Quantum ChromoDynamics

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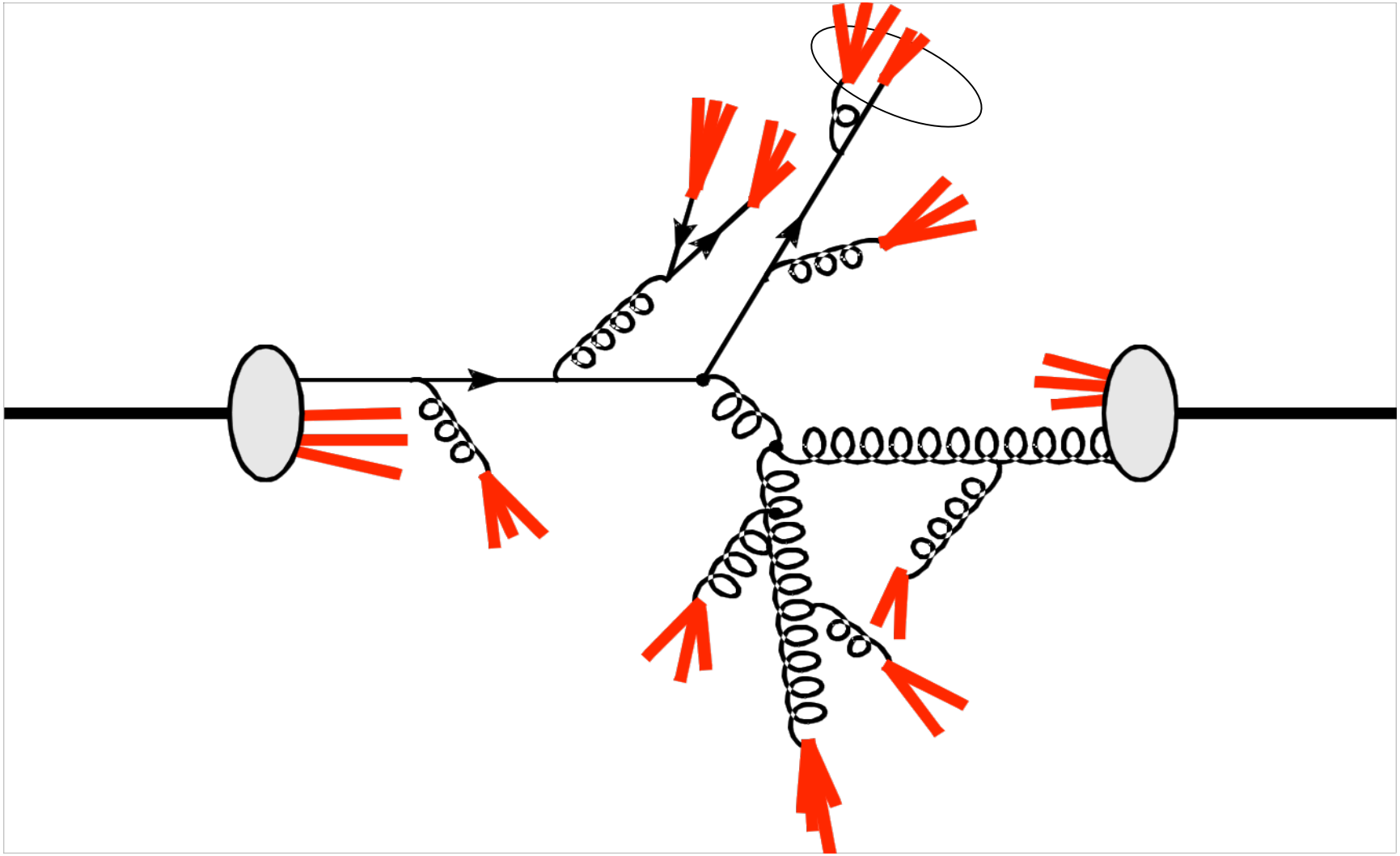
Overview



Outline

1. Basics
2. QCD Phenomenology at Tree Level
3. Higher Order corrections
4. Monte Carlo techniques

Quarks and Gluons

- Fundamental particles of the Standard Model:

$$d^a \quad s^a \quad b^a \qquad a=1,2,3$$
$$u^a \quad c^a \quad t^a$$

$$g^A \qquad A=1,2\ldots8$$

Quark and Gluon Properties

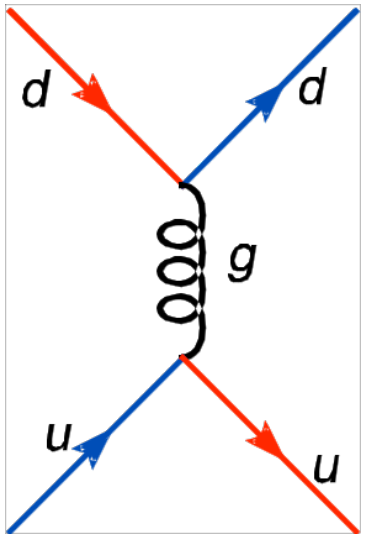
- Quarks: spin ½, three colours
  - down: d^1 d^2 d^3 $Q=-1/3$ $m_d \sim 5$–10 MeV
  - up: u^1 u^2 u^3 $Q=+2/3$ $m_u \sim 1$–5 MeV
  - strange: s^1 s^2 s^3 $Q=-1/3$ $m_s \sim 80$–150 MeV
  - charm: c^1 c^2 c^3 $Q=+2/3$ $m_c \sim 1$–1.5 GeV
  - bottom: b^1 b^2 b^3 $Q=-1/3$ $m_b \sim 4.5$ GeV
  - top: t^1 t^2 t^3 $Q=+2/3$ $m_t \sim 170$ GeV
- Masses come from electroweak symmetry breaking allowed, but not required, by QCD

- Gluons: spin 1, eight colours

  g^1 g^2 g^3 g^4 g^5 g^6 g^7 g^8 $Q=0$ $m_g=0$

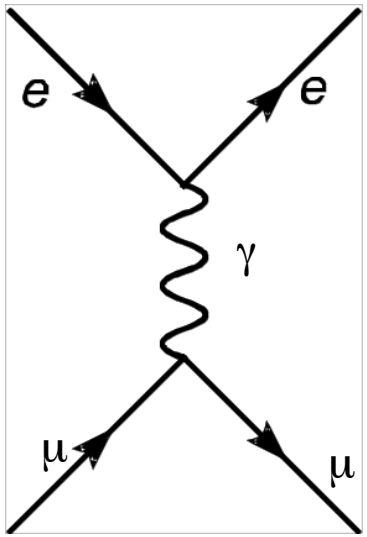
Similarities with QED

- Quark scattering



- Massless gauge boson scattering
- Colour charges

- Lepton scattering



- Massless gauge boson scattering
- Electric charges

QED is our archetypal gauge theory

Introduction to QED and Gauge Invariance

- Classical Lagrangian density for non-interacting fermion:

$$\mathcal{L}_{\text{ferm}} = \bar{f}(i\partial\!\!\!/ - m)f,$$

  is invariant under **global** changes of gauge:

$$f \to f' = \mathrm{e}^{i\theta} f,$$

Note: $\partial\!\!\!/ = \gamma^\mu a_\mu$
$\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$

- Can we make it invariant under **local** changes of gauge?

$$f(x) \to f'(x) = \mathrm{e}^{i\theta(x)} f(x).$$

- Yes. At the expense of introducing a **new field** A_μ that also transforms:

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{i}{e}(\partial_\mu \mathrm{e}^{i\theta(x)})\mathrm{e}^{-i\theta(x)},$$

- And replacing ∂_μ by the **covariant derivative**:

$$D_\mu = \partial_\mu + ie\,A_\mu.$$

- Must make new field physical by introducing kinetic term:

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

where $F_{\mu\nu}$ is the "field strength tensor", given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

- Classical QED Lagrangian density therefore

$$\mathcal{L}_{\text{classical}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{f}(i\not{D} - m) f.$$

(See separate handout for details to extend this to n types of fermion. Note that their charges are independent).

- Note that there is no way to make a photon mass term $m^2 A^\mu A_\mu$ gauge invariant $\Rightarrow$ photon massless

Gauge Fixing

- Quantum Field Theory → perturbation theory → Feynman rules
- Feynman rules are read off from the action,

$$S = i \int d^4x \, \mathcal{L}.$$

- Propagators are inverse of bilinear terms, photon:

$$\Delta_{\gamma,\mu\nu}(p) \times i[p^2 g^{\nu\sigma} - p^\nu p^\sigma] = \delta^\sigma_\mu.$$

- Does not have an inverse!

- Use gauge invariance: choose to work in a fixed gauge
  - eg covariant gauge: $\partial^\mu A_\mu = 0$ at every spacetime point
- In this gauge physical results cannot be changed by adding a term to the Lagrangian (therefore also any other gauge)

$$\mathcal{L}_{\text{gauge-fixing}} = -\frac{1}{2\lambda}(\partial^\mu A_\mu)^2, \qquad (\text{arbitrary } \lambda)$$

- (cf method of undetermined Lagrange multipliers)
- λ arbitrary:
  - λ=1: Feynman gauge
  - $\lambda \to 0$: Landau gauge
- Must now solve:

$$\Delta_{\gamma,\mu\nu}(p) \times i[p^2 g^{\nu\sigma} - (1 - \frac{1}{\lambda})p^\nu p^\sigma] = \delta^\sigma_\mu,$$

$$\Delta_{\gamma,\mu\nu} = \frac{i}{p^2}\left(-g_{\mu\nu} + (1-\lambda)\frac{p_\mu p_\nu}{p^2}\right).$$

Aside on Axial Gauges

- Another possible choice: $n^\mu A_\mu = 0$ for some arbitrary n
- Propagator a lot more complicated:

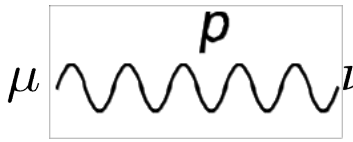
$$\Delta_{\gamma,\mu\nu} = \frac{i}{p^2}\left(-g_{\mu\nu} + \frac{n_\mu p_\nu + p_\mu n_\nu}{n\cdot p} - \frac{n^2 + \lambda\, p^2}{(n\cdot p)^2}\, p_\mu p_\nu\right)$$

  but has a big advantage:
- On-shell photons have two polarization states transverse to direction of motion

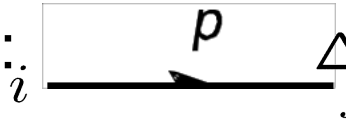
→ "physical gauge"

- Somewhat simpler with $\lambda \to 0$ and $n^2=0$ "lightcone gauge" but still complicated by spurious singularities $n\cdot p=0$

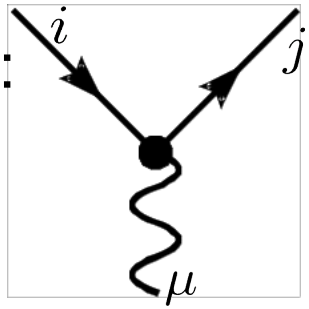
Feynman Rules of QED in Feynman gauge

- Photon propagator: 

$$\Delta_{\gamma,\mu\nu}(p) = i\frac{-g_{\mu\nu}}{p^2}$$

- Fermion propagator: 

$$\Delta_{ij}(p) = \frac{i}{(\not{p} - m)_{ji}} = i\frac{(\not{p} + m)_{ji}}{p^2 - m^2}$$

- Fermion—photon vertex: 

$$\Gamma^{\mu}_{ji} = -i\,e\,\gamma^{\mu}_{ji}$$

SU(3) and colour

- QCD identical to QED except replace gauge transform of a simple phase (U(1)) by non-Abelian group (SU(N_c))
- N_c=3 from experiment:
  - Baryon wave functions
  - Total e^+e^- cross section ($\propto N_c$)
  - π^0 decay rate ($\propto 1/N_c$)
- Nevertheless, keep N_c arbitrary when making calculations
  - useful diagnostic tool

Crash course on SU(N)

- SU(N) is the group of NxN matrices that are
  - unitary ($U^\dagger U=1$)
  - with determinant +1
- If U is an element of SU(N) that is infinitesimally close to the identity, it can be written $U=1+iG$ with G infinitesimal
- Easy to show that G is Hermitian ($G^\dagger=G$) and traceless
- **Choose** a basis set of matrices t^A with $A=1,\ldots,N^2-1$
  - arbitrary complex matrix has $2N^2$ degrees of freedom,
  - Hermitian $\Rightarrow$ N^2 constraints
  - traceless $\Rightarrow$ 1 constraint
- such that any G can be written as $G=\sum_A \varepsilon^A t^A$
- Note: $a,b,c,\ldots=1,\ldots,N$, $A,B,C,\ldots=1,\ldots,N^2-1$

Crash course on SU(N) continued

- Can show that $[t^A,t^B] \equiv t^A t^B - t^B t^A$ must be antiHermitian
- Therefore can be written as a linear combination of t^Cs

$$[t^A,t^B] \equiv i\, f^{ABC} t^C$$

(the Lie algebra of the group)

- f^{ABC} are real ("structure constants of the group")
- t^A define the fundamental representation of the group
- Any other set of objects that obey the Lie algebra form another representation of the group, eg
- $(T^A)_{BC} \equiv -if^{ABC}$ which define the adjoint representation
- Can make any finite element by infinite number of infinitesimal ones:

$$U = \lim_{N\to\infty} (1+i\theta^A t^A/N)^N = \exp(i\theta^A t^A) \equiv \exp(it\cdot\theta)$$

- Note: $U^{-1} = \exp(-it\cdot\theta)$

Colour Sums

- Actually never need to know the colour matrices, only how to sum over them:

$$\begin{aligned}
\mathrm{Tr}(t^A t^B) &= \tfrac{1}{2}\delta^{AB} \equiv T_R \delta^{AB} \\
\sum_A t^A_{ab} t^A_{bc} &= \frac{N^2-1}{2N}\delta_{ac} \equiv C_F \delta_{ac} \\
\mathrm{Tr}(T^C T^D) &= \sum_{A,B} f^{ABC} f^{ABD} = N\delta^{CD} \equiv C_A \delta^{CD}
\end{aligned}$$

- C_F and C_A are Casimir operators of fundamental and adjoint representations
- T_R=1/2
- C_F=4/3
- C_A=3

Constructing the Lagrangian of QCD

- Classical Lagrangian density for n non-interacting quarks

$$\mathcal{L}_{\text{quarks}} = \sum_{i}^{n} \bar{q}_i^a (i\partial\!\!\!/ - m_i)_{ab} q_i^b$$

- Note:
  - quark wave functions carry an N-vector 'phase'
  - $(i\partial\text{-}m_i)_{ab}$ is proportional to identity matrix in colour space
- $\mathcal{L}_{\text{quarks}}$ is invariant under **global** changes of gauge:

$$q_a \to q'_a = \mathrm{e}^{it\cdot\theta_{ab}} q_b$$

- To make it invariant under **local** changes of gauge

$$q_a(x) \to q'_a(x) = \mathrm{e}^{it\cdot\theta(x)_{ab}} q_b(x)$$

introduce covariant derivative:

$$D_{\mu,ab} = \partial_\mu 1_{ab} + i g_s (t\cdot A_\mu)_{ab}$$

$$\mathcal{L}_{\text{quarks}} = \sum_i^n \bar{q}_i^a (i\not{D} - m_i)_{ab} q_i^b \qquad D_{\mu,ab} = \partial_\mu 1_{ab} + i g_s (t \cdot A_\mu)_{ab}$$

- A_μ^A are coloured vector fields that transform so that

$$D'_{\mu,ab} q'_b(x) = \mathrm{e}^{it\cdot\theta(x)_{ab}} D_{\mu,bc} q_c(x)$$

- giving

$$t\cdot A'_\mu = \mathrm{e}^{it\cdot\theta(x)} t\cdot A_\mu \mathrm{e}^{-it\cdot\theta(x)} + \frac{i}{g_s}(\partial_\mu \mathrm{e}^{it\cdot\theta(x)})\mathrm{e}^{-it\cdot\theta(x)}$$

- (note that $t\cdot A_\mu$ in QCD ≅ A_μ in QED)

- Again have to introduce kinetic term:

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4} F^A_{\mu\nu} F^{\mu\nu}_A$$

- but have to modify field strength tensor for gauge inv.

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - g_s f^{ABC} A^B_\mu A^C_\nu$$

- Note:
  - Gluons self-interacting: three- and four-point vertices

  → Asymptotic freedom at high energies

  & Strong interactions at low energies
- Directly follow from non-Abelian nature of gauge group
- Also:
  - All quark colour charges must be same

Gauge Fixing

- Proceeds exactly as in QED
- More complicated renormalization in unphysical gauges

→ Presence of unphysical fields “ghosts”

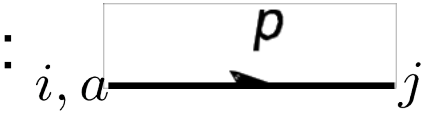
  – cancel off unphysical degrees of freedom

- Details unimportant here

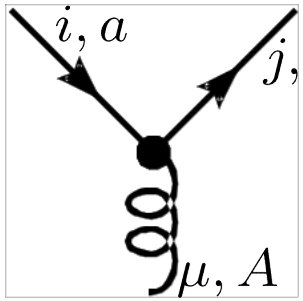
Feynman Rules of QCD in Feynman gauge

- Gluon propagator:

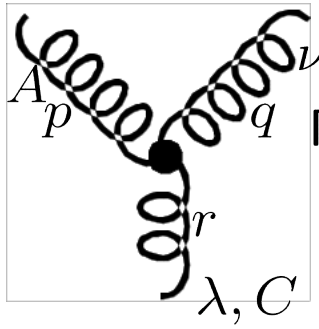
$$\Delta^{AB}_{g,\mu\nu}(p) = \delta^{AB}\, i\frac{-g_{\mu\nu}}{p^2}$$

- Quark propagator: 

$$\Delta^{ab}_{q,ij}(p) = \delta^{ab}\, i\frac{(\not{p} + m_q)_{ji}}{p^2 - m_q^2}$$

- Quark—gluon vertex: 

$$\Gamma^{\mu\, A}_{ji\, ab} = -i\, g_s\, t^A_{ba}\gamma^\mu_{ji}$$

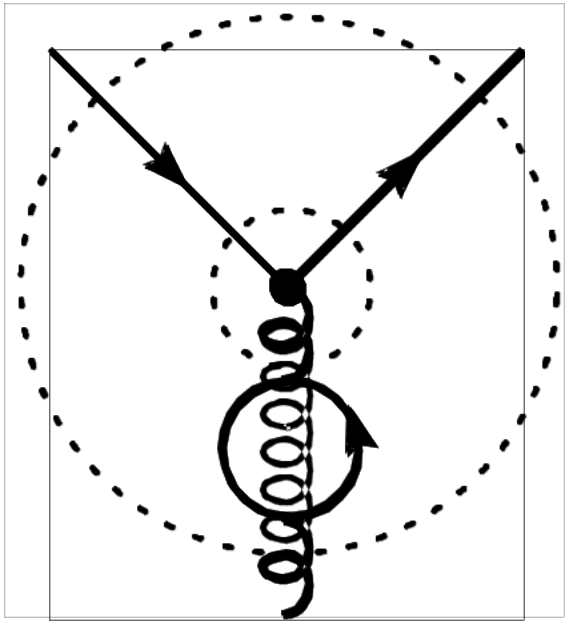
- Three—gluon vertex: 

$$p + q + r = 0$$
$$\Gamma_{ggg} = -g_s f^{ABC}[(p-q)^\lambda g^{\mu\nu} + (q-r)^\mu g^{\nu\lambda} + (r-p)^\nu g^{\lambda\mu}]$$

- Plus four-gluon vertex and ghost rules

Coupling Constant α_s

- Defined by $\alpha_s = g_s^2/4\pi$
- g_s is a parameter in the Lagrangian: physical?
- Redefine as strength of quark—gluon coupling:



- Higher orders of perturbation theory:
  - Modified by loop effects
- Is this part of the
  - vertex, or
  - the rest of the diagram
- Decide by introducing a scale μ_R
- Renormalization scale

- Gauge theory of pointlike particles: loop is divergent

Renormalization Group Equation

- Formally study by considering a dimensionless physical observable R, function of only one physical scale Q^2 (and assume not sensitive to quark masses)
- After renormalization, R can only be $R(Q^2/\mu^2,\alpha_s(\mu^2))$
- μ is not a physical scale: R must be independent of μ

$$\mu^2\frac{d}{d\mu^2}R(Q^2/\mu^2,\alpha_S)=0 \;=\; \left[\mu^2\frac{\partial}{\partial\mu^2}+\mu^2\frac{\partial\alpha_S}{\partial\mu^2}\frac{\partial}{\partial\alpha_S}\right]R$$

$$\equiv \left[\mu^2\frac{\partial}{\partial\mu^2}+\beta(\alpha_S)\frac{\partial}{\partial\alpha_S}\right]R$$

- Solved by $R(1,\alpha_s(Q^2))$
- Note:

$$\mu^2\frac{\partial\alpha_S}{\partial\mu^2}=\beta(\alpha_S)$$

  - Q^2 dependence of R entirely given by renormalization
  - μ^2 dependence of α_s can be derived from calculation of R
  - α_s small → R perturbatively calculable → $\beta(\alpha_s)$ is too

The β Function

- $\beta(\alpha_S) = -\alpha_S^2(\beta_0 + \beta_1\alpha_S + \beta_2\alpha_S^2 + \beta_3\alpha_S^3 + \ldots)$
- $\beta_0 = \dfrac{11C_A - 4T_R N_f}{12\pi}$
- N_f is number of quark flavours that can appear in loops
- Note: $\beta_0>0 \Rightarrow \beta(\alpha_s)<0$ when α_s small
  - Asymptotic freedom: interactions become weak at high energy
  - Infrared slavery: interactions become strong at low energy
- Solution for $\beta_{>0}=0$ (“leading log”)

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2)\beta_0 \ln\frac{Q^2}{\mu^2}}$$

- Higher orders essential for quantitative calculation, but not for qualitative understanding

Choosing μ^2

- Physical quantities do not depend on μ^2
- Truncating perturbation theory leads to μ^2 dependence
- eg, suppose $R=R_1\alpha_s+\ldots$,

  then $R(1,\alpha_S(Q^2)) \approx R_1\alpha_S(Q^2)$

$$= R_1\alpha_S(\mu^2)\left[1-\beta_0\alpha_S(\mu^2)\ln\frac{Q^2}{\mu^2}+\beta_0^2\alpha_S^2(\mu^2)\ln^2\frac{Q^2}{\mu^2}+\ldots\right]$$

- Although μ^2 arbitrary, choosing it far from Q^2 guarantees a large truncation error (converse not true)

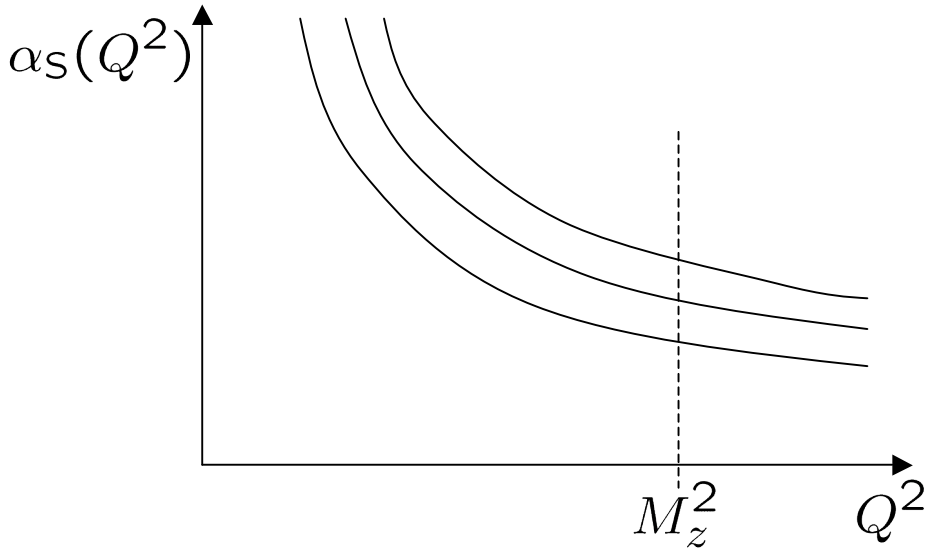
$\Rightarrow$ Should choose μ^2 close to Q^2

- Use μ^2 variation as (crude) estimator of truncation error?

Measuring α_s

- Measurement of α_s at different scales

→ measurement of which curve our universe lies on



- Choose any convenient labelling of the curves, eg $\alpha_s(M_z)$

$\Lambda_{\rm QCD}$

- Another way to label the running coupling
- Although Lagrangian of massless QCD has no scale, renormalization introduces a dimensionful parameter,

$$\Lambda^2 = \mu^2 \exp \int_{\alpha_S(\mu^2)}^{\infty} \frac{dx}{\beta(x)} \approx \mu^2 e^{-1/\beta_0 \alpha_S(\mu^2)}$$

  ("dimensional transmutation")
- At leading log, Λ is the scale at which coupling is infinite
- However, precise value depends on
  - order of perturbation theory
  - renormalization scheme
  - number of flavours N_f
  - approximations used in solving RG equation

→ Less practical than α_s(some agreed perturbative scale)

Quark masses

- m_q another parameter of Lagrangian:
  for physical calculation need physical definition
- Take to infinity and put on a balance? Confinement
- Renormalize at scale μ_R → running mass
- Perturbative evolution equation, leading log solution

$$m(Q^2) = M\left[\alpha_S(Q^2)\right]^{\frac{1}{\pi\beta_0}}$$

- *M* is RG-invariant constant (cf Λ_{QCD})
- Note: increasing $Q^2 \Rightarrow$ decreasing m^2

- Alternative scheme: pole mass
- Define m_q as pole of propagator $i\frac{(\not{p}+m_q)}{p^2-m_q^2}$ to all orders
- Useful for $Q \sim m_q$, but large higher order logs for $Q \gg m_q$

Decoupling

- If observable R is finite for massless quarks then quark mass effects are suppressed by powers of $(m_q/Q)^n$ $n \geq 1$
- Effect of very heavy quarks suppressed by $(Q/m_q)^n$ $n \geq 2$

→ Approximation of N_f massless flavours (cf β_0)

Summary

- QCD is a gauge theory
- The gauge symmetry predicts gluon self-interaction
- Becomes strongly interacting at low energies → non-perturbative
- and weakly interacting at high energies → perturbation theory
- The main tools we'll use are
  - factorization of low-energy non-perturbative effects
  - renormalization and decoupling of high-energy physics
  - perturbation theory → Feynman rules

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