SYMMETRY OF DIFFERENTIAL EQUATIONS FOR NON-POLYLOGARITHMIC LOOP INTEGRALS AND QUADRATIC RELATIONS FOR THEIR ϵ EXPANSION COEFFICIENTS.

Roman N. Lee

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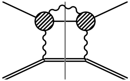
HISTORICAL REMARK

Remarkable fact

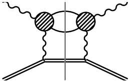
In QFT perturbative calculations, both polylogarithmic and non-polylogarithmic functions appeared in one and the same journal issue by one and the same author Racah [1934a,b] (as it is expected, polylogs appeared a bit earlier, on pages 461-476, while the elliptic integrals — on pages 477-481).

Energy loss in $e^- Z \to e^- \gamma Z$



$$\frac{\left(12\varepsilon^2+4\right)\ln(\varepsilon+p)}{3\varepsilon p}-\frac{(8\varepsilon+6p)\ln^2(\varepsilon+p)}{3\varepsilon p^2}-\frac{2}{\varepsilon p}\mathrm{Li}_2\left(-\frac{2p}{\varepsilon-p}\right)-\frac{4}{3}$$

Total cross section of $\gamma Z \to e^+ e^- Z$



$$\frac{692+468\xi+76\xi^2+108\xi^3}{27(1+\xi)^3}\mathrm{K}(\xi^2)-\frac{692+360\xi+692\xi^2}{27(1+\xi)^3}\mathrm{E}(\xi^2)$$
$$-\frac{4(1-\xi)^2}{(1+\xi)^2}\int_0^\xi\frac{\mathrm{K}(\eta^2)d\eta}{1-\eta}+\frac{16(1-\xi)^2}{(1+\xi)^2}\int_0^\xi\frac{d\zeta}{1-\zeta^2}\int_0^\zeta\frac{\mathrm{K}(\eta^2)d\eta}{1-\eta}$$

Both results have been obtained by explicit integration of spectra.

Note the contemporary form of the latter result.

1. Consider a family of integrals

$$j(n_1, \ldots, n_N) = \int \frac{d^d l_1 \ldots d^d l_L}{D_1^{n_1} \ldots D_N^{n_N}} \, .$$

Integrals are functions of kinematic variables x_i and space-time dimension $d = 4 - 2\epsilon$.

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3. Find differential equations [Kotikov, 1991, Remiddi, 1997] (and/or dimensional recurrences [Tarasov, 1996]) for master integrals

Differential equations

$$\frac{\partial}{\partial x_i}\mathbf{j} = M(\mathbf{x}, \epsilon)\mathbf{j}$$

Dimensional recurrences

$$\mathbf{j}(\epsilon + 1) = R(\mathbf{x}, \epsilon)\mathbf{j}(\epsilon)$$

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Steps ##2,4,5 can be troublesome. In what follows I will mostly talk about general solution and its properties.

- In 2013 the following observation has been made [Henn, 2013]. A careful choice of the master integrals ('canonical' masters J) leads to the differential equation in the form:

$$\partial_x J = \epsilon S(x) J, \quad S(x) = \sum_i \frac{S_i}{x - a_i}\,, \qquad (\epsilon\text{-form})$$

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Solution via polylogs

Once the ϵ-form is found, the general solution

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- The algorithm of finding the ϵ-form has been suggested in [Lee, 2015]. The algorithm consists of applying a sequence of rational transformations

$$M \to T^{-1}(MT - \partial_x T)$$

which eventually reduce the differential system to ϵ-form. Now it is implemented in three publicly available codes: `epsilon` [Prausa, 2017], `Fuchsia` [Gituliar and Magerya, 2017], `Libra` [Lee, 2020].

However, soon it was realized that rational transformation matrices $T(x, \epsilon)$ are not always sufficient. Note that if we allow for any functions in $T(x, \epsilon)$, the trivial ϵ-form $\partial J = 0$ is always achievable by choosing $T = \mathrm{Pexp}\left[\int M\, dx\right]$. Thus, the question is how to extend the class of transformations in the most natural and minimal way.

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- Sometimes several auxiliary variables $y, z, \ldots$ are needed $x = f(y) = g(z) = \ldots$. For $U = \mathrm{Pexp}\left[\epsilon \int S(x, y, \ldots) dx\right]$ to be expressible via polylogs, these variables either should not appear simultaneously in any iterated integral coming from the expansion of U or there should be universal rationalizing variable.

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- Sometimes there is no transformation to ϵ-form in the class described in two previous items. Criterion of irreducibility: [Lee and Pomeransky, 2017]. If the system has the form $\partial_x J = (A + \epsilon B)J$ then one can use $T = U_0$, where U_0 is the solution of $\partial_x U_0 = AU_0$. In particular, U_0 can be expressible via elliptic polylogs.

Systems reducible to ϵ-form.

We can construct solution via polylogs with the following benefits:

- Functional relations, argument transformations.
- Explicit asymptotics, series expansion wrt argument.
- Fast numerical computation.
- Grading by transcendental weight, known bases, use of `pslq`.

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Systems irreducible to ϵ-form.

Special treatment of each irreducible case.

- Functional relations: little is known in general.
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- There seems to be a nontrivial symmetry in all available examples of differential systems for multiloop integrals.
- One can use existing tools (in particular, `Libra`) to detect this symmetry and construct quadratic constraints for the coefficients of ϵ-expansion.
- There seem to be an especially adjusted basis where these constraints have the most simple form. For ϵ-reducible systems this is just the canonical basis. To systematically find this form for ϵ-irreducible cases some new algorithms may be needed.

Definition (rational equivalence)

We will say that two systems

$$\partial_x \mathbf{j}_1 = M_1 \mathbf{j}_1 \qquad \text{and} \qquad \partial_x \mathbf{j}_2 = M_2 \mathbf{j}_2 \qquad (*)$$

are rationally (more precisely, x-rationally) equivalent if $\exists$ a rational change of functions $j_1 = Tj_2$ which maps the first system to the second. Or, alternatively, if

$$M_2 = T^{-1}(M_1 T - \partial_x T)\,.$$

We will write $M_2 \overset{R}{\approx} M_1$ for matrices of rationally equivalent systems.

- The monodromy groups of rationally equivalent systems are isomorphic.
- Thanks to dimensional recurrences we have $M(\epsilon - 1) \overset{R}{\approx} M(\epsilon)$.
- The ϵ-reducibility means the rational equivalence to the system $\partial_x \mathbf{J} = \epsilon S(x)\mathbf{J}$ with S being independent of ϵ.

Question

Given two systems (*) how to find T or to establish that it does exists?

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3. Otherwise, search for a constant (independent of x) **invertible** matrix $T_3(\epsilon)$ such that

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Proof becomes a trivial exercise given the Proposition of [Lee and Pomeransky, 2017].

Definition (ϵ-transposition)

Let $M = M(\epsilon)$ be a matrix depending on ϵ. We call the involution

$$M(\epsilon) \to M^\star(\epsilon) = M^\intercal(-\epsilon)$$

the ϵ-transposition. We define analogously the ϵ-symmetric matrices ($M = M^\star$) and ϵ-orthogonal matrices ($U\,U^\star = 1$).

Definition (ϵ-conjugated system)

We call the differential system $\partial_x \mathbf{j} = -M^\star(x,\epsilon)\mathbf{j}$ the ϵ-conjugated to $\partial_x \mathbf{j} = M(x,\epsilon)\mathbf{j}$ (note the minus sign).

NB: for matrices independent of ϵ one can omit "ϵ" in the above notations ("ϵ-transposition" $\longrightarrow$ "transposition" etc.)

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Observation

Let

$$\partial_x \mathbf{j} = M\mathbf{j} \tag{DE}$$

be such a homogeneous differential system corresponding to some block irreducible to BT form. Then we observe on many examples that the ϵ-conjugated differential system

$$\partial_x \mathbf{j} = -M^{\star}\mathbf{j} \tag{DE$\star$}$$

is **rationally equivalent** to the original system (DE).

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SYMMETRIC ϵ- AND $(\epsilon + \frac{1}{2})$-FORM

This observation is not new for the systems which are reducible to ϵ- and/or to $(\epsilon + 1/2)$- forms. In Ref. Lee [2018] it was observed that such systems can be reduced to symmetric form $\partial_x J = \mu S(x) J$, ($S = S^T$ and $\mu = \epsilon$ or $\epsilon + \frac{1}{2}$).

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- The converse statement «If the system can be reduced to μ-form (μ is either ϵ or $\epsilon + 1/2$) and the ϵ-conjugated system is rationally equivalent to the original, then the system can be reduced to **symmetric** μ-form.» can be proved using Proposition in [Lee and Pomeransky, 2017].

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However, there are relevant examples of systems which can not be reduced neither to ϵ-form nor to $(\epsilon + 1/2)$-form — see below.

Remarks

1. Note that if $M(\epsilon) \overset{R}{\approx} -M^{\intercal}(-\epsilon)$ holds for $d = d_0 - 2\epsilon$ then it necessarily holds for any $d = d_0 + k - 2\epsilon$ ($k \in \mathbb{Z}$ can be both even and **odd**) — thanks to dimensional recurrences.
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Let us now consider the general solution of $\partial \mathbf{j} = M\mathbf{j}$ in the form of path-ordered exponent, $U(x, x_0|\epsilon) = \text{Pexp}\left[\int_{x_0}^{x} M\, dx\right]$. It is easy to see that $U^{-1\intercal}(x, x_0| - \epsilon) = \text{Pexp}\left[-\int_{x_0}^{x} M^{\star}\, dx\right]$, so, is a general solution of ϵ-conjugated differential system $\partial \mathbf{j} = M^{\star}\mathbf{j}$.

Now we use our observation: $M = T^{-1}((-M^{\star})T - \partial T)$. For path-ordered exponents it translates to

$$U(x, x_0|\epsilon) = T^{-1}(x, \epsilon)U^{-1\intercal}(x, x_0| - \epsilon)T(x_0, \epsilon)$$

Therefore, we have

$$U^{\intercal}(x, x_0| - \epsilon)T(x, \epsilon)U(x, x_0|\epsilon) = T(x_0, \epsilon)$$

Quadratic constraints

Let $\mathbf{j}_1(x,\epsilon)$ and $\mathbf{j}_2(x,\epsilon)$ be any two (possibly coinciding) solutions of the system $\partial_x \mathbf{j} = M(x,\epsilon)\mathbf{j}$. Then it is possible to find (using the available techniques) a rational matrix $T(x,\epsilon)$, such that

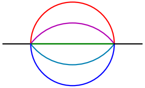
$$\mathbf{j}_1^{\mathsf{T}}(x,-\epsilon)T(x,\epsilon)\mathbf{j}_2(x,\epsilon) = \mathrm{const}$$

is independent of x. (The right-hand side can be found by considering some suitable asymptotics.)

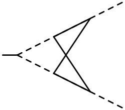
Note that the opposite sign of ϵ in $j_1^{\mathsf{T}}(x,-\epsilon)$, so this relation concerns the solutions of two **different** differential systems (related via $\epsilon \to -\epsilon$). But within perturbative calculations we are interested in the coefficients of ϵ expansion, which are the same, up to alternating sign, for $j_1(x,-\epsilon)$ and $j_1(x,\epsilon)$. Thus, expanding the above relation in ϵ, we obtain an infinite set of quadratic relations for the expansion coefficients of the solution of the original differential system.

EXAMPLES OF SYSTEMS IRREDUCIBLE TO BOTH ϵ- AND $(\epsilon + 1/2)$-FORMS.

Reducible to $\epsilon + 1/2$-form

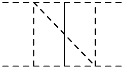


- Massive sunrises
- Two-loop vertex [von Manteuffel and Tancredi, 2017]

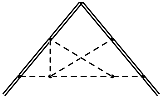


Irreducible to $\epsilon + 1/2$-form

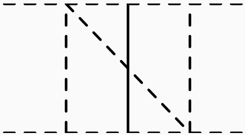
- 3-loop forward massive box from [Mistlberger, 2018] (4 masters)

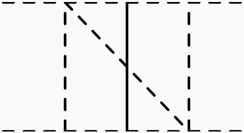


- 4-loop HQET vertex from [Brüser et al., 2020] (11 masters)

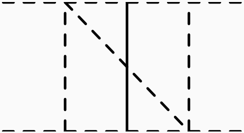


For all above families we have checked that our observation is valid.





$$M = \begin{pmatrix} -\frac{3\epsilon+2}{s} & \frac{1}{s} & 0 & 0 \\ -\frac{(3s-1)(4\epsilon+1)(5\epsilon+1)}{s\left(s^2+11s-1\right)} & \frac{-2(\epsilon+1)s^2+11(2\epsilon-1)s-6\epsilon}{s\left(s^2+11s-1\right)} & -\frac{(s-2)\epsilon}{s^2+11s-1} & \frac{(s+3)\epsilon}{s^2+11s-1} \\ \frac{(s+3)(4\epsilon+1)(5\epsilon+1)}{s\left(s^2+11s-1\right)} & -\frac{(3s+4)(4\epsilon+1)}{s\left(s^2+11s-1\right)} & \frac{-2s^2-(3\epsilon+22)s+\epsilon+2}{s\left(s^2+11s-1\right)} & \frac{(2s+1)\epsilon}{s\left(s^2+11s-1\right)} \\ \frac{(s-2)(4\epsilon+1)(5\epsilon+1)}{s\left(s^2+11s-1\right)} & -\frac{(4s-3)(4\epsilon+1)}{s\left(s^2+11s-1\right)} & -\frac{5\epsilon}{s^2+11s-1} & \frac{-2(\epsilon+1)s^2-(19\epsilon+22)s+\epsilon+2}{s\left(s^2+11s-1\right)} \end{pmatrix}$$



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Using `Libra` we find

$$W = s^2 \begin{pmatrix} 2\left(3s^2+198s-53\right)\epsilon^2+1 & -6\epsilon s^2+(2-100\epsilon)s+14\epsilon-1 & -2s((s+13)\epsilon-2) & s((6s-22)\epsilon+3) \\ 6\epsilon s^2+2(50\epsilon+1)s-14\epsilon-1 & -6s^2-4s+1 & -2s(s+2) & -s(14s+3) \\ 2s((s+13)\epsilon+2) & -2s(s+2) & 6s^2 & -8s^2 \\ -s((6s-22)\epsilon-3) & -s(14s+3) & -8s^2 & -31s^2 \end{pmatrix}$$

such that

$$M(s,\epsilon) = W^{-1}\left(-M^{\mathsf{T}}(s,-\epsilon)W - \partial_s W\right)$$

Suppose that we have found a rational transformation $W = W(x, \epsilon)$, such that

$$M(x, \epsilon) = W^{-1} \left[-M^{\star}(x, \epsilon) W - \partial_x W \right]$$

Then, if we make the transformation $M \to T^{-1}(MT - \partial_x T)$, it induces the change of W:

$$W \to T^{\star} W T$$

Question

What is the possible simplest form of W we can achive by rational transformations T?

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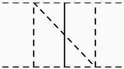
- For ϵ-reducible systems the answer is $W = 1$. It is exactly by passing to symmetric ϵ-form.
- For several other examples we find that we can make W to be constant matrix (independent of both x and ϵ!). In particular, this is true for  .

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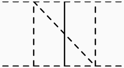
- For several other examples we find that we can make W to be constant matrix (independent of both x and ϵ!). In particular, this is true for  .

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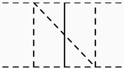
- For several other examples we find that we can make W to be constant matrix (independent of both x and ϵ!). In particular, this is true for  .

Unfortunately, we lack algorithms to answer the question in a systematic way for any given system.

Two-loop equal mass cut sunrise in $d = 2 - 2\epsilon$ dimensions can be expressed via hypergeometric functions ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1-\epsilon; y\right)$ and ${}_2F_1\left(\frac{4}{3}, \frac{2}{3}; 1-\epsilon; y\right)$ [Tarasov, 2006]. The quadratic constraint reads

$$
\begin{aligned}
{}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1-\epsilon; y\right) {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \epsilon+1; y\right) + \tfrac{(y-1)}{3\epsilon} {}_2F_1\left(\tfrac{4}{3}, \tfrac{2}{3}; 1-\epsilon; y\right) {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \epsilon+1; y\right) \\
+ \tfrac{(1-y)}{3\epsilon} {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1-\epsilon; y\right) {}_2F_1\left(\tfrac{4}{3}, \tfrac{2}{3}; \epsilon+1; y\right) = 1\,.
\end{aligned}
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\end{aligned}
$$

When expanded in ϵ it results to the following “shuffling-like” identities ($N = 0, 1, \ldots$):

$$
\sum_{n=0}^{N} (-)^n H_{\alpha,\mathbf{1}_n} H_{\alpha,\mathbf{1}_{N-n}} + [1 + (-)^N] y(1-y) \sum_{n=0}^{N+1} (-)^n \left(\partial H_{\alpha,\mathbf{1}_n}\right) H_{\alpha,\mathbf{1}_{N+1-n}} = \delta_{N0}\,.
$$

where

$$
H_{\alpha,\mathbf{1}_n}(y) = \sum_{j=0}^{\infty} \frac{(3j)!}{3^{3j}\,(j!)^3} y^j S_{\underbrace{1,\ldots,1}_{n}}(j)
$$

Multiloop sunrise integrals in $d = 2$ in coordinate scape are expressed via functions

$$\mathrm{IKM}[\{a_0, b_0\}_{m_0}, \{a_1, b_1\}_{m_1}, \ldots, s] = \int dx\, x^s \prod_k [I_0(m_k x)]^{a_k} [K_0(m_k x)]^{b_k} \,.$$

Examples of quadratic relations: Broadhurst-Robert-like relations

Multiloop sunrise integrals in $d = 2$ in coordinate scape are expressed via functions

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Our approach allows one to obtain quadratic relations for those functions. E.g., for two-loop sunrise we obtain

$$\mathrm{IKM}\left[\{2,1\}_m, \{0,1\}_1, 1\right] \mathrm{IKM}\left[\{3,0\}_m, \{0,1\}_1, 3\right] - \mathrm{IKM}\left[\{2,1\}_m, \{0,1\}_1, 3\right] \mathrm{IKM}\left[\{3,0\}_m, \{0,1\}_1, 1\right] = \frac{4\left(1-5m^2\right)}{\left(1-m^2\right)^2\left(1-9m^2\right)^2} .$$

The right-hand side has been calculated from the limit $m \to 0$.
At 3 loops we, e.g. have

$$9\mathrm{IKM}\left(\{3,1\}_{\frac{1}{4}}, \{0,1\}_1, 1\right) \mathrm{IKM}\left(\{3,1\}_{\frac{1}{4}}, \{0,1\}_1, 3\right) - 16\mathrm{IKM}\left(\{3,1\}_{\frac{1}{4}}, \{0,1\}_1, 1\right)^2 = 20 ,$$

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In Ref. [Broadhurst and Roberts, 2018] remarkable quadratic relations have been conjectured

$$\sum_{k,l} \mathrm{IKM}\left[\{\tilde{n}, N-\tilde{n}\}_1, k\right] D_{kl}(N, \tilde{n}, n) \mathrm{IKM}\left[\{n, N-n\}_1, l\right] = \pi^{N+1-\tilde{n}-n} B(N, \tilde{n}, n)\,,$$

where $D(N, \tilde{n}, n)$ and $B(N, \tilde{n}, n)$ are rational numerical matrices. Recently, these relations have been proved in Ref. [Fresán et al., 2020], except that matrix D was defined differently. Within our approach we have been able to do the same (with yet another definition of D).

- The differential equations for multiloop integrals seem to always exhibit the symmetry with respect to ϵ-conjugation

$$\partial_x \mathbf{j} = M(x,\epsilon)\mathbf{j} \qquad \longleftrightarrow \qquad \partial_x \mathbf{j} = -M^{\intercal}(x,-\epsilon)\mathbf{j}$$

  These two systems appear to be rationally equivalent. We already have tools to find the rational transformation W connecting the two systems.
- This symmetry leads to an infinite set of nontrivial quadratic constraints for the coefficients of ϵ expansion of solutions.
- For the systems reducible to ϵ-form we can achieve $W = 1$. To find the simplest form of the matrix W for ϵ-irreducible systems we need to invent new algorithms.

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