Elliptic KZB Associator and Open-String Integrals at Genus One

André Kaderli

Max Planck Institute for Gravitational Physics (AEI), Potsdam
Humboldt University (HU), Berlin

based on arXiv:1912.09927 [Broedel AK '19] and arXiv:2007.03712 [Broedel, AK Schlotterer '20]

in collaboration with J. Brödel and O. Schlotterer

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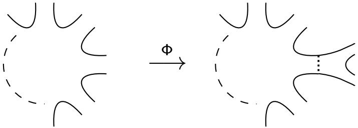

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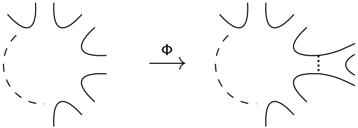

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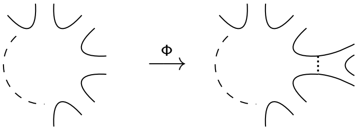

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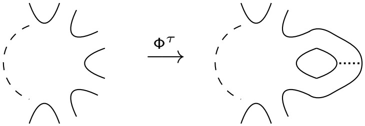
- **Recursion in n and genus g via elliptic KZB associator Φ^τ (eMZVs)** [Broedel AK '19] [Broedel, AK Schlotterer '20]

$$\boldsymbol{Z}_{n+2}(\alpha') \xrightarrow{\Phi^\tau} \boldsymbol{Z}_n^\tau(\alpha')$$



Outline

1. Elliptic Multiple Polylogarithms (eMPLs)
   - Definition and Regularisation
   - Elliptic Multiple Zeta Values (eMZVs)

2. Elliptic KZB Associator
   - Construction
   - Properties
   - Regularised Boundary Values and Associator Equation

3. Open-String Integrals at Genus One
   - Generating series: Z_n^τ-Integrals
   - Augmentation of Z_n^τ-Integrals with Auxiliary Point z_0: $Z_{0,n}^\tau$-Integrals
   - Properties of $Z_{0,n}^\tau$-Integrals
   - Recursion and Geometric Interpretation
   - Two-Point Example

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- Integration kernels defined by **Eisenstein-Kronecker series**

$$F(z,\eta,\tau)=\frac{\theta_1'(0,\tau)\theta_1(z+\eta,\tau)}{\theta_1(z,\tau)\theta_1(\eta,\tau)}=\frac{1}{\eta}\sum_{k\geq 0} g^{(k)}(z,\tau)\eta^k$$

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- For example $(\partial_z\zeta(z,\tau)=-\wp(z,\tau))$

$$g^{(0)}(z,\tau)=1$$

$$g^{(1)}(z,\tau)=\zeta(z,\tau)-2\zeta(1/2,\tau)z\xrightarrow{z\to k}\frac{1}{z-k}\,,\quad k\in\mathbb{Z}$$

$$g^{(2)}(z,\tau)=\frac{1}{2}\left(g^{(1)}(z,\tau)\right)^2-\frac{1}{2}\wp(z,\tau)$$

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$$\tilde{\Gamma}\left(\begin{smallmatrix}1\\0\end{smallmatrix};z,\tau\right)=\log(1-e^{2\pi iz})-\pi iz+4\pi\sum_{k,l>0}\frac{1}{2\pi k}\left(1-\cos(2\pi kz)\right)q^{kl}$$

$$\tilde{\Gamma}\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix};z,\tau\right)=\tilde{\Gamma}\left(\begin{smallmatrix}0\\0\end{smallmatrix};z,\tau\right)\tilde{\Gamma}\left(\begin{smallmatrix}1\\0\end{smallmatrix};z,\tau\right)-\tilde{\Gamma}\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix};z,\tau\right)$$

Elliptic Multiple Zeta Values (eMZVs)

- Denote for a word $\boldsymbol{w} = \boldsymbol{x}_{k_1} \cdots \boldsymbol{x}_{k_r} \in \boldsymbol{X}^*$ in alphabet $\boldsymbol{X} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \dots\}$

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$$\omega(1; \tau) = \omega(\boldsymbol{x}_1; \tau) = 0 \,,$$
$$\omega(2m; \tau) = \omega(\boldsymbol{x}_{2m}; \tau) = -2\zeta_{2m} \,,$$
$$\omega(0, 1; \tau) = \omega(\boldsymbol{x}_0 \boldsymbol{x}_1; \tau) = -\frac{\pi i}{2} + 2 \sum_{k,l \geq 0} \frac{q^{kl}}{k}$$

Elliptic KZB Equation

- Consider **generating series of eMPLs** associated to $\boldsymbol{X} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \dots\}$

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- Therefore, $\boldsymbol{\Phi}^{\tau}$ is the **generating series of regularised eMZVs**

The Associator Equation

- Let $\boldsymbol{S}(z)$ be some solution of the elliptic KZB equation

$$\frac{\partial}{\partial z}\boldsymbol{S}(z) = \left(\sum_{k\geq 0} g^{(k)}(z,\tau)\,\boldsymbol{x}_k\right)\boldsymbol{S}(z)$$

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The Associator Equation

- Let $\boldsymbol{S}(z)$ be some solution of the elliptic KZB equation

$$\frac{\partial}{\partial z}\boldsymbol{S}(z) = \left(\sum_{k\geq 0} g^{(k)}(z,\tau)\boldsymbol{x}_k\right)\boldsymbol{S}(z)$$

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&= \lim_{z\to 1}(\tilde{\boldsymbol{\Gamma}}(1-z))^{-1}\boldsymbol{S}(z)\\
&= \lim_{z\to 1}(-2\pi i(1-z))^{-\boldsymbol{x}_1}\boldsymbol{S}(z) = \boldsymbol{C}_1^\tau
\end{aligned}$$

Generating series: Z_n^τ-Integrals – Definition

- $\boldsymbol{\mathcal{A}}_{\text{open},n}(\alpha') = \boldsymbol{\mathcal{A}}_{\text{YM},n} \left(\boldsymbol{F}_n^{g=0}(\alpha') + \int d\tau \, \boldsymbol{F}_n^{g=1}(\alpha', \tau) + \dots \right)$

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$$Z_n^\tau(1,\sigma) = \int_{0=z_1<z_2<\cdots<z_n<1} \prod_{i=2}^n dz_i \, \mathrm{KN}_{12\ldots n}^\tau \, \varphi^\tau(1,\sigma)$$

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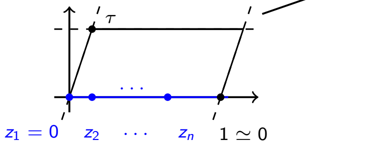

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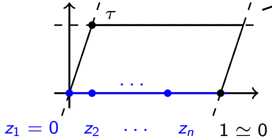

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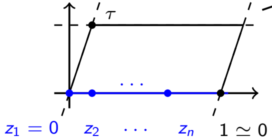

$$\text{KN}_{12\ldots n}^\tau = \prod_{1\le i<j\le n} \exp\left(-s_{ij}\,\mathcal{G}_{ij}^\tau\right)$$

$$s_{i\ldots j} = -\alpha'(k_i + \dots + k_j)^2$$

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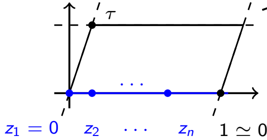

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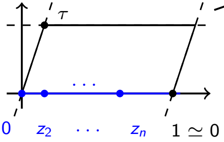
$$\varphi^\tau(1,2,\dots,n) = F_{12}(\eta_{2\dots n}) F_{23}(\eta_{3\dots n}) \dots F_{n-2,n-1}(\eta_{n-1,n}) F_{n-1,n}(\eta_n)$$

$$Z_n^\tau(1,\sigma) = \int_{0=z_1<z_2<\dots<z_n<1} \prod_{i=2}^n \mathrm{d}z_i\, \mathrm{KN}_{12\dots n}^\tau\, \varphi^\tau(1,\sigma)$$



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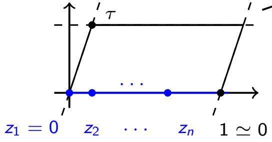

$$\text{KN}^\tau_{12\dots n} = \prod_{1\leq i<j\leq n}\exp\left(-s_{ij}\,\mathcal{G}^\tau_{ij}\right)$$

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Generating series: Z_n^τ-Integrals – Definition

- $\mathcal{A}_{\text{open},n}(\alpha') = \mathcal{A}_{\text{YM},n}\left(\boldsymbol{F}_n^{g=0}(\alpha') + \int d\tau\, \boldsymbol{F}_n^{g=1}(\alpha',\tau) + \dots\right)$
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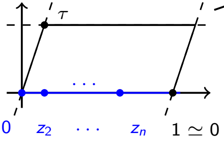

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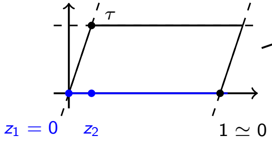
Generating series: Z_n^τ-Integrals – Examples

- Two-point example $\sigma = (2)$:

$$Z_2^\tau(1,2) = \int_{0=z_1<z_2<1} dz_2\, \mathrm{KN}_{12}^\tau\, \varphi^\tau(1,2)$$

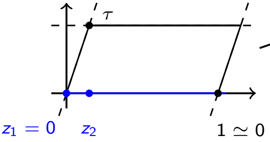


$$\mathrm{KN}_{12}^\tau = \exp\left(-s_{12}\,\mathcal{G}_{12}^\tau\right) \qquad \varphi^\tau(1,2) = F_{12}(\eta_2)$$

Generating series: Z^τ_n-Integrals – Examples

- Two-point example $\sigma = (2)$:

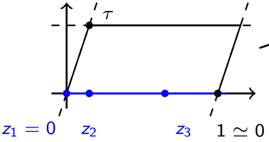
$$Z^\tau_2(1,2) = \int_{0=z_1<z_2<1} dz_2 \, \mathrm{KN}^\tau_{12} \, \varphi^\tau(1,2)$$



$$\mathrm{KN}^\tau_{12} = \exp\left(-s_{12}\,\mathcal{G}^\tau_{12}\right) \qquad \varphi^\tau(1,2) = F_{12}(\eta_2)$$

- Three-point example $\sigma = (2,3),(3,2)$:

$$\boldsymbol{Z}^\tau_3 = \begin{pmatrix} Z^\tau_3(1,2,3) \\ Z^\tau_3(1,3,2) \end{pmatrix} = \int_{0=z_1<z_2<z_3<1} dz_3 dz_2 \, \mathrm{KN}^\tau_{123} \begin{pmatrix} \varphi^\tau(1,2,3) \\ \varphi^\tau(1,3,2) \end{pmatrix}$$

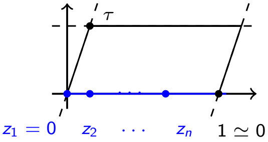


$$\varphi^\tau(1,\sigma) = F_{1\sigma(2)}(\eta_{23}) F_{\sigma(2)\sigma(3)}(\eta_{\sigma(3)})$$

$$\mathrm{KN}^\tau_{123} = \exp\left(-s_{12}\,\mathcal{G}^\tau_{12} - s_{13}\,\mathcal{G}^\tau_{13} - s_{23}\,\mathcal{G}^\tau_{23}\right)$$

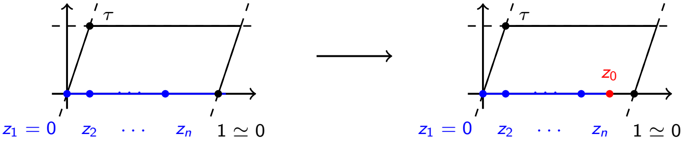
Augmentation: $Z^\tau_{0,n}$-Integrals – Definition

- Augment $Z^\tau_n(1,\sigma) = \int_{0=z_1<z_2<\cdots<z_n<1} \prod_{i=2}^n dz_i \, \mathrm{KN}^\tau_{12\ldots n} \, \varphi^\tau(1,\sigma)$ with additional puncture z_0 and associated auxiliary momentum k_0:



Augmentation: $Z^\tau_{0,n}$-Integrals – Definition

- Augment $Z^\tau_n(1,\sigma) = \int_{0=z_1<z_2<\dots<z_n<1} \prod_{i=2}^n dz_i \ \mathrm{KN}^\tau_{12\dots n}\, \varphi^\tau(1,\sigma)$ with additional puncture z_0 and associated auxiliary momentum k_0:



- Augmented Z^τ_n-integrals: $Z^\tau_{0,n}$-integrals

$$Z^\tau_{0,n}(1,\sigma_1;0,\sigma_0) = \int_{0=z_1<z_2<\dots<z_n<z_0} \prod_{i=2}^n dz_i \ \mathrm{KN}^\tau_{012\dots n}\, \varphi^\tau(1,\sigma_1)\, \varphi^\tau(0,\sigma_0)$$

where $\sigma=(\sigma_1,\sigma_0)$ and $\mathrm{KN}^\tau_{012\dots n} = \mathrm{KN}^\tau_{12\dots n} \prod_{i=1}^n \exp\left(-s_{0i}\, \mathcal{G}^\tau_{0i}\right)$

Augmentation: $Z^{\tau}_{0,n}$-Integrals – Examples

- Two-point example $\sigma = (2)$

$$Z^{\tau}_{2}(1,2) \quad \longrightarrow \quad \boldsymbol{Z}^{\tau}_{0,2} = \begin{pmatrix} Z^{\tau}_{0,2}(1,2;0) \\ Z^{\tau}_{0,2}(1;0,2) \end{pmatrix}$$

Augmentation: $Z^{\tau}_{0,n}$-Integrals – Examples

- Two-point example $\sigma = (2)$

$$Z_2^{\tau}(1,2) \longrightarrow \boldsymbol{Z}^{\tau}_{0,2} = \begin{pmatrix} Z^{\tau}_{0,2}(1,2;0) \\ Z^{\tau}_{0,2}(1;0,2) \end{pmatrix}$$

e.g. $Z^{\tau}_{0,2}(1,2;0) = \int_0^{z_0} \mathrm{d}z_2 \ \mathrm{KN}^{\tau}_{012} \ \underbrace{\varphi^{\tau}(1,2)}_{F(z_1-z_2,\eta_2,\tau)} \ \underbrace{\varphi^{\tau}(0)}_{1}$

Augmentation: $Z^\tau_{0,n}$-Integrals – Examples

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- Three-point example $\sigma \in \{(2,3),(3,2)\}$

$$\boldsymbol{Z}_3^\tau = \begin{pmatrix} Z_3^\tau(1,2,3) \\ Z_3^\tau(1,3,2) \end{pmatrix} \longrightarrow \boldsymbol{Z}^\tau_{0,3} = \begin{pmatrix} Z^\tau_{0,3}(1,2,3;0) \\ Z^\tau_{0,3}(1,3,2;0) \\ Z^\tau_{0,3}(1,2;0,3) \\ Z^\tau_{0,3}(1,3;0,2) \\ Z^\tau_{0,3}(1;0,2,3) \\ Z^\tau_{0,3}(1;0,3,2) \end{pmatrix}$$

e.g. $Z^\tau_{0,3}(1,2,3;0) = \int_{0=z_1<z_2<z_3<z_0} \mathrm{d}z_3 \mathrm{d}z_2 \; \mathrm{KN}^\tau_{0123} \; \underbrace{\varphi^\tau(1,2,3)}_{F(z_1-z_2,\eta_2+\eta_3,\tau)F(z_2-z_3,\eta_3,\tau)} \; \underbrace{\varphi^\tau(0)}_{1}$

Augmentation: $Z^{\tau}_{0,n}$-Integrals – Examples

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$$Z^{\tau}_{2}(1,2) \longrightarrow \boldsymbol{Z}^{\tau}_{0,2} = \begin{pmatrix} Z^{\tau}_{0,2}(1,2;0) \\ Z^{\tau}_{0,2}(1;0,2) \end{pmatrix}$$

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Elliptic KZB Equation and Boundary Values of $\boldsymbol{Z}^{\tau}_{0,n}$

- Length-$n!$ vector $\boldsymbol{Z}^{\tau}_{0,n}$ satisfies an elliptic KZB equation

$$\frac{\partial}{\partial z_0}\boldsymbol{Z}^{\tau}_{0,n} = \sum_{k\geq 0} g^{(k)}(z_0,\tau)\,\boldsymbol{x}_k\,\boldsymbol{Z}^{\tau}_{0,n}$$

where matrices $\boldsymbol{x}_k$ are explicitly known, linear in α' and of degree $k-1$ in η_i

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- Regularised boundary values contain basis of $(n+2)$-point, tree-level and n-point, one-loop string integrals

$(n+2)$-point, tree-level and BCJ

$$\boldsymbol{C}^{\tau}_0 = \lim_{z_0\to 0}(-2\pi i z_0)^{-\boldsymbol{x}_1}\boldsymbol{Z}^{\tau}_{0,n} = -e^{s_{01\ldots n}\omega(0,1;\tau)}\begin{pmatrix}\boldsymbol{Z}_{n+2}\\ \vdots\end{pmatrix} = \boldsymbol{U}_n\,\boldsymbol{Z}_{n+2}$$

Elliptic KZB Equation and Boundary Values of $Z^\tau_{0,n}$

- Length-$n!$ vector $\boldsymbol{Z}^\tau_{0,n}$ satisfies an elliptic KZB equation

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$(n+2)$-point, tree-level and BCJ

$$\boldsymbol{C}^\tau_0 = \lim_{z_0\to 0}(-2\pi i z_0)^{-\boldsymbol{x}_1}\boldsymbol{Z}^\tau_{0,n} = -e^{s_{01\ldots n}\omega(0,1;\tau)}\begin{pmatrix}\boldsymbol{Z}_{n+2}\\ \vdots\end{pmatrix} = \boldsymbol{U}_n\,\boldsymbol{Z}_{n+2}$$

$$\boldsymbol{C}^\tau_1 = \lim_{z_0\to 1}(-2\pi i(1-z_0))^{-\boldsymbol{x}_1}\boldsymbol{Z}^\tau_{0,n} \simeq \begin{pmatrix}\boldsymbol{Z}^\tau_n\\ \vdots\end{pmatrix} \quad \text{s.t.} \quad \boldsymbol{P}_n\,\boldsymbol{C}^\tau_1 = \boldsymbol{Z}^\tau_n$$

basis trafo: $\boldsymbol{U}^{-1}\boldsymbol{x}_1\boldsymbol{U} = -\mathrm{diag}(\underbrace{s_{01},\ldots,s_{01}}_{(n-1)!},\ldots)$

n-point, one-loop with $s^{\text{1-loop}}_{1j} = s_{1j} + s_{0j}$ and shuffles

String Integral Recursion from Associator Equation of $\mathbf{\Phi}^\tau$

- Since $\boldsymbol{C}_0^\tau = \boldsymbol{U}_n \boldsymbol{Z}_{n+2}$ and $\boldsymbol{P}_n \boldsymbol{C}_1^\tau = \boldsymbol{Z}_n^\tau$ (with $s_{1j}^{\text{1-loop}} = s_{1j} + s_{0j}$)

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- Can use $\boldsymbol{C}_1^\tau = \boldsymbol{\Phi}^\tau \boldsymbol{C}_0^\tau$ to obtain **recursion for string integrals** (in genus and n)

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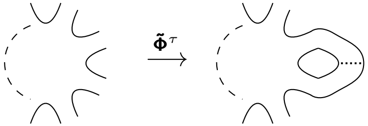

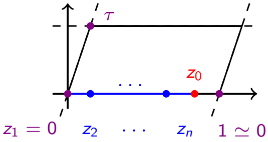

unintegrated punctures:
integrated punctures:
integration domain:

$$\boldsymbol{Z}^{\tau}_{0,n}$$

$$\big\downarrow \partial_{z_0}$$

$$\boldsymbol{X}_k$$

$$\big\downarrow$$

$$\tilde{\boldsymbol{\Phi}}^{\tau} = \boldsymbol{P}_n \boldsymbol{\Phi}^{\tau} \boldsymbol{U}_n$$

unintegrated punctures: ●●
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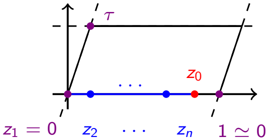
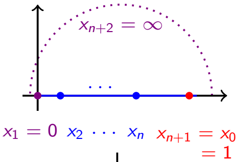
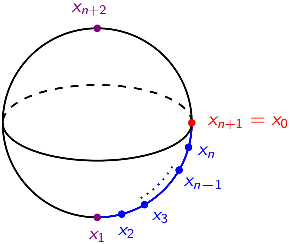


$x_i = \frac{z_i}{z_0}$

$z_0 \to 0$





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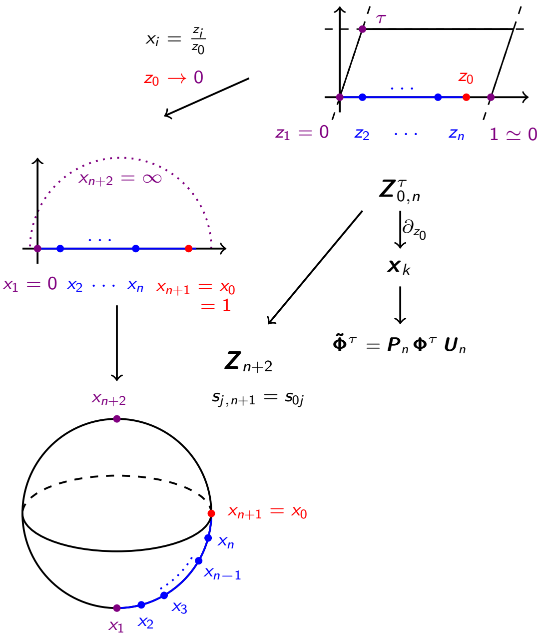

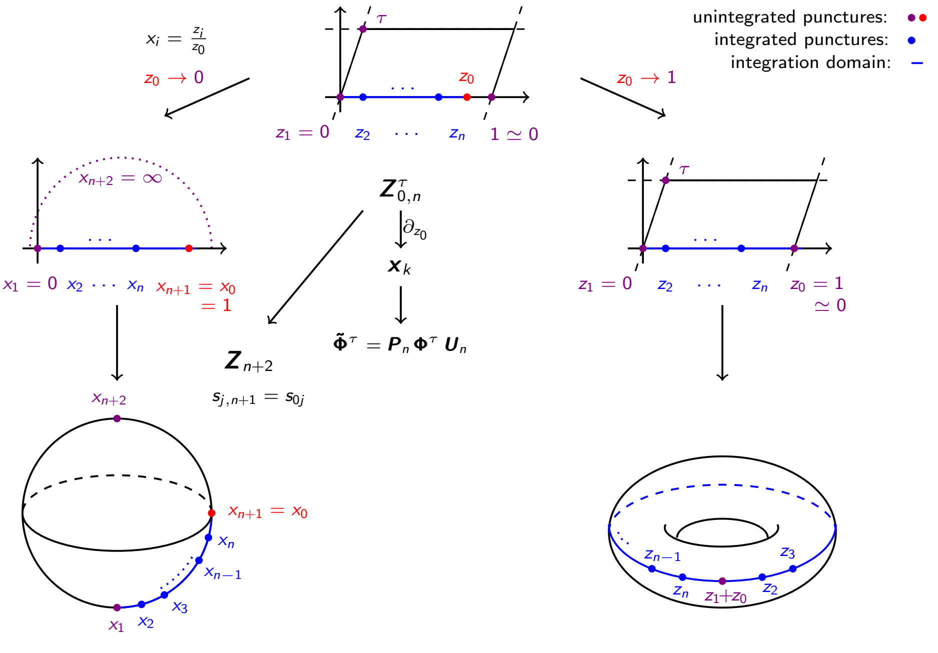
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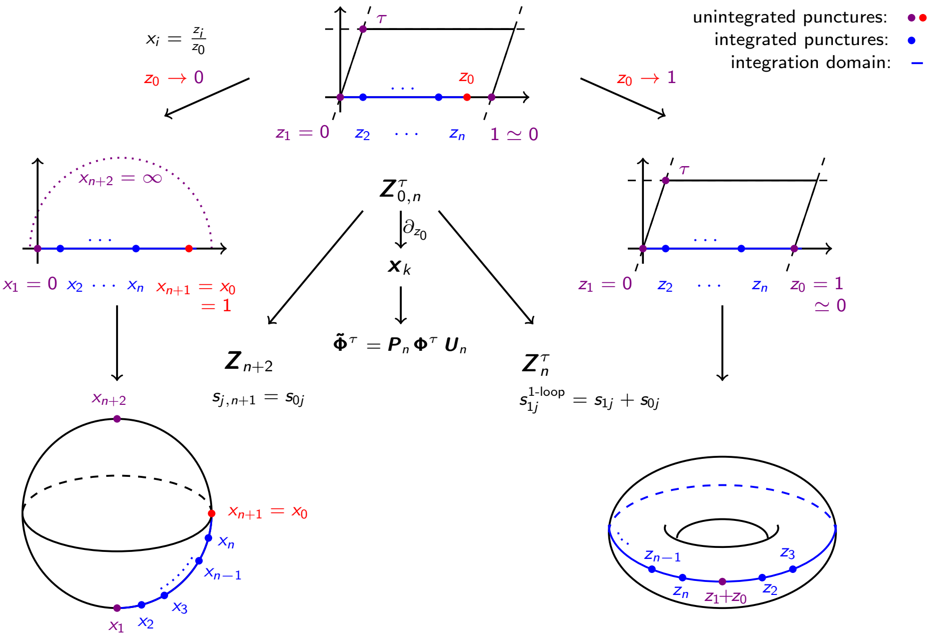
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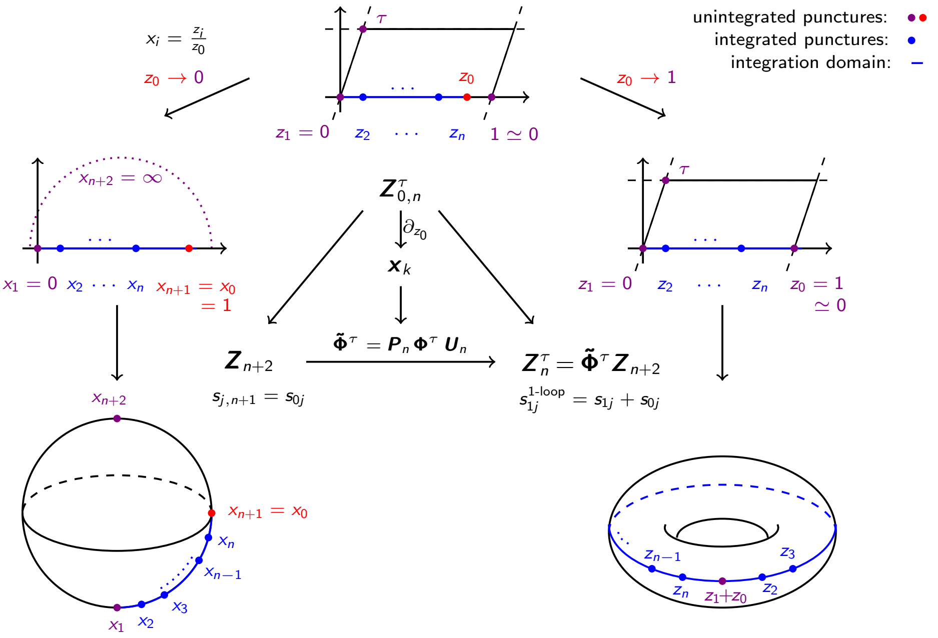
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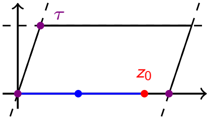
String Integral Recursion – Two-Point Example



$$\boldsymbol{Z}^{\tau}_{0,2} = \int_0^{z_0} \mathrm{d}z_2 \; \mathrm{KN}^{\tau}_{012} \begin{pmatrix} F(z_1 - z_2, \eta_2, \tau) \\ F(z_0 - z_2, \eta_2, \tau) \end{pmatrix}$$

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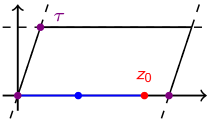

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$$\downarrow (1)$$

$$\boldsymbol{x}_k$$

$$\downarrow$$

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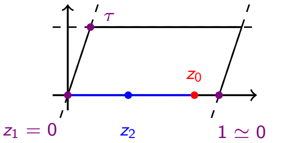

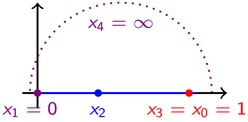

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(2) ↙ $\qquad$ (1) ↓

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↓

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(2) $z_0 \to 0$: $\boldsymbol{C}^{\tau}_0 = \boldsymbol{U}_2 \boldsymbol{Z}_4$

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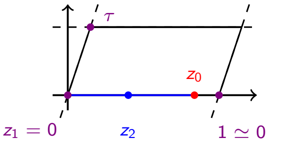

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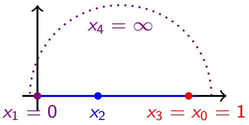

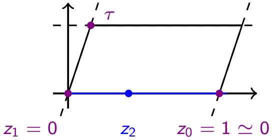

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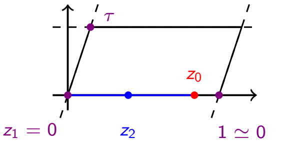

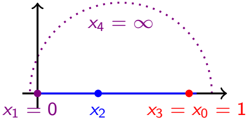

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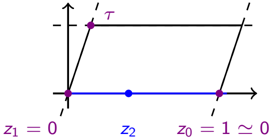

$$\boldsymbol{Z}_4 = \frac{\Gamma(1-s_{12})\Gamma(1-s_{02})}{\Gamma(1-s_{12}-s_{02})} \xrightarrow[(4)]{\tilde{\boldsymbol{\Phi}}^{\tau} = \boldsymbol{P}_2 \boldsymbol{\Phi}^{\tau} \boldsymbol{U}_2} \boldsymbol{Z}^{\tau}_2 = \int_0^1 \mathrm{d}z_2 \, e^{-(s_{12}+s_{02})\mathcal{G}^{\tau}_{12}} F(z_{12}, \eta_2, \tau) = \tilde{\boldsymbol{\Phi}}^{\tau} \boldsymbol{Z}_4$$

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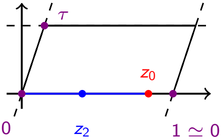

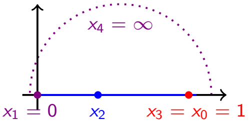

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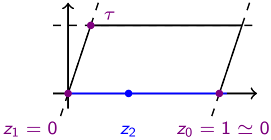

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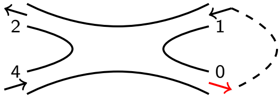

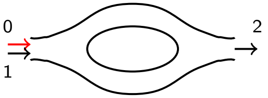

$$s^{\text{1-loop}}_{12} = s_{12} + s_{02}$$

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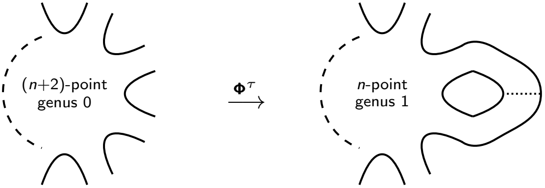
  - can be used to calculate (α'-expansion of) one-loop string integrals
  - implements a **gluing mechanism of worldsheets**



Outlook (and wishful thoughts)

- Generalise to higher genera: complete recursion in genus

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\cdots & \xrightarrow{\boldsymbol{\Phi}} & F^{g=0}_{n+1} & \xrightarrow{\boldsymbol{\Phi}} & F^{g=0}_{n+2} & \xrightarrow{\boldsymbol{\Phi}} & F^{g=0}_{n+3} & \xrightarrow{\boldsymbol{\Phi}} & \cdots
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Notes – Algebra of the Elliptic KZB System

- Elliptic KZB system on the twice-punctured torus with modular parameter τ (fixed puncture $z_1 = 0$ and variable puncture z_0)

$$\partial_{z_0} \boldsymbol{Z}^{\tau}_{0,n} = \left(\sum_{k=0}^{\infty} g^{(k)}_{01} \boldsymbol{x}_k \right) \boldsymbol{Z}^{\tau}_{0,n}$$

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with $n! \times n!$–matrices $\boldsymbol{x}_k$, $\tilde{\boldsymbol{\epsilon}}_k$, $\boldsymbol{b}_k$ and where $(k \geq 2)$

$$G_0(\tau) = -1, \qquad G_{2k}(\tau) = \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^k}, \qquad G_{2k+1} = 0$$

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- $\boldsymbol{P}_n \boldsymbol{C}^{\tau}_1 = \boldsymbol{Z}^{\tau}_n$ satisfies elliptic KZB system on once-punctured torus [Mafra Schlotterer '19]

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- Schwarz integrability condition $\partial_{z_0}\partial_\tau \boldsymbol{Z}^\tau_{0,n} = \partial_\tau\partial_{z_0}\boldsymbol{Z}^\tau_{0,n}$ leads to

$$\boldsymbol{b}_k = \boldsymbol{x}_{k-1}$$

$$[\boldsymbol{x}_\ell, \tilde{\boldsymbol{\epsilon}}_0] = \sum_{j=0}^{\lfloor \ell/2 \rfloor -1} \binom{\ell}{j} \frac{(\ell-1-2j)}{(j+1)} [\boldsymbol{x}_j, \boldsymbol{x}_{\ell-1-j}]$$

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- However $\boldsymbol{x}_k \neq \mathrm{ad}^k_{\boldsymbol{x}}(\boldsymbol{y})$ for some $\boldsymbol{x}, \boldsymbol{y}$ (as opposed to [Enriquez '14])

- Motivation for conjecture ($\tilde{\epsilon}_k + \delta_{k\neq 0}\, \mathbf{x}_{k-1}$ for $k = 0, 4, 6, 8, \ldots$ satisfies Tsunogai's derivation algebra):

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$$Z_{n,0}^{\tau}$$

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since for $k > 1$ $\lim_{z_0 \to 1} g_{01}^{(2k)} = -G_{2k}$ and $\lim_{z_0\to 1} g_{01}^{(2k+1)} = 0$

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since for $k > 1$ $\lim_{z_0\to 1} g_{01}^{(2k)} = -G_{2k}$ and $\lim_{z_0\to 1} g_{01}^{(2k+1)} = 0$, commutativity leads to ($P_n E_n = \mathbb{I}_{(n-1)!}$)

$$P_n\, \tilde{\epsilon}_0\, E_n = \epsilon_0$$

$$P_n(\tilde{\epsilon}_k + x_{k-1}) E_n = \epsilon_k\,, \quad k = 4, 6, 8, \ldots$$

Notes – Two-Point Example (in Detail)

- Consider $\boldsymbol{Z}^{\tau}_{0,2} = \begin{pmatrix} Z^{\tau}_{0,2}(1,2;0) \\ Z^{\tau}_{0,2}(1;0,2) \end{pmatrix} = \int_0^{z_0} \mathrm{d}z_2 \ \mathrm{KN}^{\tau}_{012} \begin{pmatrix} F(z_1 - z_2, \eta, \tau) \\ F(z_0 - z_2, \eta, \tau) \end{pmatrix}$

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- Satisfies elliptic KZB eq.

$$\frac{\partial}{\partial z_0} \boldsymbol{Z}^{\tau}_{0,2} = \sum_{k \geq 0} g^{(k)}(z_0, \tau) \boldsymbol{x}_k \, \boldsymbol{Z}^{\tau}_{0,2}$$

with $(k \geq 1)$

$$\boldsymbol{x}_0 = \begin{pmatrix} -s_{02}\partial_\eta & -s_{02}/\eta \\ s_{12}/\eta & s_{12}\partial_\eta \end{pmatrix}$$

$$\boldsymbol{x}_k = \eta^{k-1} \begin{pmatrix} 0 & (-1)^{k-1} s_{02} \\ s_{12} & 0 \end{pmatrix} + \delta_{k,1} \begin{pmatrix} -(s_{01} + s_{02}) & 0 \\ 0 & -(s_{01} + s_{12}) \end{pmatrix}$$

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- Eigenvalue decomposition of $\boldsymbol{x}_1$:

$$\begin{pmatrix} -(s_{01} + s_{02}) & s_{02} \\ s_{12} & -(s_{01} + s_{12}) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -\frac{s_{12}}{s_{02}} \end{pmatrix} \begin{pmatrix} -s_{01} & 0 \\ 0 & -s_{012} \end{pmatrix} \frac{1}{s_{02} + s_{12}} \begin{pmatrix} s_{12} & s_{02} \\ s_{02} & -s_{02} \end{pmatrix}$$

- $\frac{\partial}{\partial z_0} \boldsymbol{Z}^{\tau}_{0,2} = \sum_{k\geq 0} g^{(k)}(z_0,\tau) \boldsymbol{x}_k \boldsymbol{Z}^{\tau}_{0,2}$ with

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- Lower boundary value $\boldsymbol{C}^{\tau}_0 = \lim_{z_0\to 0}(-2\pi i z_0)^{-\boldsymbol{x}_1} \boldsymbol{Z}^{\tau}_{0,2} = \boldsymbol{U}_2 \boldsymbol{Z}_4$:

$$\boldsymbol{C}^{\tau}_0 = e^{s_{012}\omega(0,1;\tau)} \begin{pmatrix} 1 \\ -\frac{s_{12}}{s_{02}} \end{pmatrix} \frac{1}{s_{12}} \frac{\Gamma(1-s_{12})\Gamma(1-s_{02})}{\Gamma(1-s_{12}-s_{02})}$$

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- Upper boundary value $\boldsymbol{C}^{\tau}_1 = \lim_{z_0\to 1}(-2\pi i(1-z_0))^{-\boldsymbol{x}_1} \boldsymbol{Z}^{\tau}_{0,2}$ s.t. $\boldsymbol{P}_2 \boldsymbol{C}^{\tau}_1 = \boldsymbol{Z}^{\tau}_2$:

$$e^{-s_{01}\omega(0,1;\tau)} \frac{1}{s_{12}+s_{02}} \begin{pmatrix} s_{12} & s_{02} \end{pmatrix} \boldsymbol{C}^{\tau}_1 = \int_0^1 \mathrm{d}z_2 \, e^{-(s_{12}+s_{02})\mathcal{G}^{\tau}_{12}} F(z_{12},\eta,\tau)$$

- $\frac{\partial}{\partial z_0} \boldsymbol{Z}^{\tau}_{0,2} = \sum_{k\geq 0} g^{(k)}(z_0,\tau) \boldsymbol{x}_k \boldsymbol{Z}^{\tau}_{0,2}$ with

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- Recursion from associator equation $\boldsymbol{C}^{\tau}_1 = \boldsymbol{\Phi}^{\tau} \boldsymbol{C}^{\tau}_0$:

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- Recursion from associator equation $\boldsymbol{C}_1^\tau = \boldsymbol{\Phi}^\tau \, \boldsymbol{C}_0^\tau$:

$$\int_0^1 \mathrm{d}z_2 \, e^{-(s_{12}+s_{02})\,\mathcal{G}_{12}^\tau} \, F(z_2,\eta,\tau) = \boldsymbol{P}_2 \, \boldsymbol{\Phi}^\tau \, \boldsymbol{U}_2 \, \frac{1}{s_{12}} \frac{\Gamma(1-s_{12})\Gamma(1-s_{02})}{\Gamma(1-s_{12}-s_{02})}$$

with

$$\boldsymbol{\Phi}^\tau = \sum_{\boldsymbol{w}\in\boldsymbol{X}^*} \boldsymbol{w}\,\omega(\boldsymbol{w}^t) = 1 + \boldsymbol{x}_0 - 2\zeta_2\boldsymbol{x}_2 + \frac{1}{2}\boldsymbol{x}_0\boldsymbol{x}_0 + [\boldsymbol{x}_1,\boldsymbol{x}_0]\omega(0,1;\tau) + \ldots$$

where $\frac{\partial}{\partial z_0}\boldsymbol{Z}_{0,2}^\tau = \sum_{k\geq 0} g^{(k)}(z_0,\tau)\boldsymbol{x}_k \, \boldsymbol{Z}_{0,2}^\tau$ and $\boldsymbol{X} = \{\boldsymbol{x}_0,\boldsymbol{x}_1,\ldots\}$

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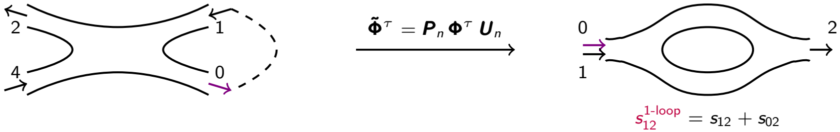
$$\int_0^1 \mathrm{d}z_2 \, e^{-(s_{12}+s_{02})\,\mathcal{G}_{12}^\tau} \, F(z_2,\eta,\tau) = \boldsymbol{P}_2 \, \boldsymbol{\Phi}^\tau \, \boldsymbol{U}_2 \, \frac{1}{s_{12}} \frac{\Gamma(1-s_{12})\Gamma(1-s_{02})}{\Gamma(1-s_{12}-s_{02})}$$

with

$$\boldsymbol{\Phi}^\tau = \sum_{\boldsymbol{w}\in \boldsymbol{X}^*} \boldsymbol{w}\,\omega(\boldsymbol{w}^t) = 1 + \boldsymbol{x}_0 - 2\zeta_2 \boldsymbol{x}_2 + \frac{1}{2}\boldsymbol{x}_0\boldsymbol{x}_0 + [\boldsymbol{x}_1,\boldsymbol{x}_0]\omega(0,1;\tau) + \ldots$$

where $\frac{\partial}{\partial z_0}\boldsymbol{Z}_{0,2}^\tau = \sum_{k\geq 0} g^{(k)}(z_0,\tau)\boldsymbol{x}_k \, \boldsymbol{Z}_{0,2}^\tau$ and $\boldsymbol{X} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots\}$

- Geometric interpretation: gluing of external states



Notes – z-Independence

- $\mathbf{\Phi}^{\tau}$'s independence of z: Let S_1, S_2 be non-trivial solutions of elliptic KZB eq. $\partial_z S(z) = \nabla_{\boldsymbol{x}} S(z)$, where $\left(\nabla_{\boldsymbol{x}} = \sum_{k\geq 0} g^{(k)}(z,\tau)\, \boldsymbol{x}_k\right)$ and consider $S_1^{-1} S_2 = C$ then

$$\begin{aligned}
\frac{\partial}{\partial z}(S_1 C) &= \frac{\partial}{\partial z} S_2 \\
\Rightarrow (\nabla_{\boldsymbol{x}} S_1) C + S_1 \frac{\partial}{\partial z} C &= \nabla_{\boldsymbol{x}} S_2 \\
\Rightarrow \nabla_{\boldsymbol{x}} S_2 + S_1 \frac{\partial}{\partial z} C &= \nabla_{\boldsymbol{x}} S_2 \\
\Rightarrow \frac{\partial}{\partial z} C &= 0
\end{aligned}$$

Hence, $\mathbf{\Phi}^{\tau} = (\tilde{\mathbf{\Gamma}}_{\boldsymbol{x}}(1-z))^{-1} \tilde{\mathbf{\Gamma}}(z)$ is z-independent

Notes – PDE of Generating Series of eMPLs

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$\downarrow$

elliptic KZB associator

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