Modular Graph Functions
– genus-one and beyond –

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Focus of this talk

- **String theory amplitudes for closed oriented strings**
    - ⋆ given by a topological expansion in the string coupling g_s

$$g_s^{-2} \quad \overset{z_1 \qquad z_4}{\underset{z_2 \qquad z_3}{\bigcirc}} + g_s^0 \; \bigcirc + g_s^2 \; \bigcirc \bigcirc + \cdots$$

    - ⋆ For genus h integrate over points z_i for incoming or outgoing strings
    - ⋆ Integrate over the moduli space $\mathcal{M}_h$ of compact Riemann surfaces

- **Modular graph function for a Riemann surface of genus one**
    - ⋆ maps a graph to an $SL(2, \mathbb{Z})$-invariant function on half-plane $\mathcal{H}_1$
    - ⋆ generalizes real-analytic Eisenstein series
    - ⋆ related to single-valued elliptic polylogarithms (see Zerbini's talk)

- **Generalization to Riemann surfaces of higher genus**
    - ⋆ maps a graph to a function on $\mathcal{M}_h$
    - ⋆ generalizes invariants of Kawazumi and Zhang (2008)

Genus zero

- **Genus-zero amplitudes are given by integrals of the type**

$$\prod_{i=1}^{N-3} \int_{\mathbb{C}} d^2 z_i |z_i - 1|^{-2s_{i\,N-1}} |z_i|^{-2-2s_{i\,N-2}} \prod_{j \neq i}^{N-3} |z_i - z_j|^{-s_{ij}}$$

  ⋆ kinematic variables $4s_{ij} = \alpha'(k_i + k_j)^2$ for massless momenta k_i
  ⋆ Meromorphic in s_{ij} with simple poles at non-negative integers
  ⋆ The four-graviton amplitude is proportional to the invariant $\mathcal{R}^4$

$$\frac{1}{s_{12}\, s_{13}\, s_{14}} \frac{\Gamma(1 - s_{12})\, \Gamma(1 - s_{13})\, \Gamma(1 - s_{14})}{\Gamma(1 + s_{12})\, \Gamma(1 + s_{13})\, \Gamma(1 + s_{14})}$$

- **Low energy expansion for** $|s_{ij}| \ll 1$ and $\sigma_n = s_{12}^n + s_{13}^n + s_{14}^n$ with $\sigma_1 = 0$

$$\mathcal{A}_4^{(0)}(s_{ij}) \sim \frac{3}{\sigma_3} + 2\zeta(3) + \sigma_2 \zeta(5) + \frac{2}{3}\zeta(3)^2 \sigma_3 - \frac{1}{2}\sigma_2^2 \zeta(7) + \cdots$$

  ⋆ For all N the coefficients are "single-valued" multiple-zeta values.
  (conjectured in Schlotterer, Stieberger 2012; Stieberger 2013; Stieberger, Taylor 2014)
  (proofs in Schlotterer, Schnetz; Brown, Dupont; Vanhove, Zerbini 2018)

Genus-one

- **Torus** $\Sigma = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ **with modulus** $\tau = \tau_1 + i\tau_2 \in \mathcal{H}_1$, $\tau_1, \tau_2 \in \mathbb{R}, \tau_2 > 0$
  - $\star$ Integral over N points $z_i \in \Sigma$

$$\mathcal{B}_N^{(1)}(s_{ij}|\tau) = \prod_{k=1}^{N} \int_\Sigma \frac{d^2 z_k}{\tau_2} \exp\left\{ \sum_{1 \leq i < j \leq N} s_{ij}\, g(z_i - z_j|\tau) \right\}$$

  - $\star$ Scalar Green function (in terms of $z = u + v\tau$ with $u, v \in \mathbb{R}/\mathbb{Z}$)

$$g(z|\tau) = \sum_{m,n\in\mathbb{Z}}' \frac{\tau_2}{\pi|m\tau + n|^2} e^{2\pi i(mu - nv)}$$

  - $\star$ $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$ is invariant under the modular group $SL(2,\mathbb{Z})$
  - $\star$ has simple and double poles at $s_{ij} \in \mathbb{N}$; holomorphic in s_{ij} for $|s_{ij}| < 1$

- **String amplitude for** $N = 4$ **is proportional to** (Green, Schwarz, 1982)

$$\int_{\mathcal{M}_1} \frac{d^2\tau}{\tau_2^2} \mathcal{B}_4^{(1)}(s_{ij}|\tau) \qquad \mathcal{M}_1 = PSL(2,\mathbb{Z})\backslash\mathcal{H}_1$$

  - $\star$ Analytic continuation in s_{ij} was proven to exist (ED & Phong 1994)
  - $\star$ $N > 4$ involves factors of $\partial_z g(z|\tau)$ (Green, Mafra, Schlotterer 2013)

Graphical Representation of Taylor series of $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$

- **Absolute convergence of** $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$ **for** $|s_{ij}| < 1$ **and fixed** τ
  - $\star$ allows for Taylor expansion in the variables s_{ij}
    - $=$ physically corresponds to the "low energy expansion"

- **Represented by Feynman graphs** (Green, Russo, Vanhove 2008)
  - $\star$ Each marked point z_i on Σ is represented by a vertex $\bullet$
  - $\star$ Each Green function by an edge between vertices z_i and z_j

$$\underset{z_i \qquad\quad z_j}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet} \quad = \quad g(z_i - z_j|\tau)$$

  - $\star$ Each vertex is integrated over Σ
  - $\star$ To a graph with w edges we assign *weight* w

- **Reducibility** : A graph which becomes disconnected
  - $\star$ upon cutting one edge vanishes by $\int_\Sigma g = 0$
  - $\star$ upon removing one vertex factorizes into the product of its components

Modular graph functions

- **To each graph is associated a real-analytic modular function**
  - ⋆ since $\mathcal{B}_N^{(1)}(s_{ij}|\tau)$ is a modular function, so are its Taylor coefficients in s_{ij}
  - ⋆ Organize by the number of loops = depth of the graph

$w = 2$

$w = 3$

$w = 4$

$w = 5$

one-loop   two-loops   three-loops
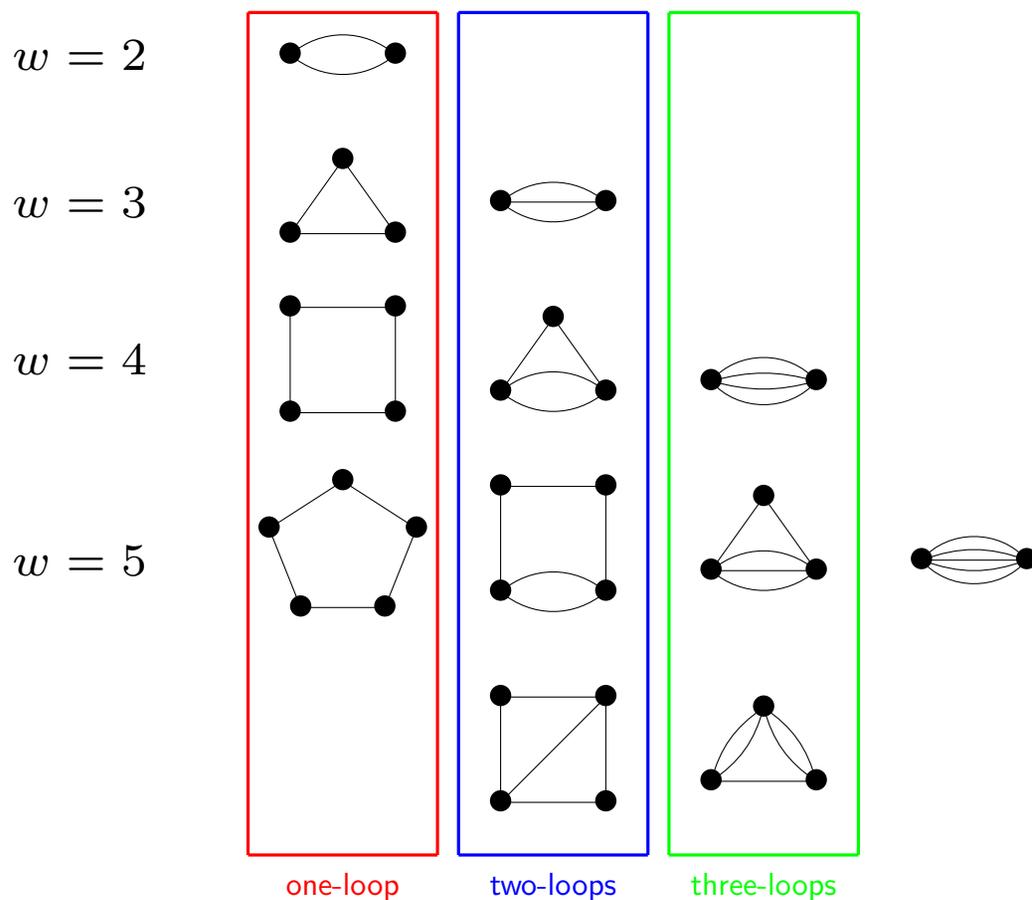
One-loop modular graph functions

- **One-loop weight w graph**
  - ⋆ has w vertices and w edges

$$\prod_{i=1}^{w} \int_{\Sigma} \frac{d^2 z_i}{\tau_2} \, g(z_i - z_{i+1}|\tau) = \sum_{p \in \Lambda'} \frac{\tau_2^w}{\pi^w |p|^{2w}} = E_w(\tau)$$

  - ⋆ with $\Lambda = \mathbb{Z} + \tau \mathbb{Z}$ and $\Lambda' = \Lambda \setminus \{0\}$ with $p = m + \tau n, \ (m, n) \in \mathbb{Z}^2$

- **coincides with a real-analytic Eisenstein series E_w**
  - ⋆ invariant under the modular group $SL(2, \mathbb{Z})$ acting on τ
  - ⋆ Eigenfunction of the Laplace-Beltrami operator on $\mathcal{H}_1$

$$\Delta E_w = w(w-1) E_w \qquad \Delta = 4\tau_2^2 \partial_\tau \partial_{\bar\tau}$$

  - ⋆ Laurent polynomial in τ_2 at cusp $\tau \to i\infty$

$$E_w = \frac{2\zeta(2w)}{\pi^w} \tau_2^w + \frac{2\Gamma(w - \frac{1}{2})\zeta(2w-1)}{\Gamma(w)\,\pi^{w-\frac{1}{2}}} \, \tau_2^{1-w} + \mathcal{O}(e^{-2\pi\tau_2})$$
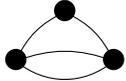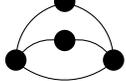
Two-loop modular graph functions

- **Two-loop graphs evaluate to a multiple Kronecker-Eisenstein series**

$$C_{a_1,a_2,a_3}(\tau) = \sum_{p_1,p_2,p_3 \in \Lambda'} \delta\Big(\sum_{r=1}^{3} p_r\Big) \prod_{r=1}^{3} \Big(\frac{\tau_2}{\pi|p_r|^2}\Big)^{a_r}$$

  ⋆ weight $w = a_1 + a_2 + a_3$
  ⋆ invariant under $SL(2,\mathbb{Z})$ and under permutations of a_1, a_2, a_3
  ⋆ Laurent polynomial of degree $(w, 1-w)$ in τ_2 at cusp $\tau \to i\infty$

- **Satisfy "inhomogeneous eigenvalue equations"** (ED, Green, Vanhove 2015)
  ⋆ with eigenvalues of the form $s(s-1)$ and $s \in \mathbb{N}$
  ⋆ the inhomogeneous terms are linear or bilinear in Eisenstein series

  e.g. $C_{1,1,1} =$ $(\Delta - 0)C_{1,1,1} = 6E_3$

  $C_{2,1,1} =$ $(\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2$

  $C_{3,1,1} =$ $(\Delta - 6)C_{3,1,1} = 3C_{2,2,1} + 16E_5 - 4E_2E_3$

  $C_{2,2,1} =$ $(\Delta - 0)C_{2,2,1} = 8E_5$

Modular graph functions at higher loop order

- **Expansion near the cusp** $\tau \to i\infty$
  - ⋆ Laurent polynomial in τ_2 of degree $(w, 1-w) + \mathcal{O}(e^{-2\pi\tau_2})$ whose coefficients
  - ⋆ are single-valued multiple zeta-values (conjectured ED, Green, Gürdogan, Vanhove 2015)
  - ⋆ include "irreducible" multiple zeta-values (Zerbini 2015)

- **Modular graph functions satisfy algebraic identities of uniform weight**

$$\text{e.g.} \qquad = 24C_{2,1,1} + 3E_2^2 - 18E_4$$

$$= 60C_{3,1,1} + 10E_2C_{1,1,1} - 48E_5 + 16\zeta(5)$$
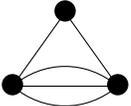
$$= \tfrac{15}{2}C_{3,1,1} + 3E_2E_3 - \tfrac{69}{10}E_5 + \tfrac{7}{40}\zeta(5)$$
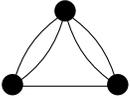
$$= 2C_{3,1,1} - \tfrac{2}{5}E_5 + \tfrac{3}{10}\zeta(5)$$

- **Laplace-Beltrami operator for 3 loops and higher**
  - ⋆ generically no longer maps the space of modular graph forms into itself

Arbitrary number of loops and exponents

- **Modular graph forms** (ED & Green 2016)
  A decorated graph (Γ, A, B) *with* V *vertices and* R *edges consists of*
    - ⋆ *connectivity matrix with components* Γ_{vr}, $v = 1, \cdots, V$, $r = 1, \cdots, R$
    - ⋆ *decoration of the edges by "exponents"* $a_r, b_r \in \mathbb{N}$ *for* $r = 1, \cdots, R$
      $A = [a_1, \cdots, a_R]$ *and* $B = [b_1, \cdots, b_R]$
  To the decorated graph (Γ, A, B) *we associate a function on* $\mathcal{H}_1$

$$\mathcal{C}_\Gamma \begin{bmatrix} A \\ B \end{bmatrix} (\tau) = \sum_{p_1, \cdots, p_R \in \Lambda'} \left(\prod_{r=1}^{R} \frac{(\tau_2/\pi)^{a_r}}{(p_r)^{a_r} (\bar{p}_r)^{b_r}} \right) \prod_{v=1}^{V} \delta \left(\sum_{r=1}^{R} \Gamma_{vr} p_r \right)$$

- **Transformation under** $SL(2, \mathbb{Z})$

$$\mathcal{C}_\Gamma \begin{bmatrix} A \\ B \end{bmatrix} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) = (\gamma\bar{\tau} + \delta)^\mu \, \mathcal{C}_\Gamma \begin{bmatrix} A \\ B \end{bmatrix} (\tau) \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z})$$

⋆ modular weight $(0, \mu)$ with $\mu = \sum_r (b_r - a_r)$: "modular graph form"
⋆ for $\mu \neq 0$, there is no canonical normalization of τ_2-factors
⋆ $A = B \Rightarrow \mu = 0$ recover modular graph functions for Green function g

Algebraic and differential identities

- **Momentum conservation at vertex v implies algebraic identities**

$$\sum_{r=1}^{R} \Gamma_{vr}\, \mathcal{C}_\Gamma \begin{bmatrix} A - S_r \\ B \end{bmatrix} = \sum_{r=1}^{R} \Gamma_{vr}\, \mathcal{C}_\Gamma \begin{bmatrix} A \\ B - S_r \end{bmatrix} = 0$$

  where $A = [a_1 \; \cdots \; a_R]$, $B = [b_1 \; \cdots \; b_R]$ and $S_r = [0_{r-1} \; 1 \; 0_{R-r}]$

- **The Maass operator $\nabla = 2i\tau_2^2 \partial_\tau$ implies differential identities**

$$\nabla\, \mathcal{C}_\Gamma \begin{bmatrix} A \\ B \end{bmatrix} = \sum_{r=1}^{R} a_r\, \mathcal{C}_\Gamma \begin{bmatrix} A + S_r \\ B - S_r \end{bmatrix}$$

- **Algorithm for identities on modular graph functions and forms**
  - ⋆ to prove an identity $F = 0$, solve the weaker condition $\nabla^w F = 0$
  - ⋆ using *holomorphic subgraph reduction* (ED & Green 2016; ED & Kaidi 2016)
  - $\Longrightarrow$ All algebraic identities between modular functions of weight $w \leq 6$

  (MATHEMATICA program Gerken 2020; see also Basu 2015; Kleinschmidt, Verschinin 2017)

- **Identities from generating functions and iterated integrals**

  (Gerken, Kleinschmidt, Schlotterer 2019-20; Broedel, Schlotterer, Zerbini 2018)

Single-valued elliptic multiple polylogarithms

• **Severing a vertex of a modular graph function** (ED, Green, Gürdogan, Vanhove)
    ⋆ produces a real-valued (single-valued) elliptic function on Σ
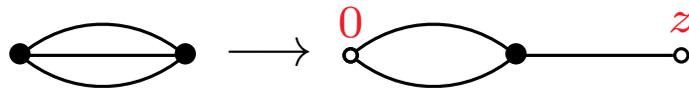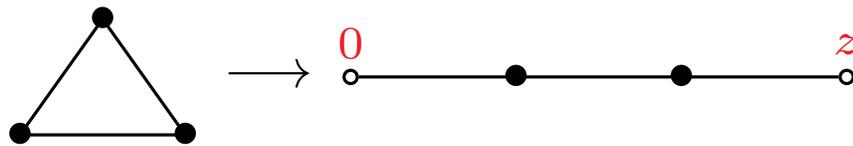
                  ○ = un-integrated vertex

• **Linear chain graphs are single-valued elliptic polylogarithms** (Zagier 1990)
    ⋆ generalizing the Bloch-Wigner dilogarithm

• **Higher loop graphs produce single-valued elliptic multiple polylogarithms**
    ⋆ Modular graph functions may be viewed as special value at $z = 0$

Higher genus

- **How to generalize the genus-one formula to higher genus ?**
  - ⋆ recall the genus-one generating function

$$\mathcal{B}_N^{(1)}(s_{ij}|\tau) = \prod_{i=1}^{N} \int_{\Sigma_1} \frac{d^2 z_i}{\tau_2} \, \exp\left\{ \sum_{1 \le i < j \le N} s_{ij}\, g(z_i - z_j) \right\}$$

- **Compact Riemann surface Σ of genus $h \ge 2$ without boundary**
  - ⋆ we need a scalar Green function $G(z_i, z_j)$
  - ⋆ and a measure $d\mu_N$ on Σ^N

$$\mathcal{B}_N^{(h)}(s_{ij}|\Sigma) = \int_{\Sigma^N} d\mu_N \, \exp\left\{ \sum_{1 \le i < j \le N} s_{ij}\, G(z_i, z_j) \right\}$$

Compact Riemann surfaces Σ of genus h

- **Homology and cohomology**
  - $\star$ One-cycles $H_1(\Sigma, \mathbb{Z}) \approx \mathbb{Z}^{2h}$ with intersection pairing $\mathfrak{J}(\cdot, \cdot) \to \mathbb{Z}$
  - $\star$ Canonical basis $\mathfrak{J}(\mathfrak{A}_I, \mathfrak{A}_J) = \mathfrak{J}(\mathfrak{B}_I, \mathfrak{B}_J) = 0$, $\mathfrak{J}(\mathfrak{A}_I, \mathfrak{B}_J) = \delta_{IJ}$ for $1 \leq I, J \leq h$
  - $\star$ Canonical dual basis of holomorphic one-forms ω_I in $H^{(1,0)}(\Sigma)$

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad\qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ}$$

  - $\star$ Period matrix Ω obeys Riemann relations $\Omega^t = \Omega$, $\text{Im}\,(\Omega) > 0$

- **Modular group** $Sp(2h, \mathbb{Z}) : H_1(\Sigma, \mathbb{Z}) \to H_1(\Sigma, \mathbb{Z})$ **leaves** $\mathfrak{J}(\cdot, \cdot)$ **invariant**

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad M^t \mathfrak{J} M = \mathfrak{J} \qquad \begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix} \to M \begin{pmatrix} \mathfrak{B} \\ \mathfrak{A} \end{pmatrix}$$

  - $\star$ action on 1-forms ω and period matrix Ω given by

$$\begin{aligned} \omega &\to \omega \, (C\Omega + D)^{-1} \\ \Omega &\to (A\Omega + B) \, (C\Omega + D)^{-1} \end{aligned}$$

Modular graph functions for arbitrary genus

- **Canonical metric on Σ = pull-back of flat metric on Jacobian $J(\Sigma)$**
  - ⋆ modular invariant and smooth canonical volume form on Σ

$$\kappa = \frac{i}{2h} \sum_{I,J} Y_{IJ}^{-1}\, \omega_I \wedge \overline{\omega_J} \qquad\qquad Y = \mathrm{Im}\,\Omega \qquad\qquad \int_\Sigma \kappa = 1$$

- **The Arakelov Green function $G(w,z)$ is defined by**

$$\partial_{\bar{w}} \partial_w G(w,z) = -\pi \delta(w,z) + \pi \kappa(w) \qquad\qquad \int_\Sigma \kappa\, G = 0$$

- **"Natural" generating function for higher genus modular graph functions**

$$\mathcal{C}_N^{(h)}(s_{ij}|\Sigma) = \int_{\Sigma^N} \prod_{i=1}^N \kappa(z_i)\ \exp\left\{ \sum_{1 \le i < j \le N} s_{ij}\, G(z_i, z_j) \right\}$$

  - ⋆ Integrals absolutely convergent for $|s_{ij}| < 1$; analytic near $s_{ij} = 0$
  - ⋆ Depend only on Σ, not on specific Ω chosen to represent Σ
  - ⋆ Taylor coeffs in s_{ij} give *modular graph functions* (ED, Green, Pioline 2017)

Genus-two string amplitude

- **Actually genus 2 string amplitude does NOT correspond to** $\mathcal{C}_4^{(2)}(s_{ij}|\Sigma)$
  - ⋆ volume form κ is unique on Σ
  - ⋆ but κ^N is not unique on Σ^N for $N \geq 2$

- **Genus-two four-graviton string amplitude given by** (ED & Phong 2005)

$$\mathcal{B}_4^{(2)}(s_{ij}|\Sigma) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \bar{\mathcal{Y}}}{(\det Y)^2} \exp \left\{ \sum_{1 \leq i < j \leq 4} s_{ij} \, G(z_i, z_j) \right\}$$

- **Measure given by a holomorphic** $(1,0)^{\otimes 4}$ **form** $\mathcal{Y}$ **on** Σ^4

$$\mathcal{Y} = (s_{14} - s_{13})\Delta(z_1, z_2) \wedge \Delta(z_3, z_4) + 2 \text{ cycl perms of } (2,3,4)$$

  ⋆ where Δ is a holomorphic $(1,0)^{\otimes 2}$ form on Σ^2

$$\Delta(z_i, z_j) = \omega_1(z_i) \wedge \omega_2(z_j) - \omega_2(z_i) \wedge \omega_1(z_j)$$

  ⋆ Volume form $\mathcal{Y} \wedge \bar{\mathcal{Y}}/(\det Y)^2$ and $\mathcal{B}_4^{(2)}(s_{ij}|\Sigma)$ are $Sp(4, \mathbb{Z})$-invariant.

Taylor expansion of genus two amplitude

- **Low energy expansion of the genus-two four graviton** $\mathcal{B}_4^{(2)}$

$$\mathcal{B}_4^{(2)}(s_{ij}|\Sigma) = 32\sigma_2 + 64\sigma_3\varphi(\Sigma) + 32\sigma_4\psi(\Sigma) + \mathcal{O}(s_{ij}^5)$$

  ⋆ recall $\sigma_n = s_{12}^n + s_{13}^n + s_{14}^n$

- $\varphi(\Sigma)$ **is the Kawazumi-Zhang invariant for genus two** (ED & Green 2013)

$$\varphi(\Sigma) = \int_{\Sigma^2} |\nu(x,y)|^2 G(x,y) \qquad \nu(x,y) = \frac{i}{2}Y^{IJ}\omega_I(x)\overline{\omega}_J(y)$$

  ⋆ introduced as a spectral invariant (Kawazumi 2008; Zhang 2008)
  ⋆ related to the Faltings invariant (Faltings 1984; De Jong 2010)

- **New genus-two modular graph functions at every order in** s_{ij}**, e.g.**

$$\psi(\Sigma) = \int_{\Sigma^4} \frac{|\Delta(1,2)\Delta(3,4)|^2}{(\det Y)^2}\Big(G(1,4) + G(2,3) - G(1,3) - G(2,4)\Big)^2$$

Modular geometry and differential equation

- **Siegel half space** $\mathcal{H}_h = \{\Omega \in \mathbb{C}^{h^2},\, \Omega^t = \Omega,\, Y = \mathrm{Im}\,(\Omega) > 0\} = Sp(2h, \mathbb{R})/U(h)$
  - ⋆ with $Sp(2h, \mathbb{R})$ invariant Kähler metric

  $$ds^2 = \sum_{I,J,K,L} Y_{IJ}^{-1}\, d\bar{\Omega}_{JK}\, Y_{KL}^{-1}\, d\Omega_{LI}$$

  - ⋆ and $Sp(2h, \mathbb{R})$ invariant Laplace-Beltrami operator on $\mathcal{H}_h$

  $$\Delta = \sum_{I,J,K,L} 4\, Y_{IK}\, Y_{JL}\, \bar{\partial}^{IJ}\, \partial^{KL} \qquad \partial^{IJ} = \frac{1}{2}(1 + \delta^{IJ})\frac{\partial}{\partial \Omega_{IJ}}$$

  - ⋆ Genus-two moduli space $\mathcal{M}_2 = \mathcal{H}_2/Sp(4, \mathbb{Z})$ (minus diagonal Ω)

- $\varphi(\Sigma)$ **satisfies inhomogeneous eigenvalue equation on** $\overline{\mathcal{M}_2}$

  $$(\Delta - 5)\varphi = -2\pi\delta_{SN}$$

  - ⋆ δ_{SN} has support on separating node (into two genus-one surfaces)
  - ⋆ proven by methods from complex structure deformations theory
  - ⋆ allows one to integrate $5\int_{\mathcal{M}_2}\varphi = \int_{\mathcal{M}_2}(\Delta\varphi + 2\pi\delta_{SN}) = 2\pi^3/9$
  - √ check with $D^6\mathcal{R}^4$ prediction from $SL(2, \mathbb{Z})$ duality of Type IIB
    (ED, Green, Pioline, Russo 2014; see also Kawazumi 2008)
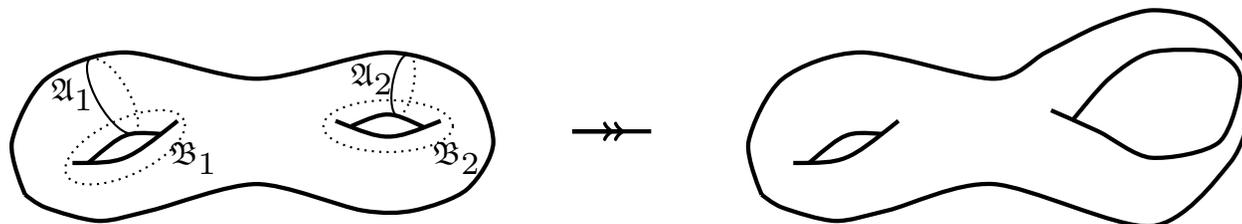
Degenerations of genus-two Riemann surfaces

- **Locally parametrize** $\mathcal{M}_2 = \mathcal{H}_2/Sp(4,\mathbb{Z})$ **by the period matrix**

$$\Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix} \qquad \tau, \sigma, v \in \mathbb{C} \qquad \det\left(\operatorname{Im}\Omega\right) > 0$$

⋆ *Separating degeneration* $v \to 0$

⋆ *Non-separating degeneration* $\sigma \to i\infty$

⋆ *Tropical limit* $\operatorname{Im}\Omega \to \infty$ *keeping ratios fixed = field theory limit*

Non-separating degeneration

- Σ **degenerates to torus** Σ_1 **of modulus** τ **with punctures** p_a, p_b
    - $\star$ keep the cycles $\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{A}_2$ fixed, and let $\mathfrak{B}_2 \to \infty$ as $\mathrm{Im}\,(\sigma) \to \infty$

- **Modular group** $Sp(4, \mathbb{Z})$ **reduces to** $SL(2, \mathbb{Z}) \ltimes \mathbb{Z}^3$ **Fourier-Jacobi group**
    - $=$ the subgroup that leaves $\mathfrak{B}_2$ invariant
    - $\star$ The degeneration parameter σ is not invariant under $SL(2, \mathbb{Z})$
    - $\star$ e.g. Siegel modular forms decompose into Jacobi forms (Eichler & Zagier 1985)

- **There exists a real-valued** $SL(2, \mathbb{Z}) \ltimes \mathbb{Z}^3$**-invariant parameter** $t > 0$

$$t \equiv \frac{\det\,(\mathrm{Im}\,\Omega)}{\mathrm{Im}\,\tau} = \mathrm{Im}\,\sigma - \frac{(\mathrm{Im}\,v)^2}{\mathrm{Im}\,\tau} \qquad \Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix}$$

   - $\star$ the non-separating node is characterized by $t \to \infty$

Non-separating degeneration cont'd

- **Theorem**   **Expansion near the separating node of** (ED, Green, Pioline 2018)
  - $\star$ Expand $\mathcal{B}$ in powers of s_{ij} = weight

$$\mathcal{B}(s_{ij}|\Sigma) = \sum_{w=0}^{\infty} \frac{1}{w!} \mathcal{B}_w(s_{ij}|\Sigma)$$

$$\mathcal{B}_w(s_{ij}|\Sigma) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \overline{\mathcal{Y}}}{(\det Y)^2} \left(\sum_{i<j} s_{ij} G(z_i, z_j) \right)^w$$

  - $\star$ then $\mathcal{B}_w$ is given by a Laurent polynomial of degree $(w, -w)$ in t

$$\mathcal{B}_w(s_{ij}|\Omega) = \sum_{k=-w}^{w} \mathcal{B}_w^{(k)}(s_{ij}|v, \tau)\, t^k + \mathcal{O}(e^{-2\pi t})$$

  - $\star$ where the coefficients are invariant under $SL(2, \mathbb{Z}) \ltimes \mathbb{Z}^2 \subset Sp(4, \mathbb{Z})$
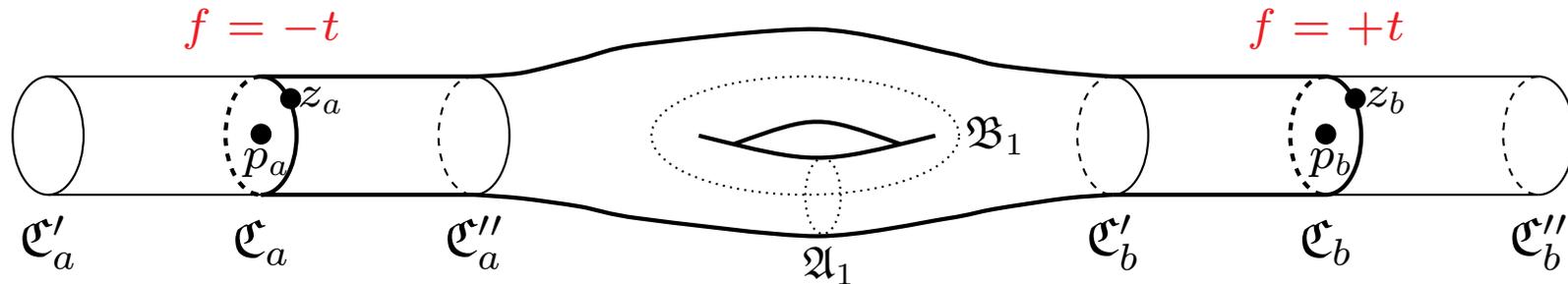
$$\mathcal{B}_w^{(k)} \left(s_{ij} \left| \frac{v + m\tau + n}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right. \right) = \mathcal{B}_w^{(k)}(s_{ij}|v, \rho)$$

  - $\star$ and are elliptic modular graph functions $\sim$ single-valued elliptic polylogs

Ingredients in the proof

- **Funnel construction of the non-separating degeneration** (see Fay 1973)
    - ⋆ start from genus-one surface with two punctures p_a, p_b
    - ⋆ identify cycles $\mathfrak{C}_a \approx \mathfrak{C}_b$, $\mathfrak{C}'_a \approx \mathfrak{C}'_b$ and $\mathfrak{C}''_a \approx \mathfrak{C}''_b$ all homologous to $\mathfrak{A}_2$
    - ⋆ the cycle $\mathfrak{B}_2$ is a curve connecting z_a to z_b.

- ⋆ Key is the existence of a Morse function $f(z) = g(z - p_b|\tau) - g(z - p_a|\tau)$

    - ⋆ For large enough t: cycles prescribed by $f(\mathfrak{C}_a) = -t$ and $f(\mathfrak{C}_b) = +t$
    - ⋆ The cycles $\mathfrak{C}_a, \mathfrak{C}_b$ have exponentially vanishing coordinate radius
        so that $z \in \mathfrak{C}_a$ satisfies $|z - p_a|^2 \approx e^{-2\pi t}$
    - ⋆ Power dependence in t arises from singular behavior of f near p_a, p_b
    - ⋆ Allows to extract t-derivative (cfr RG methods in quantum field theory)

Genus-two five-string amplitude

- **Genus-two amplitude for five external massless states**
  - ⋆ constructed by chiral splitting and pure spinors (ED, Mafra, Pioline, Schlotterer 2020)
  - ⋆ for external NS states constructed in RNS (ED, Schlotterer, in progress)
  - ⋆ several independent kinematical invariants
  - ⋆ $D^2\mathcal{R}^5 \sim D^4\mathcal{R}^4$ given by constant integrand on $\mathcal{M}_2$
  - ⋆ $D^4\mathcal{R}^5 \sim D^6\mathcal{R}^4$ given by Kawazumi-Zhang invariant
  - ⋆ $D^6\mathcal{R}^5 \sim D^8\mathcal{R}^4$ given by weight-two modular graph functions

$$\mathcal{Z}_1 = 8 \int_{\Sigma^2} \kappa(1)\kappa(2)\mathcal{G}(1,2)^2 \qquad \mathcal{Z}_2 = 4 \int_{\Sigma^3} |\nu(1,2)|^2 \kappa(3)\mathcal{G}(1,3)\mathcal{G}(2,3)$$

$$\mathcal{Z}_4 = 2 \int_{\Sigma^2} |\nu(1,2)|^2 \mathcal{G}(1,2)^2 \qquad \mathcal{Z}_3 = 2 \int_{\Sigma^4} |\nu(1,3)\nu(2,4)|^2 \mathcal{G}(1,2)\mathcal{G}(3,4)$$

- **Satisfy a remarkably simple algebraic identity** (ED, Mafra, Pioline, Schlotterer 2020)

$$\mathcal{Z}_1 + \mathcal{Z}_2 + \mathcal{Z}_3 + \mathcal{Z}_4 = \varphi^2$$

  - ⋆ implies Basu's differential identity (Basu 2018)

- **Further new modular graph function for** $(D^6\mathcal{R}^5)'$ **not related to** $D^8\mathcal{R}^4$

Integrating over genus-one moduli – transcendentality

- **Tree-level Type II amplitude has uniform transcendent weight**

$$\mathcal{A}_4^{(0)}(s_{ij}) = \frac{3}{\sigma_3} \exp\left\{ -\sum_{n=1}^{\infty} \frac{2\zeta(2n+1)}{2n+1} \sigma_{2n+1} \right\}$$

⋆ $\sigma_m = s_{12}^m + s_{13}^m + s_{14}^m$

⋆ assigning weight $w[\zeta(n)] = n$ and $w[s_{ij}] = -1$ implies $w[\mathcal{A}_4^{(0)}(s_{ij})] = 3$

- **Genus-one amplitude obtained by integration on moduli space**

$$\mathcal{A}^{(1)}(s_{ij}) = \int_{\mathcal{M}_1} \frac{d^2\tau}{\tau_2^2} \mathcal{B}^{(1)}(s_{ij}|\tau)$$

⋆ Inherits the OPE pole singularities of $\mathcal{B}^{(1)}(s_{ij}|\tau)$ at $s_{ij} \in \mathbb{N}$.

⋆ Further singularities from non-uniform behavior of G at cusp $\tau \to i\infty$;
     produce poles and branch cuts in s_{ij} as expected from unitarity.

⋆ Isolate a small neighborhood of the cusp $\tau_2 > L \gg 1$

$$\mathcal{A}^{(1)}(s_{ij}) = \mathcal{A}_{\tau_2 < L}^{(1)}(s_{ij}) + \mathcal{A}_{\tau_2 > L}^{(1)}(s_{ij})$$

⋆ by construction all L-dependence must cancel in the sum.

The non-analytic part $\mathcal{A}^{(1)}_{\tau_2 > L}$

- **To all orders in s_{ij} the orders L^0 and $\ln L$ may be computed exactly**

$$\mathcal{A}^{(1)}_{\tau_2 > L}(s_{ij}) = \mathcal{A}_*(L; s = s_{12}, t = s_{14}) + \text{ 5 permutations}$$

⋆ To all orders in s, t the function $\mathcal{A}_*(L; s, t)$ is given by (ED & Green 2019)

$$\mathcal{A}_*(L; s, t) = \sum_{N=2}^{\infty} \frac{s^N}{N!} \mathcal{R}^{(N)}_{12}(L; s, t) + \sum_{M,N=2}^{\infty} \frac{s^{M+N}}{M!\,N!} \mathcal{R}^{(M,N)}_{12;34}(L; s, t)$$

⋆ The coefficient $\mathcal{R}_{12}$ is given by (corresponding formula for $\mathcal{R}_{12;34}$)

$$\mathcal{R}^{(N)}_{12} = \sum_{k=1}^{\infty} \sum_{\ell=0}^{k-1} C_{k,\ell} S(N,k) s^{k-\ell+1} t^\ell \Big(\ln(-4\pi L s) + \Psi(k - \ell + 1) - 2\Psi(k + 6) \Big)$$

where $C_{k,\ell} \in \mathbb{Q}$ and $S(N, k)$ are multiple zeta values

$$S(N, k) = \sum_{m_1, \cdots, m_N \neq 0} \frac{\delta(\sum_r m_r)}{|m_1 \cdots m_N|(|m_1| + \cdots |m_N|)^k}$$

- $\ln(-s)$ **part may be obtained via factorization onto tree-level**
  ⋆ but the remaining analytic part cannot be determined that way.

The analytic part $\mathcal{A}^{(1)}_{\tau_2 < L}$

• **To compute the analytic part, Taylor expand in** s_{ij}

$$\mathcal{B}^{(1)}(s_{ij}|\tau) = \sum_{p,q=0}^{\infty} \mathcal{B}_{(p,q)}(\tau) \frac{\sigma_2^p \, \sigma_3^q}{p! \, q!} \qquad w[\mathcal{B}_{(p,q)}] = 2p + 3q$$

  – Extract L^0 and $\ln L$ dependence from integral over τ.

• **Exploit algebraic and differential identities to simplify** $\mathcal{B}_{(p,q)}$ **for weight** ≤ 6
  (ED, Green, Vanhove 2015-16; ED, Kaidi 2016; Broedel, Sclotterer, Zerbini 2018)
  ⋆ Expose Eisenstein series, and Laplacian of modular functions
  ⋆ Exploit Stokes's theorem to integrate Laplacian
  ⋆ e.g. at weight 6,

$$6 \, \mathcal{B}_{(3,0)} = \Delta\left(-9C_{2,2,1,1} + 18E_3^2 + 9E_2E_4 + 6C_{4,1,1} + 156C_{3,2,1} + 41C_{2,2,2}\right)$$

$$+ 72E_2C_{2,1,1} - 12E_3^2 - 36E_2E_4 - 2652E_6$$

$$27 \, \mathcal{B}_{(0,2)} = \Delta\left(9C_{2,2,1,1} - 18E_3^2 - 9E_2E_4 - 6C_{4,1,1} + 258C_{3,2,1} + 64C_{2,2,2}\right)$$

$$- 36E_2C_{2,1,1} + 483E_3^2 + 30\zeta(3)E(3) + 18E_2E_4 + 6E_2^3 - 3186E_6 + 3\zeta(3)^2$$

Transcendentality counting for $D^8 \mathcal{R}^4$

- $D^8 \mathcal{R}^4$ **is the lowest order at which non-analyticities appear**
  - ⋆ All L-dependence cancels in the sum (by construction)

$$\mathcal{A} = \frac{4\zeta(3)s^4}{15} \left[-\ln(-2\pi s) + \frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta'(3)}{\zeta(3)} - \gamma + \frac{63}{20} \right] + 2 \text{ perms of } s, t, u$$

RULES towards conserving transcendental weight
  - √ The argument $-2\pi s$ of the log consistently has weight 0;
  - √ Assign weight 1 to $\ln(-2\pi s)$;
  - √ Assign weight 1 to the combination $\frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta'(3)}{\zeta(3)} - \gamma$
  - √ Assign weight 1 to harmonic sums which result in $\frac{63}{20}$
      as is familiar from QFT (Kotikov, Lipatov 2002; Beccaria, Forini 2009)

- **Corresponding analysis for** $D^{10}\mathcal{R}^4$ **and** $D^{12}\mathcal{R}^4$ **confirms that**
    they obey uniform transcendentality

Summary and outlook

- **Low energy expansion of string amplitudes reveals a rich structure of**

  ⋆ Modular graph functions for Riemann surfaces of genus-one and beyond;
  ⋆ Relation with Kawazumi-Zhang and Faltings invariants;
  ⋆ Systematics of algebraic and differential identities for genus one;
  ⋆ The first identities between genus-two modular graph functions.

- **Integration over moduli space for genus one** (cfr. ED & Green 2019)
  ⋆ Consistent assignment of transcendental weight possible at genus one.

- **Simple, but striking, observation at order** $D^{12}\mathcal{R}^4$
  ⋆ recall two invariants, σ_2^3 and σ_3^2 from explicit calculation we find

$$\tfrac{1}{18}\zeta(3)^2\sigma_3^2 + 0 \cdot \zeta(3)^2\sigma_2^3$$

  ⋆ First term can arise from "square of tree-level", but not the second term !
  ⋆ Why ? Is there a pattern ?