Elliptic polylogarithms and superstring amplitudes

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Outline

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Multiple polylogarithms

- $X = \{x_0, x_1\}$ formal alphabet, X^* set of possible words $w = x_{i_1} \cdots x_{i_n}$ in non-commutative letters x_0, x_1.
- In $\mathbb{C}$ consider $l_0 := [0, i\infty]$, $l_1 := [1, i\infty]$, $U := \mathbb{C} \setminus \{l_0, l_1\}$ $\rightsquigarrow \pi_1(U, x) = 1$ and $\log(z)$ well-defined on U.

Multiple polylogarithms (of one variable)

Family of holomorphic functions on U indexed by words $w \in X^*$, defined by setting $L_{x_0^n}(z) := (\log(z))^n/n!$ and then for any other w by

$$L_{x_{i_1} \cdots x_{i_n}}(z) := \int_0^z \frac{dz'}{z' - i_1} L_{x_{i_2} \cdots x_{i_n}}(z').$$

Rmk 1: $$L_{x_0^{k_r - 1} x_1 \cdots x_0^{k_1 - 1} x_1}(z) = (-1)^r \sum_{0 < n_1 < \cdots < n_r} \frac{z^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}}.$$

Rmk 2: Multiple polylogs are *multi-valued functions* on $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$.

Multiple zeta values

Multiple zeta values

For $k_1, \ldots, k_r \in \mathbb{N}$, $k_r \geq 2$, we set

$$\zeta(k_1, \ldots, k_r) := \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} = L_{x_0^{k_r-1}x_1 \cdots x_0^{k_1-1}x_1}(1)$$

- Ring $\mathcal{Z}$ of multiple zeta values, conjecturally graded by weight $k_1 + \cdots + k_r$
- $L_w(z)$ has at most logarithmic divergence at $z = 1$
  $\rightsquigarrow$ regularized special value $L_w(1) \in \mathcal{Z}$ for any $w \in X^*$

The KZ equation

Theorem

The formal series $L(z) := \sum_{w \in X^*} L_w(z)\, w \in \mathbb{C}\langle\langle X^* \rangle\rangle$ is the unique holomorphic solution on U of the differential equation

$$\frac{\partial}{\partial z} L(z) = \left(\frac{x_0}{z} + \frac{x_1}{z-1} \right) L(z)$$

such that $L(z) \sim \exp\left(x_0 \log(z)\right)$ as $z \to 0$.

In particular, $L(1)$ (regularized value) is the Drinfel'd associator.

Theorem (F. Brown)

There is a unique real-analytic solution $\mathcal{L}(z) \in \mathbb{C}\langle\langle X^* \rangle\rangle$ on $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$ of the KZ-equation s.t. $\mathcal{L}(z) \sim \exp\left(x_0 \log |z|^2\right)$ as $z \to 0$.

Single-valued multiple polylogarithms

Definition

If we write $\mathcal{L}(z) =: \sum_{w \in X^*} \mathcal{L}_w(z)\, w$, we call $\mathcal{L}_w(z)$ **single-valued multiple polylogarithms**.

- Single-valued multiple polylogs are given by $\mathcal{Z}$-linear combinations of products $L_{w_1}(z)\overline{L_{w_2}(z)}$.
- $\mathcal{L}_{x_0}(z) = L_{x_0}(z) + \overline{L_{x_0}(z)} = \log(z) + \overline{\log(z)} = \log|z|^2$.
- The map $\mathrm{sv} : L_w(z) \to \mathcal{L}_w(z)$ respects shuffle identities. We call it the *single-valued projection*.

Single-valued multiple zeta values

Regularised special values $\mathcal{L}_w(1)$ belong to $\mathcal{Z}$, but span a conjecturally smaller subring $\mathcal{Z}^{\mathrm{sv}}$. We call them **single-valued multiple zeta values**. Conjecturally, the single-valued projection restricts to well-defined $\mathrm{sv} : L_w(1) \to \mathcal{L}_w(1) \rightsquigarrow$ we denote $\zeta^{\mathrm{sv}}(k_1, \ldots, k_r) := \mathrm{sv}(\zeta(k_1, \ldots, k_r))$

Examples: $\zeta^{\mathrm{sv}}(2k) = 0$, $\zeta^{\mathrm{sv}}(2k+1) = 2\,\zeta(2k+1)$

Perturbative expansion of string amplitudes

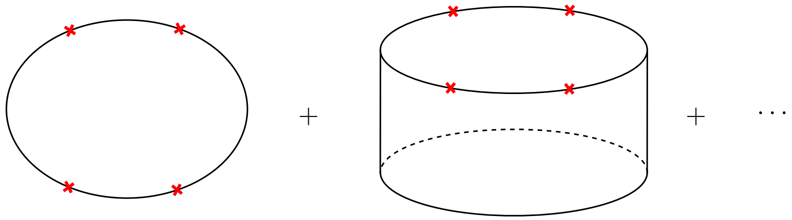

Figure: Four open oriented strings

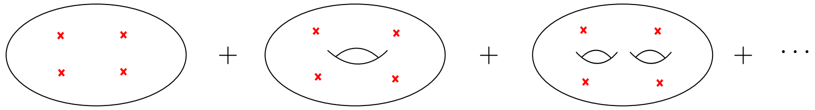

Figure: Four closed oriented strings

Building blocks of genus zero amplitudes

Set $N+3$ =number of string states, $\rho, \sigma \in \mathfrak{S}_N$ permutations, $s_{ij} := \alpha'(k_i \cdot k_j)$ Mandelstam variables

Open string building blocks:

$$Z^{(N)}_{\rho,\sigma}(\boldsymbol{s}) := \int_{0 \le x_{\sigma(1)} \le \cdots \le x_{\sigma(N)} \le 1} \frac{\prod_{i=1}^{N} dx_i \prod_{1 \le i < j \le N+2} |x_i - x_j|^{s_{ij}}}{x_{\rho(1)}(1 - x_{\rho(N)}) \prod_{i=2}^{N} (x_{\rho(i)} - x_{\rho(i-1)})}$$

Closed string building blocks:

$$J^{(N)}_{\rho,\sigma}(\boldsymbol{s}) := \int_{(\mathbb{P}^1_{\mathbb{C}})^N} \frac{\prod_{i=1}^{N} d^2 z_i \prod_{1 \le i < j \le N+2} |z_i - z_j|^{2 s_{ij}}}{z_{\rho(1)} \overline{z}_{\sigma(1)} (1 - z_{\rho(N)})(1 - \overline{z}_{\sigma(N)}) \prod_{i=2}^{N} (z_{\rho(i)} - z_{\rho(i-1)})(\overline{z}_{\sigma(i)} - \overline{z}_{\sigma(i-1)})}$$

The 4-point case ($N = 1$)

Open strings: the Veneziano amplitude

$$Z^{(1)}_{\mathrm{id},\mathrm{id}}(s,t) = \int_{[0,1]} x^{s-1}(1-x)^{t-1}dx = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$$

$$\rightsquigarrow Z^{(1)}_{\mathrm{id},\mathrm{id}}(s,t) = \frac{s+t}{st}\exp\left(\sum_{n\geq 2}\frac{(-1)^n\zeta(n)}{n}\left(s^n+t^n-(s+t)^n\right)\right)$$

Closed strings: the Virasoro-Shapiro amplitude

$$J^{(1)}_{\mathrm{id},\mathrm{id}}(s,t) = \int_{\mathbb{P}^1_{\mathbb{C}}} |z|^{2s-2}|1-z|^{2t-2}\frac{dzd\bar{z}}{(-2\pi i)} = \frac{\Gamma(s)\Gamma(t)\Gamma(1-s-t)}{\Gamma(1-s)\Gamma(1-t)\Gamma(s+t)}$$

$$\rightsquigarrow$$

$$J^{(1)}_{\mathrm{id},\mathrm{id}}(s,t) = \frac{s+t}{st}\exp\left(-2\sum_{n\geq 1}\frac{\zeta(2n+1)}{(2n+1)}\left(s^{2n+1}+t^{2n+1}-(s+t)^{2n+1}\right)\right)$$

KLT formula: $J^{(1)}_{\mathrm{id},\mathrm{id}}(s,t) = \frac{\sin(\pi s)\sin(\pi t)}{\pi\ \sin(\pi(s+t))}\left(Z^{(1)}_{\mathrm{id},\mathrm{id}}(s,t)\right)^2$

Single-valued projection: $\mathrm{sv}\left(Z^{(1)}_{\mathrm{id},\mathrm{id}}(s,t)\right) = J^{(1)}_{\mathrm{id},\mathrm{id}}(s,t)$ coefficientwise

State of the art

- KLT relations
- Coefficients of the small α'-expansion of $Z^{(N)}_{\rho,\sigma}(\boldsymbol{s})$ belong to $\mathcal{Z}$
- Coefficients of the small α'-expansion of $J^{(N)}_{\rho,\sigma}(\boldsymbol{s})$ belong to $\mathcal{Z}^{\mathrm{sv}}$
- "$\mathrm{sv}(Z^{(N)}_{\rho,\sigma}(\alpha'\boldsymbol{s})) = J^{(N)}_{\rho,\sigma}(\alpha'\boldsymbol{s})$"
- Recursive relations between open string building blocks, originating from KZ equation

Kawai, Lewellen, Tye, Stieberger, Broedel, Mafra, Schlotterer, Terasoma, Schnetz, Brown, Dupont, Vanhove, Zerbini, ...

Outline

Cohomology and periods

- M smooth compact orientable manifold
- $H_n^{\mathrm{Sing}}(M, \mathbb{Q})$ n-th singular homology group ($\mathbb{Q}$-vector space)
- $H^n_{\mathrm{Sing}}(M, \mathbb{Q}) := (H_n^{\mathrm{Sing}}(M, \mathbb{Q}))^*$ n-th singular cohomology group
- $H^n_{\mathrm{dR,an}}(M, \mathbb{C})$ n-th de Rham cohomology group
- Stokes $\rightsquigarrow$ $[\omega] : [\sigma] \to \int_\sigma \omega$ well-defined $\rightsquigarrow$ can view elements of $H^n_{\mathrm{dR,an}}$ as elements of H^n_{Sing}
- de Rham: $H^n_{\mathrm{dR,an}}(M, \mathbb{C}) \tilde{\to} H^n_{\mathrm{Sing}}(M, \mathbb{Q}) \otimes \mathbb{C}$

Algebraic version of this story:

- X smooth algebraic variety defined over $\mathbb{Q}$
- $H^n_{\mathrm{dR,alg}}(X, \mathbb{Q})$ n-th algebraic de Rham cohomology group
- Grothendieck: $H^n_{\mathrm{dR,alg}}(X, \mathbb{Q}) \otimes \mathbb{C} \tilde{\to} H^n_{\mathrm{Sing}}(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C}$
- Entries of the matrix representing this iso are called **periods**
- The same holds for relative cohomology $\rightsquigarrow$ periods in the sense of Kontsevich-Zagier

de Rham theorem for fundamental groups

Rmk: $H_1^{\mathrm{Sing}}(M,\mathbb{Q}) \simeq \pi_1^{\mathrm{Ab}}(M,x) \simeq \mathbb{Q}[\pi_1(M,x)]/J^2$,
$J := \ker(\mathbb{Q}\pi_1(M,x) \to \mathbb{Q})$ the *augmentation ideal*
Problem: Look for functions on $\mathbb{Q}\pi_1(M,x)$, i.e. *homotopy functionals*.
Idea: Use integrals!
Obstacle: For ω smooth 1-form, $\int \omega$ homotopy functional $\Leftrightarrow$ ω closed
$\rightsquigarrow \int \omega$ only detects elements in $H_1^{\mathrm{Sing}}(M,\mathbb{Q})$
Solution (Chen): Iterated integrals!
$\omega_1,\ldots,\omega_r$ closed smooth 1-forms, if $\int \omega_1 \cdots \omega_r$ is homotopy functional then it defines $\mathbb{C}$-valued function on $\mathbb{Q}\pi_1(M,x)/J^{r+1}$

de Rham theorem for $\pi_1(M,x)$ (Chen)

Integration induces
$\{\text{Homotopy invariant iterated integrals}\} \tilde{\to} \mathcal{O}(\pi_1^{\mathrm{un}}(M,x)) \otimes \mathbb{C}$

We're more interested in the "relative version" for $\pi_1(M,x,y)$ (paths from x to y)

Models, and multiple polylogarithms

Rmk: Looking for all homotopy invariant iterated integrals using all 1-forms is hard!
Shortcut: Use a model! Identify subcomplex $A^\bullet$ of complex of smooth diff. forms $\Omega^\bullet$ s.t. $H^n(A^\bullet,\mathbb{C}) \simeq H^n(\Omega^\bullet,\mathbb{C}) = H^n_{\mathrm{dR,an}}(M,\mathbb{C})$

Theorem (Chen)

Integration induces
$\{$Homotopy invariant iterated integrals of A^1-forms$\} \xrightarrow{\sim} \mathcal{O}(\pi_1^{\mathrm{un}}(M,x)) \otimes \mathbb{C}$

A rational model for $\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}$

$\mathbb{Q} \oplus (\mathbb{Q}\frac{dz}{z} \oplus \mathbb{Q}\frac{dz}{z-1}) \hookrightarrow \Omega^\bullet(\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\})$ is a quasi-isomorphism
$\rightsquigarrow$ functions on $\pi_1^{\mathrm{un}}(\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\})$ given by (homotopy invariant) iterated integrals of dz/z and $dz/z-1$, i.e. **multiple polylogarithms**, which therefore give all the "periods of $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\},0,z)$"

Compact Riemann surfaces

- X_g compact Riemann surface of genus $g \geq 1$ (donuts)
- $\pi_1(X_g, x) = \langle A_1, \ldots, A_g, B_1, \ldots, B_g \mid \prod_i A_i B_i A_i^{-1} B_i^{-1} = 1 \rangle$
- $H_1^{\mathrm{Sing}}(X_g, \mathbb{Q}) = \mathbb{Q}^{2g}$
- $\Omega^1_{\mathrm{alg}}(X_g, \mathbb{C}) :=$ meromorphic differentials on X_g
- $\omega \in \Omega^1_{\mathrm{alg}}(X_g, \mathbb{C})$ "1st kind" if holomorphic ($\rightsquigarrow H^{1,0}(X_g)$)
- $\omega \in \Omega^1_{\mathrm{alg}}(X_g, \mathbb{C})$ "2nd kind" if meromorphic with no residues
- $H^1_{\mathrm{dR,alg}}(X_g, \mathbb{C}) := \frac{\{\text{2nd kind differentials}\}}{d\mathcal{M}(X)}$
- $H^1_{\mathrm{dR,alg}}(X_g, \mathbb{C}) \tilde{\to} H^1_{\mathrm{Sing}}(X_g, \mathbb{Q}) \otimes \mathbb{C}$ via integration:
  $\int \omega : \sigma \to \int_\sigma \omega$ well-defined by residue theorem!

"Algebraic" version of Chen's theorem for curves (Hain)

{"Iterated integrals of 2nd kind"} $\otimes\, \mathbb{C} \tilde{\to} \mathcal{O}(\pi_1^{\mathrm{un}}(X_g, x)) \otimes \mathbb{C}$

Configuration spaces

- X possibly punctured compact Riemann surface
- $C(X,n) := \{(x_1, \dots x_n) \in X^n \mid x_i \neq x_j\}$ configuration space of n distinct points on X
- $\pi_1(C(X,n),x) \rightsquigarrow$ theory of braid groups
- Periods of $\pi_1(C(\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\},n),0,z) = \pi_1(\mathfrak{M}_{0,n+3},x)$ given by multiple polylogs in several variables $\sum_{n_1<\cdots<n_r} \frac{z_1^{n_1}\cdots z_r^{n_r}}{n_1^{k_1}\cdots n_r^{k_r}}$

Why important for us?
Configuration spaces of Riemann surfaces are very natural geometric objects related to computation of **string theory amplitudes**
What's known?
Kriz and Totaro described the cohomology rings, **but** the description is not explicit for $g \geq 2 \rightsquigarrow$ hard to build models $\rightsquigarrow$ hard to construct periods of fundamental groups

Outline

Complex (one-punctured) tori

- $\tau \in \mathbb{H}$ (i.e. $\mathrm{Im}(\tau) > 0$)
- $\mathbb{T}_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ complex tori $\leftrightsquigarrow$ genus-one Riemann surfaces
- $\mathbb{T}_\tau^* := \mathbb{T}_\tau \setminus \{0\}$ one-punctured genus-one Riemann surfaces
- $\pi_1(\mathbb{T}_\tau, x) = \mathbb{Z}^2$, $\pi_1(\mathbb{T}_\tau^*, x) = \mathbb{Z} * \mathbb{Z}$
- $H^1(\mathbb{T}_\tau, \mathbb{Q}) = H^1(\mathbb{T}_\tau^*, \mathbb{Q}) = \mathbb{Q}^2$
- $H^2(\mathbb{T}_\tau, \mathbb{Q}) = \mathbb{Q}$, $H^2(\mathbb{T}_\tau^*, \mathbb{Q}) = 0$
- $\mathcal{P}(z) := \frac{1}{z^2} + \sum_{k\geq 2}(k+1)G_{k+2}(\tau)z^n$, $G_k(\tau) := \sum_{m,n} \frac{1}{(m\tau+n)^k}$, $\mathcal{O}(\mathbb{T}_\tau^*)$ polynomials in $\mathcal{P}(z)$ and $\mathcal{P}'(z)$
- $H^1_{\mathrm{dR,alg}}(\mathbb{T}_\tau, \mathbb{C}) \simeq H^1_{\mathrm{dR,alg}}(\mathbb{T}_\tau^*, \mathbb{C}) \simeq \mathbb{C}[dz] + \mathbb{C}[\mathcal{P}(z)dz]$, $\rightsquigarrow$ "periods" $(1, \tau)$ and "quasi-periods" $(G_2(\tau), 2\pi i + \tau G_2(\tau))$
- $H^1_{\mathrm{dR,an}}(\mathbb{T}_\tau, \mathbb{C}) \simeq H^1_{\mathrm{dR,an}}(\mathbb{T}_\tau^*, \mathbb{C}) \simeq \mathbb{C}[dz] + \mathbb{C}[d\bar{z}]$
- $H^1_{\mathrm{dR,an}}(\mathbb{T}_\tau, \mathbb{C}) \simeq H^1_{\mathrm{dR,alg}}(\mathbb{T}_\tau, \mathbb{C})$ via $\frac{\pi}{\mathrm{Im}(\tau)} d\bar{z} = \left(\frac{\pi}{\mathrm{Im}(\tau)} - G_2(\tau)\right) dz - \mathcal{P}(z)dz + df(z)$

The Kronecker function

The Kronecker function

For $z \in \mathbb{T}_\tau^*$ consider $\zeta := e^{2\pi i z}$, $q := e^{2\pi i \tau}$. Define $\theta(z) := q^{1/12}(\zeta^{1/2} - \zeta^{-1/2}) \prod_{j\geq 1}(1 - q^j\zeta)(1 - q^j\zeta^{-1})$ and set $F(z,\alpha) := \frac{\theta'(0)\theta(z+\alpha)}{\theta(z)\theta(\alpha)}$

Rmk 1: Multi-valued function of z on $\mathbb{T}_\tau^*$, because $F(z+1,\alpha) = F(z,\alpha)$ but $F(z+\tau,\alpha) = \exp(-2\pi i\alpha)F(z,\alpha)$, simple pole at $z=0$

Rmk 2: Writing $F(z,\alpha) =: \sum_{n\geq 0} g^{(n)}(z)\alpha^{n-1}$, we find

$g^{(0)}(z) = 1 \qquad g^{(1)}(z) = \frac{1}{z} - \sum_{n\geq 1} G_{n+1}(\tau)z^n = \zeta(z) - G_2(\tau)z$,

$g^{(n)}(z) \in \mathcal{O}(\mathbb{T}_\tau^*)[g^{(1)}(z)]$, holomorphic at $z=0$

Elliptic multiple polylogarithms (first definition)

$$\tilde{\Gamma}(n_1,\ldots,n_r \,|\, z) := \int_0^z g^{(n_1)}(z)dz \cdots g^{(n_r)}(z)dz \qquad (z \in \mathbb{C})$$

Natural def. in string theory, limit $\tau \to i\infty \rightsquigarrow$ genus-0 multiple polylogs

The Brown-Levin approach

Consider the formal 1-form $\Omega(z,\alpha) := \exp\left(2\pi i\alpha\frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\right)F(z,\alpha)dz$.

$\Omega(z,\alpha)$ is well-defined on $\mathbb{T}^*_\tau$, because $\frac{2\pi i\,\mathrm{Im}(z+\tau)}{\mathrm{Im}(\tau)} = \frac{2\pi i\,\mathrm{Im}(z)}{\mathrm{Im}(\tau)} + 2\pi i$ compensates for the monodromy of F, and is real analytic

Theorem (Brown-Levin)

Let $\Omega(z,\alpha) =: \sum_{n\geq 0}\omega^{(n)}\alpha^{n-1}$, and let $\nu := 2\pi i\, d(\mathrm{Im}(z)/\mathrm{Im}(\tau)) \rightsquigarrow$ $\mathbb{Q}$-model $A^\bullet(\mathbb{T}^*_\tau) := \mathbb{Q}\oplus(\mathbb{Q}\nu\oplus\mathbb{Q}\omega^{(0)}\oplus\mathbb{Q}\omega^{(1)}\oplus\cdots)\hookrightarrow\Omega^\bullet(\mathbb{T}^*_\tau)$ (in particular, $[\nu]$ and $[\omega_0]=[dz]$ basis of $H^1_{\mathrm{dR,an}}(\mathbb{T}^*_\tau,\mathbb{C})$)

Rmk: Similar construction for $C(\mathbb{T}^*_\tau, n)$ using $\omega^{(n)}(z_i - z_j)$

Theorem / second definition (Brown-Levin)

$\omega_{BL}(x_0,x_1) := \nu x_0 + \Omega(z,-\mathrm{ad}_{x_0})(x_1)$ $\mathrm{Lie}_{\mathbb{C}}[x_0,x_1]^\wedge$-valued, then periods of $\pi_1^{\mathrm{un}}(\mathbb{T}^*_\tau,0,z)$ (elliptic multiple polylogs) are the coefficients of $1+\int_0^z\omega_{BL}(x_0,x_1)+\int_0^z\omega_{BL}(x_0,x_1)\omega_{BL}(x_0,x_1)+\ldots$

Fact: Can be written in terms of $\mathrm{Im}(z)$ and $\tilde{\Gamma}(n_1,\ldots,n_r\,|\,z)$!

The Levin-Racinet approach

Consider the formal 1-form $H(z,\alpha) := \exp(-\alpha g^{(1)}(z))F(z,\alpha)dz$.
$H(z,\alpha)$ is well-defined (and holomorphic) on $\mathbb{T}^*_\tau$, because $g^{(1)}(z+\tau) = g^{(1)}(z) - 2\pi i$ compensates for the monodromy of F

Recall: $\frac{\pi}{\mathrm{Im}(\tau)}d\bar{z} = \left(\frac{\pi}{\mathrm{Im}(\tau)} - G_2(\tau)\right)dz - \mathcal{P}(z)dz + df(z)$
$\rightsquigarrow [2\pi i\, d(\mathrm{Im}(z)/\mathrm{Im}(\tau))] = [G_2(\tau)dz + \mathcal{P}(z)dz]$
Primitives are $2\pi i\,\mathrm{Im}(z)/\mathrm{Im}(\tau)$ and $-g^{(1)}(z)$, respectively!

Let $H(z,\alpha) =: \sum_{n\geq 0}\eta^{(n)}\alpha^{n-1}$

- $\eta^{(n)}$ 2-nd kind differential forms
- By Hain's Theorem, periods of $\pi_1^{\mathrm{un}}(\mathbb{T}^*_\tau, z_1, z_2)$ are all the homotopy invariant iterated integrals constructed with $\eta^{(n)}$

Alternative construction

$\omega_{LR}(x_0,x_1) := (G_2(\tau)+\mathcal{P}(z))dz\,x_0 + H(z,-\mathrm{ad}_{x_0})(x_1)$, then periods of $\pi_1^{\mathrm{un}}(\mathbb{T}^*_\tau, z_1, z_2)$ are the coefficients of
$1 + \int_{z_1}^{z_2}\omega_{LR}(x_0,x_1) + \int_{z_1}^{z_2}\omega_{LR}(x_0,x_1)\omega_{LR}(x_0,x_1) + \ldots$

Other approaches

"Classical" (Bloch, Zagier, Beilinson, Levin, Brown, ...)

Regularized infinite averages of genus-zero multiple polylogarithms (many variables) on $(\mathbb{T}_\tau^*)^n = (\mathbb{C}^*/q^{\mathbb{Z}} =^n \rightsquigarrow$ holomorphic multi-valued functions on $C(\mathbb{T}_\tau^*, n)$, can be written in terms of $\tilde{\Gamma}(n_1, \ldots, n_r \,|\, z)$, generate all periods of $\pi_1^{\mathrm{un}}(C(\mathbb{T}_\tau^*, n), 0, z)$ (Brown-Levin).
Related to this: "ELi-functions" (Adams, Bogner, Weinzierl, ...)

"Algebraic" (Broedel, Duhr, Dulat, Tancredi, ...)

Iterated integrals using algebraically defined integration kernels with at most simple poles $\rightsquigarrow$ generate same space as $\tilde{\Gamma}(n_1, \ldots, n_r \,|\, z)$.
Related to constructing all primitives of rational fcts on elliptic curves

Elliptic multiple zeta values

A-elliptic multiple zeta values (Enriquez)

$$I^A(n_1, \dots, n_r \,|\, \tau) := \tilde{\Gamma}(n_1, \dots, n_r \,|\, 1)$$

B-elliptic multiple zeta values (Enriquez)

$$I^B(n_1, \dots, n_r \,|\, \tau) := \tau^{r-n_1-\cdots-n_r} I^A(n_1, \dots, n_r \,|\, -1/\tau)$$

- $A = [0,1]$ and $B = [0,\tau]$ standard cycles of $\mathbb{T}^*_\tau$, I^A iterated integrals along A, I^B iterated integrals along B
- Coefficients of "elliptic associator" associated with "KZB-equation" (genus-one analogue of Drinfeld associator and KZ equation)
- $I^A(n_1, \dots, n_r \,|\, \tau) = \sum_{j \geq 0} a_j q^j$, $a_j \in \mathcal{Z}[2\pi i]$
- $I^B(n_1, \dots, n_r \,|\, \tau) = \sum_{k=1-n_1-\cdots-n_r}^{r} \sum_{j \geq 0} b_{k,j} (2\pi i \tau)^k q^j$, $b_{k,j} \in \mathcal{Z}$
- $I^\bullet(n_1, \dots, n_r \,|\, \tau)$ can be written as (special) combinations of iterated integrals $\int_\tau^{i\infty} G_{k_1}(\tau')d\tau' \cdots G_{k_s}(\tau')d\tau'$

Genus-one amplitudes

Genus-one Green's function $G(z_i, z_j) := -\log|\theta(z)|^2 + \frac{2\pi \mathrm{Im}(z_i - z_j)}{\mathrm{Im}(\tau)}$

Closed string integral prototype:

$$\int_{\mathfrak{M}_{1,1}} \int_{C(\mathbb{T}^*_\tau, N)} \exp\Big(\sum_{1\leq i<j\leq N+1} s_{ij} G(z_i, z_j)\Big) f(z_1, \bar{z}_1, \dots, z_N, \bar{z}_N) d\mu$$

where $z_{N+1} \equiv 0$, f made out of $\partial_z G$ and $\partial_{\bar{z}} G$

Open string integral prototype:

$$\int_{i\mathbb{R}^+} \int_{0\leq z_1\leq \dots \leq z_N \leq 1} \exp\Big(\sum_{1\leq i<j\leq N+1} s_{ij} G(z_i, z_j)\Big) g(z_1, \dots, z_N) d\mu$$

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where $z_{N+1} \equiv 0$, f made out of $\partial_z G$

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Genus-one Green's function $G(z_i, z_j) := -\log|\theta(z)|^2 + \frac{2\pi \mathrm{Im}(z_i - z_j)}{\mathrm{Im}(\tau)}$

Closed string integral prototype:

$$\int_{C(\mathbb{T}^*_\tau, N)} \exp\left(\sum_{1\le i<j\le N+1} s_{ij} G(z_i, z_j)\right) d\mu$$

$$= \sum_{l_{ij}\ge 0} \prod_{ij} \frac{s_{ij}^{l_{ij}}}{l_{ij}!} \int_{C(\mathbb{T}^*_\tau, N)} \prod_{1\le i<j\le N+1} G(z_i, z_j)^{l_{ij}} d\mu$$

Open string integral prototype:

$$\int_{0\le z_1\le\cdots\le z_N\le 1} \exp\left(\sum_{1\le i<j\le N+1} s_{ij} G(z_i, z_j)\right) d\mu$$

$$= \sum_{l_{ij}\ge 0} \prod_{ij} \frac{s_{ij}^{l_{ij}}}{l_{ij}!} \int_{0\le z_1\le\cdots\le z_N\le 1} \prod_{1\le i<j\le N+1} G(z_i, z_j)^{l_{ij}} d\mu$$

Genus-one amplitudes

Genus-one Green's function $G(z_i, z_j) := -\log|\theta(z)|^2 + \frac{2\pi \mathrm{Im}(z_i - z_j)}{\mathrm{Im}(\tau)}$

Closed string integral prototype:

$$\int_{C(\mathbb{T}^*_\tau, N)} \exp\Big(\sum_{1 \le i < j \le N+1} s_{ij} G(z_i, z_j) \Big) d\mu$$

$$= \sum_{l_{ij} \ge 0} \prod_{ij} \frac{s_{ij}^{l_{ij}}}{l_{ij}!} \int_{C(\mathbb{T}^*_\tau, N)} \prod_{1 \le i < j \le N+1} G(z_i, z_j)^{l_{ij}} d\mu$$

Blue integrals are called **modular graph functions**

Open string integral prototype:

$$\int_{0 \le z_1 \le \cdots \le z_N \le 1} \exp\Big(\sum_{1 \le i < j \le N+1} s_{ij} G(z_i, z_j) \Big) d\mu$$

$$= \sum_{l_{ij} \ge 0} \prod_{ij} \frac{s_{ij}^{l_{ij}}}{l_{ij}!} \int_{0 \le z_1 \le \cdots \le z_N \le 1} \prod_{1 \le i < j \le N+1} G(z_i, z_j)^{l_{ij}} d\mu$$

State of the art

- Coefficients of open string integrals are A-elliptic multiple zeta values
- Modular graph functions new interesting class of real-analytic modular functions, contains real-analytic Eisenstein series, many algebraic and differential relations, asymptotic limit involves single-valued MZVs (see D'Hoker's talk)
- Single-valued-like projection from "symmetrized open integrals on B-cycle" (holomorphic graph functions) to modular graph functions
- Modular graph functions are combinations of holomorphic and anti-holomorphic elliptic MZVs
- **Conjecture:** Limits of N-point genus-one integrals related to $N+2$-point genus-zero integrals (known in open case or $N=2$)
- Recursions based on KZB-equation (see Kaderli's talk)
- No known KLT-like relations
- Partial results on moduli space integrals for closed strings

Green, Russo, Vanhove, Broedel, Matthes, Mafra, Schlotterer, D'Hoker, Zerbini, Zagier, Gurdogan, Brown, Basu, Kaidi, Kleinschmidt, Gerken, Kaderli...

Outline

String amplitudes

D'Hoker, Green, Pioline: Higher-genus analogues of modular graph functions!
New interesting class of modular ($Sp_{2g}(\mathbb{Z})$) invariant functions containing Zhang-Kawazumi invariant textcolorred(see D'Hoker's talk)

	Open Strings	Closed Strings
$g = 0$	MZVs	svMZVs
$g = 1$	Elliptic MZVs	Modular graph functions
$g = 2$	?	Modular graph functions
$g > 2$	?	Modular graph functions?

Hope / expectation: open and closed string related by KLT relations (single-valued projections) at higher genus

Higher-genus analogues of polylogarithms

Enriquez: higher-genus analogue of KZB form

- Induces connection (on non-trivial bundle) which is flat, multi-valued, regular singular at one point
- Reduces to Kronecker function at genus-one
- Not explicit in higher genus

First step: single-valued (flat) version?
Two possible ways: 1) real analytic (Brown-Levin)
2) meromorphic (Levin-Racinet)
Second way easier (but lose regular singularity)
Possible to deduce one from the other?

Second step: use it to generate periods of $\pi_1^{\mathrm{un}}(X_g, x, y)$, i.e.
higher-genus analogues of polylogs
Question: do we get all of them?

(Work in progress with B. Enriquez)

THE END

Thanks for your attention!