omega\, d\Omega$$

- Analytic continuation:
  - First consider integer d
- Four dimensions:

  k = ω(1;sinϕ sinθ, cosϕ sinθ, cosθ)
- Five dimensions:

  k = ω(1;sinψ sinϕ sinθ, cosψ sinϕ sinθ, cosϕ sinθ, cosθ)
- d dimensions:

  k = ω(1;…,cosθ)

Phase Space Integrals (contd)

- Angular integrals

$$d\Omega_1 = d\phi = 2\pi$$

$$d\Omega_2 = d\phi \sin\theta \, d\theta = 4\pi$$

$$d\Omega_3 = d\psi \sin\phi \, d\phi \sin^2\theta \, d\theta = 2\pi^2$$

$$d\Omega_d = \Omega_{d-1} \sin^{d-1}\theta \, d\theta = 2\pi^{(d+1)/2}/\Gamma[(d+1)/2]$$

Regularization

- Soft divergence:

$$\int_0 \omega^{1-2\epsilon} d\omega \, \frac{1}{\omega^2} = \int_0 \frac{d\omega}{\omega^{1+2\epsilon}} \sim -\frac{1}{2\epsilon} \qquad \epsilon < 0$$

- Collinear divergence:

$$\int_0 \sin^{1-2\epsilon}\theta \, d\theta \, \frac{1}{\theta^2} \sim \int_0 \frac{d\theta}{\theta^{1-2\epsilon}} \sim -\frac{1}{2\epsilon} \qquad \epsilon < 0$$

$\sigma(e^+e^- \to qq\bar{q}arg)$

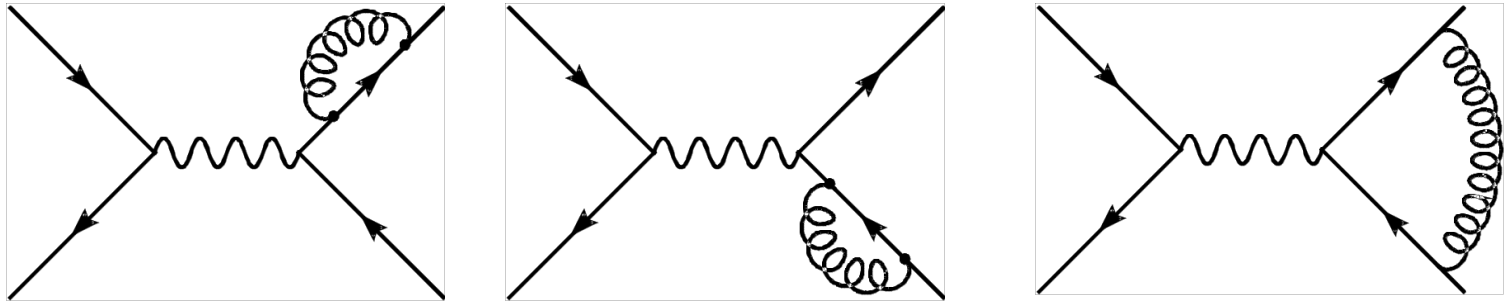
- Final result:

$$\sigma_{q\bar{q}g} = \sigma_0 \, C_F \frac{\alpha_s}{2\pi} H(\epsilon) \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} + \mathcal{O}(\epsilon) \right)$$

- Positive divergent

$\sigma(e^+e^- \to \text{qqbar})$ at one loop

- Only other process that can contribute at this order
- Three diagrams



- $\mathcal{M}_1 \propto \alpha_s \mathcal{M}_0$

→ $|\mathcal{M}_1|^2$ higher order still,
but interference $\Re\text{e}\{\mathcal{M}_0^* \mathcal{M}_1\}$ contributes at $\mathcal{O}(\alpha_s)$

$$\sigma_{q\bar{q}} = \sigma_0 \, C_F \frac{\alpha_s}{2\pi} H(\epsilon) \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon) \right)$$

Loop integrals

- In quantum mechanics, must sum over all unobserved quantum numbers at amplitude level

→ Integrate over **all** gluon momenta

$$\int d^d k$$

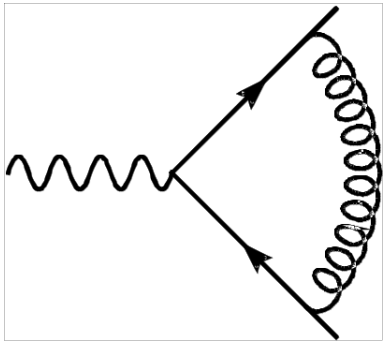
Massless quark self-energy

- Has to be zero in dimensional regularization
- However, divergent in both infrared and ultraviolet
- Split integration region in two, at arbitrary point Λ

→ $\Lambda^{-2\varepsilon}/\varepsilon - \Lambda^{-2\varepsilon}/\varepsilon$

- Subtle use of dimensional regularization

Vertex Correction



- Also divergent in infrared and ultraviolet
- Ultraviolet divergence exactly cancels the one from the self-energy (cf Ward identities)

→ total amplitude ultraviolet-finite

but infrared-divergent

$$\sigma_{q\bar{q}} = \sigma_0 \, C_F \frac{\alpha_s}{2\pi} H(\epsilon) \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon) \right)$$

Total Cross Section

$$\sigma_{q\bar{q}g} = \sigma_0\, C_F \frac{\alpha_s}{2\pi} H(\epsilon) \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} + \mathcal{O}(\epsilon) \right)$$

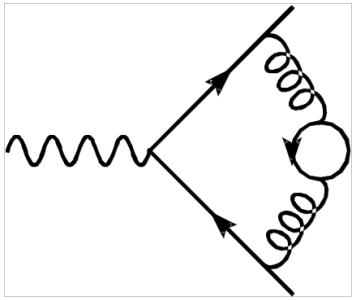
$$\sigma_{q\bar{q}} = \sigma_0\, C_F \frac{\alpha_s}{2\pi} H(\epsilon) \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon) \right)$$

- All divergences cancel, so regularization can be removed and we obtain a scheme-independent result

$$\begin{aligned} \sigma_{e^+e^-\to\text{hadrons}} &= \sigma_0 \left(1 + C_F \frac{\alpha_S}{2\pi} \frac{3}{2} \right) \\ &= \sigma_0 \left(1 + \frac{\alpha_S}{\pi} \right) \end{aligned}$$

Aside on Renormalization

- Next order is ultraviolet divergent, from diagrams like



- Need to renormalize at scale μ_R
- Implemented in dimensional regularization as

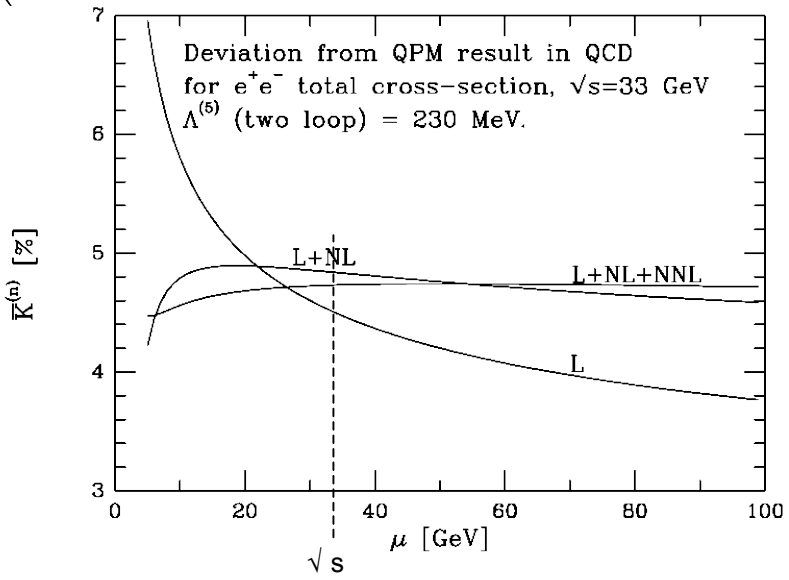
$$\alpha_s(\mu) = \alpha_s + \beta_0 F(\varepsilon) (\mu/\mu_R)^{2\varepsilon} 1/\varepsilon \alpha_s^2$$

- $F(\varepsilon)$ defines the renormalization scheme:
  - $F(\varepsilon)=1$: MS scheme
  - $F(\varepsilon)=(4\pi)^{\varepsilon}/\Gamma(1-\varepsilon)=1+(\ln 4\pi-\gamma_E)\varepsilon$: MSbar scheme

Higher orders

- Corrections known up to $\mathcal{O}(\alpha_s^3)$

$$\sigma_{e^+e^-\to\text{hadrons}} = \sigma_0 \left(1 + \frac{\alpha_S(\mu_R)}{\pi} + C_2\left(\frac{\mu_R^2}{s}\right)\left(\frac{\alpha_S(\mu_R)}{\pi}\right)^2 + C_3\left(\frac{\mu_R^2}{s}\right)\left(\frac{\alpha_S(\mu_R)}{\pi}\right)^3\right)$$



- Small if $\mu_R \sim \sqrt{s}$

→ leading order sufficient to understand phenomenology

α_s measurements

S.Bethke hep-ex/0211012

- R(LEP) = 20.775 ± 0.027
- $R_0(M_Z)$ = 19.984

→ α_s = 0.124 ± 0.004

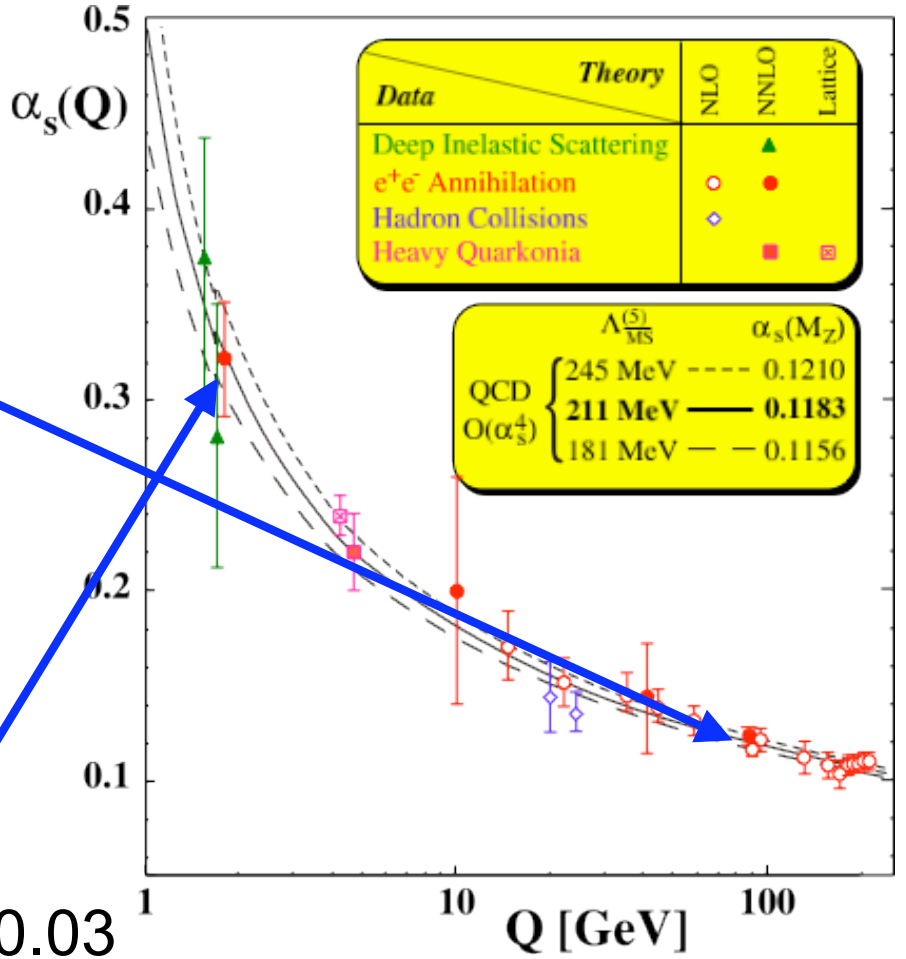
- R(PETRA) = 3.88 ± 0.03
- R_0(34 GeV) = 3.69

→ α_s = 0.162 ± 0.026

→ $\alpha_s(M_z)$ = 0.134 ± 0.018

- $\alpha_s(M_\tau$ = 1.77 GeV) = 0.33 ± 0.03

→ $\alpha_s(M_z)$ = 0.118 ± 0.004



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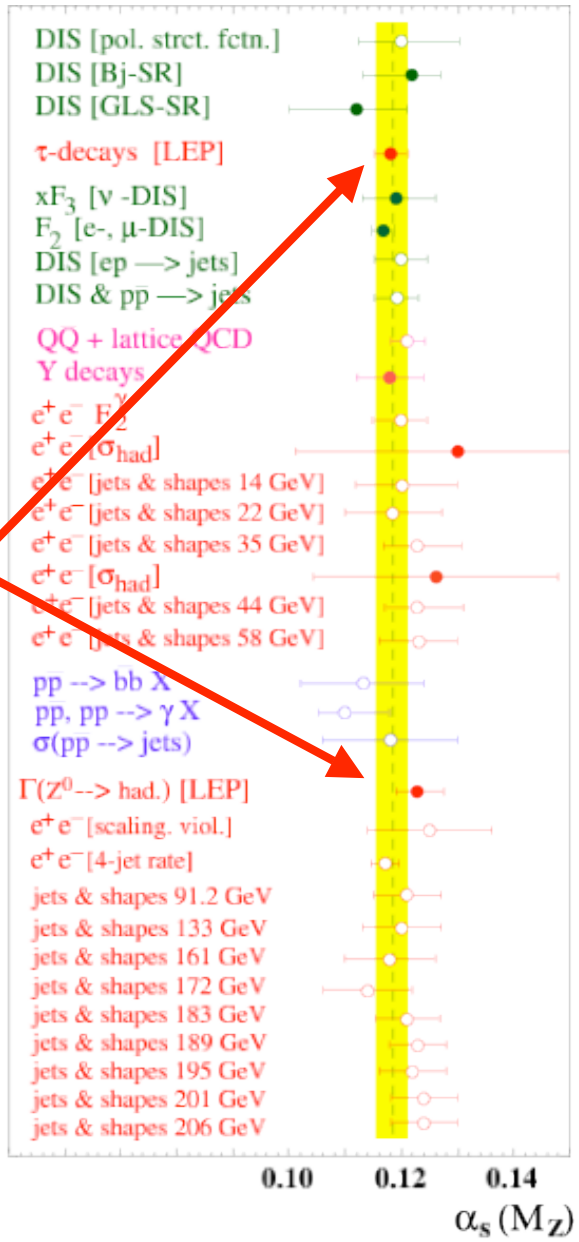


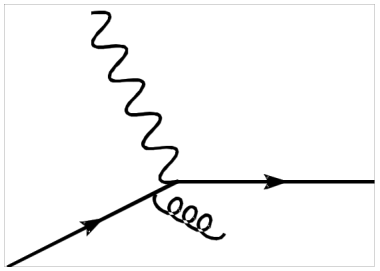
Deep Inelastic Scattering Revisited at NLO

- Parton Model based on assumption of non-interacting partons frozen within the hadron
- We know that partons are not non-interacting:

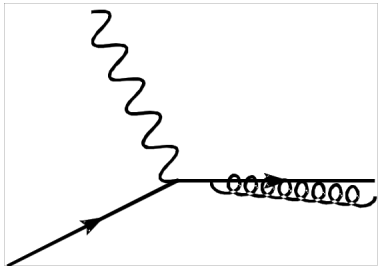
  will QCD corrections spoil parton model?
- No: find only **logarithmic** violations of scaling
- $\mathcal{O}(\alpha_s)$ corrections

Divergent Contributions to Quark Scattering

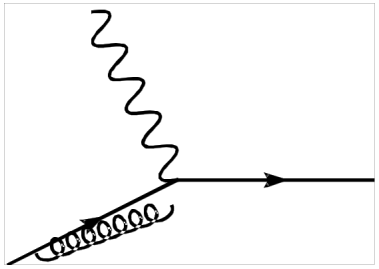
1) Soft



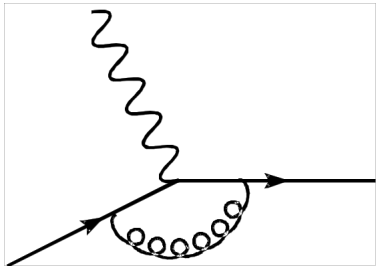
2) Collinear to outgoing



3) Collinear to incoming



4) Virtual

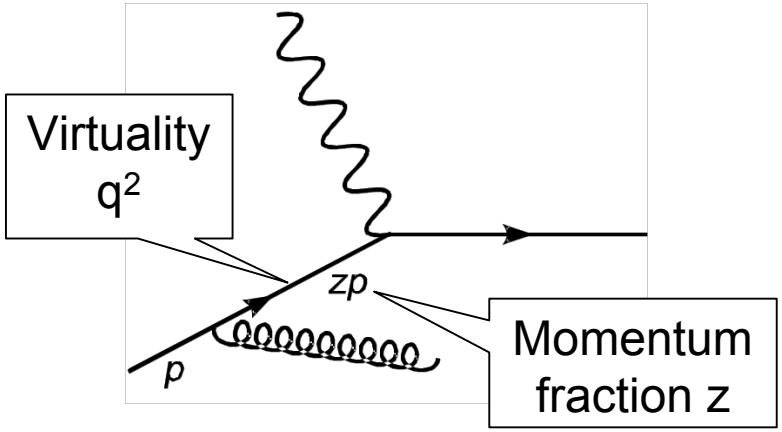


(1), (2), and (4): indistinguishable from tree-level configuration: divergences cancel

(3): incoming momentum fraction $\eta > x$, (4): $\eta = x$

→ divergence does not cancel!

Initial State Collinear Limit



- $d\sigma_{q\to qg} = d\sigma_{q\to q} \times \frac{\alpha_s}{2\pi} \times C_F \frac{1+z^2}{1-z} \frac{dz}{z} \frac{dq^2}{q^2}$
- $\hat{P}_{qq}(z) = C_F \frac{1+z^2}{1-z}$ is **unregularized** DGLAP ("Altarelli—Parisi") splitting function
- Note: this form is universal

Altarelli—Parisi Splitting Function

- $\hat{P}_{qq}(z) = C_F \frac{1+z^2}{1-z}$ is unregularized DGLAP splitting function
- Note, singular as z→1, soft emission, $\hat{P}_{qq}(z) \to C_F \frac{2}{1-z}$
- Virtual contribution contains compensating singularity, but exactly at z = 1
- Define $P_{qq}(z)$ to be the sum of real and virtual

$$\begin{aligned} P_{qq}(z) &= C_F \frac{1+z^2}{1-z} + C_F \delta(1-z) \left\{ \frac{3}{2} - \int_0^1 dz' \frac{2}{1-z'} \right\} \\ &\equiv C_F \left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right) \end{aligned}$$

- Exercise: Calculate $\int_0^1 dz\, P_{qq}(z)$ and comment on its significance

Crash course on Plus Distributions

- For any **function** *f(x)*, define **distribution** $f(x)_+$ as

$$f(x)_+ \equiv f(x) - \delta(1-x)\int_0^1 dx'\, f(x')$$

- Most useful when *f(x)* diverges at $x \to 1$
- Note that if *g(x)* is a smooth function as $x \to 1$, then

$$\int_0^1 g(x) f(x)_+ \, dx = \int_0^1 [g(x) - g(1)] f(x) dx$$

  is finite
- and $\int_a^1 g(x) f(x)_+ \, dx = \int_a^1 [g(x) - g(1)] f(x) dx - \int_0^a g(1) f(x)\, dx$

Contribution to F_2

- The total contribution to F_2 is…

$$F_2(x,Q^2) = x\sum_q e_q^2 \int_x^1 \frac{d\eta}{\eta} f_q(\eta) \left[\delta\left(1-\frac{x}{\eta}\right) + \frac{\alpha_s}{2\pi}\left\{P_{qq}\left(\frac{x}{\eta}\right)\int_0 \frac{dq^2}{q^2} + \bar{R}_{qq}\left(\frac{x}{\eta}\right)\right\}\right]$$

- Must be integrated over all virtualities
- Divergent

Factorization

- q^2 integral infrared divergent

  Comes from long timescales

→ part of hadronic wave function

- Introduce factorization scale μ_F

  $q^2 < \mu_F$ → hadron wave function: absorb into $f_q(\eta)$

→ becomes $f_q(\eta, \mu_F)$

- Contribution to F_2 is then

$$F_2(x,Q^2) = x\sum_q e_q^2 \int_x^1 \frac{d\eta}{\eta} f_q(\eta,\mu_F^2) \left[\delta\left(1-\frac{x}{\eta}\right) + \frac{\alpha_s}{2\pi}\left\{P_{qq}\left(\frac{x}{\eta}\right)\log\frac{Q^2}{\mu_F^2} + R_{qq}\left(\frac{x}{\eta}\right)\right\}\right]$$

- Physical cross sections: μ_F-independent
- Finite order perturbation theory: μ_F-dependent

Evolution Equations

- Perturbation theory cannot predict parton distribution functions, but
- Can use μ_F independence of F_2 to derive μ_F dependence of parton distribution function

$$\mu_F^2 \frac{\partial f_q(x,\mu_F^2)}{\partial \mu_F^2} = \frac{\alpha_s(\mu_F^2)}{2\pi} \int_x^1 \frac{dy}{y} f_q(y,\mu_F^2)\, P_{qq}\left(\frac{x}{y}\right) + \ldots$$

or $$\mu_F^2 \frac{\partial f_q(x,\mu_F^2)}{\partial \mu_F^2} = \frac{\alpha_s(\mu_F^2)}{2\pi} P_{qq} \otimes f_q(\mu_F^2)$$

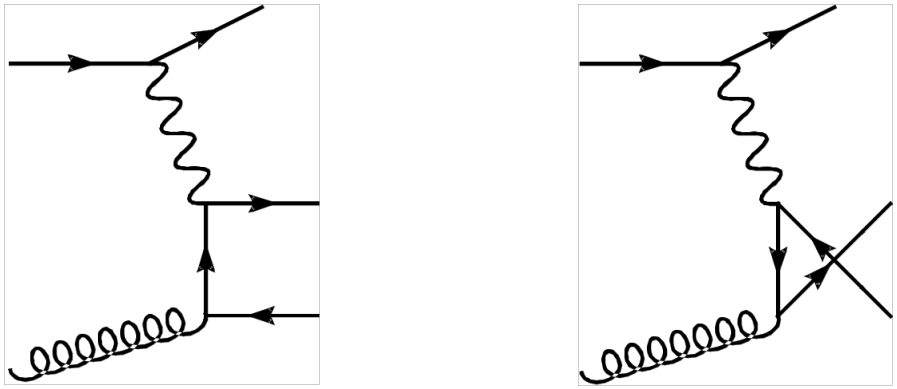
"DGLAP evolution"

- Note: pdf at given x and μ_F depends on all higher x and lower μ_F values

→ Structure function fits at low Q^2 → predictions at high Q^2

Gluon Distribution

- Gluons can also participate in hard scattering:



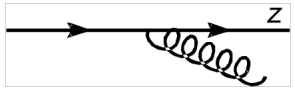
→ more initial-state singularities have to be factorized into parton distribution functions

→ evolution equation becomes set of coupled equations

$$\mu_F^2 \frac{\partial f_q(x,\mu_F^2)}{\partial \mu_F^2} = \frac{\alpha_s(\mu_F^2)}{2\pi} P_{qq} \otimes f_q(\mu_F^2) + \frac{\alpha_s(\mu_F^2)}{2\pi} P_{qg} \otimes f_g(\mu_F^2)$$

$$\mu_F^2 \frac{\partial f_g(x,\mu_F^2)}{\partial \mu_F^2} = \frac{\alpha_s(\mu_F^2)}{2\pi} P_{gq} \otimes f_q(\mu_F^2) + \frac{\alpha_s(\mu_F^2)}{2\pi} P_{gg} \otimes f_g(\mu_F^2)$$

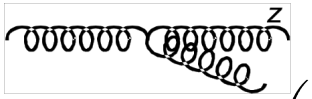
- $P_{ab}(z)$ (related to) probability of finding parton of type *a* inside parton of type *b*



$$P_{qq}(z) = C_F \left(\frac{1+z^2}{1-z} \right)_+$$

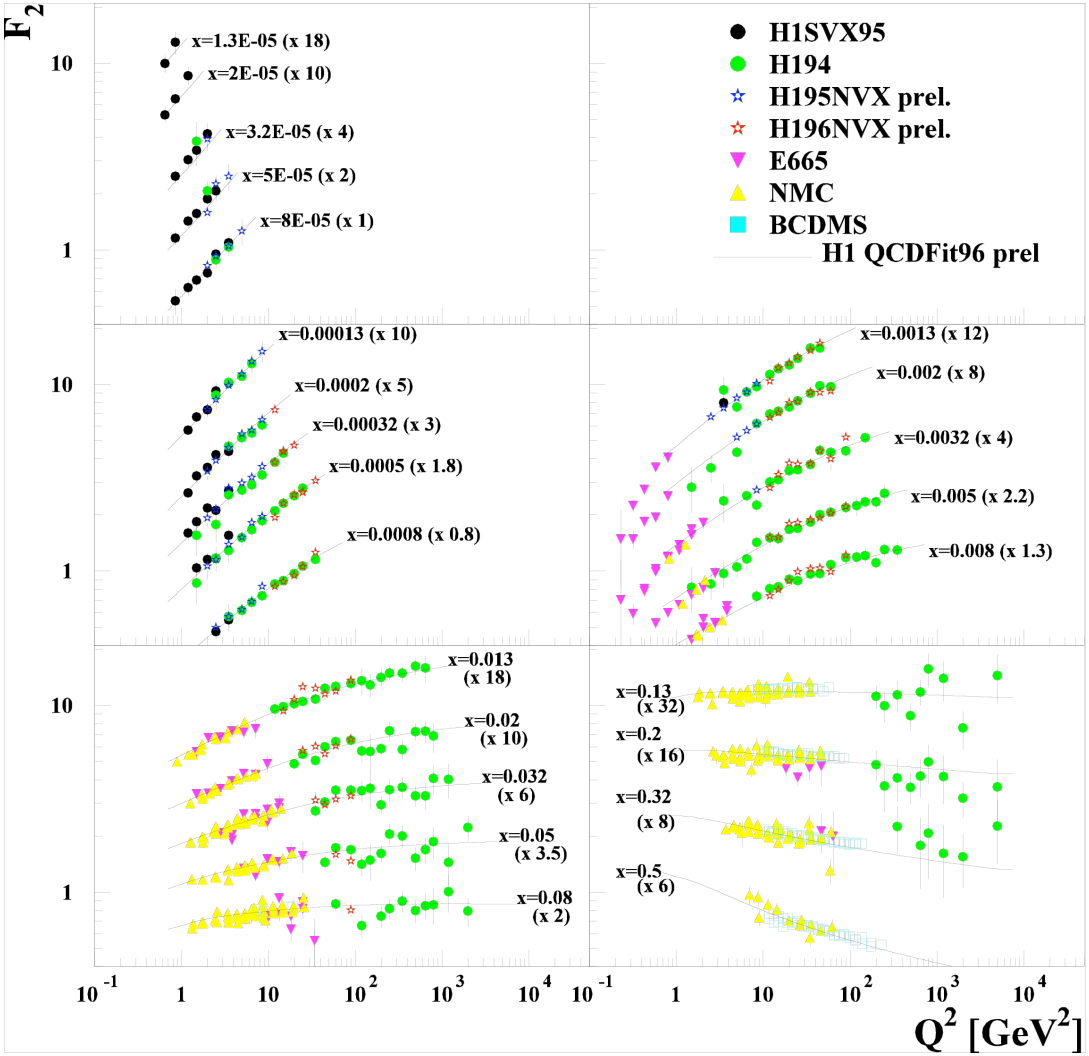
$$P_{gq}(z) = C_F \frac{1+(1-z)^2}{z}$$

$$P_{qg}(z) = T_R(z^2+(1-z)^2)$$



$$P_{gg}(z) = C_A \left(\frac{2}{(1-z)_+} + \frac{2}{z} - 2(2-z(1-z)) \right) + 2\pi\beta_0 \delta(1-z)$$

Global Fits



- Note: Q^2 evolution, violation of Bjorken scaling, entirely due to factorization
- cf renormalization in one-scale observable

Summary

- NLO calculations are hard!
- Because real and virtual corrections are each divergent
- Need a regularization scheme

- Jet cross sections should not disturb this cancellation: infrared safety

- Processes with incoming partons have extra divergences: factorize into universal parton distribution functions

- Violations of Bjorken scaling driven by DGLAP evolution