GiNaC goes elliptic
(numerical evaluation of elliptic multiple polylogarithms)

Stefan Weinzierl

Institut für Physik, Universität Mainz

in collaboration with M. Walden
and based on previous work with
L. Adams, Ch. Bogner, I. Hönemann, K. Tempest and A. Schweitzer

I:	**Review**:	**Multiple polylogarithms in GiNaC**
II:	**Background**:	**Elliptic curves**
III:	**Implementation**:	**Evaluation of elliptic multiple polylogarithms**

Introduction

Physics is about numbers

A mathematical physicist thinks about

$$\zeta_3$$

An experimental physicist thinks about

$$1.2020569031 \pm 2 \cdot 10^{-10}$$

GiNaC

GiNaC was initiated in 1999 by Ch. Bauer, A. Frink and R. Kreckel at the University of Mainz.

Despite it's name, it is a computer algebra system. Allows symbolic calculations in C++.

Shipped with major linux distributions (Ubuntu, Debian, Fedora, ...).

Available at http://www.ginac.de.

GiNaC contains a sub-package to evaluate multiple polylogarithms with arbitrary precision.

J. Vollinga, S.W., (2004)

Multiple polylogarithms

Definition based on iterated integrals:

$$G(z_1, \ldots, z_k; y) = \int_0^y \frac{dt_1}{t_1 - z_1} \int_0^{t_1} \frac{dt_2}{t_2 - z_2} \ldots \int_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}, \qquad z_k \neq 0$$

Definition based on nested sums:

$$\text{Li}_{m_1, m_2, \ldots, m_k}(x_1, x_2, \ldots, x_k) = \sum_{n_1 > n_2 > \ldots > n_k > 0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot \ldots \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

Conversion:

$$\text{Li}_{m_1, \ldots, m_k}(x_1, \ldots, x_k) = (-1)^k G_{m_1, \ldots, m_k}\left(\frac{1}{x_1}, \frac{1}{x_1 x_2}, \ldots, \frac{1}{x_1 \ldots x_k}; 1 \right)$$

Short hand notation:

$$G_{m_1, \ldots, m_k}(z_1, \ldots, z_k; y) = G(\underbrace{0, \ldots, 0}_{m_1 - 1}, z_1, \ldots, z_{k-1}, \underbrace{0 \ldots, 0}_{m_k - 1}, z_k; y)$$

Trailing zeros

We may slightly extend the definition of multiple polylogarithms and allow trailing zeros:

$$G(\underbrace{0,\ldots,0}_{r};y) = \frac{1}{r!}\ln^r(y)$$

and define recursively

$$G(z_1,z_2,\ldots,z_r;y) = \int\limits_0^y \frac{dt_1}{t_1-z_1} G(z_2,\ldots,z_r;t_1).$$

Example

```
ginsh - GiNaC Interactive Shell (GiNaC V1.7.10)
  __,  _______  Copyright (C) 1999-2020 Johannes Gutenberg University Mainz,
 (__) *       | Germany.  This is free software with ABSOLUTELY NO WARRANTY.
  ._) i N a C | You are welcome to redistribute it under certain conditions.
<-------------' For details type `warranty;'.

Type ?? for a list of help topics.
> Digits=50;
50
> evalf(G({0,0,1},1));
-1.202056903159594285399738161511449990764986292340498881794
```

Why arbitrary precision?

The PSLQ algorithm can be used to find integer relatios among a set of transcendental numbers.

Ferguson, Bailey, (1992)

This can be used to fix and to simplify boundary constants for Feynman integrals.

This requires that the transcendental numbers are known to sufficient high precision (a few hundred digits).

Special cases

The multiple polylogarihms contain as subsets the classical polylogarithms

$$\text{Li}_n(x),$$

Nielsen's generalized polylogarithms

$$S_{n,p}(x) = \text{Li}_{n+1,1,...,1}(x, \underbrace{1, ..., 1}_{p-1}),$$

the harmonic polylogarithms of Remiddi and Vermaseren

$$H_{m_1,...,m_k}(x) = \text{Li}_{m_1,...,m_k}(x, \underbrace{1, ..., 1}_{k-1}).$$

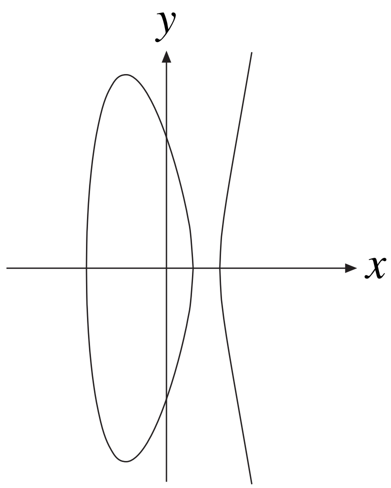
Part II

Elliptic curves

Elliptic curves

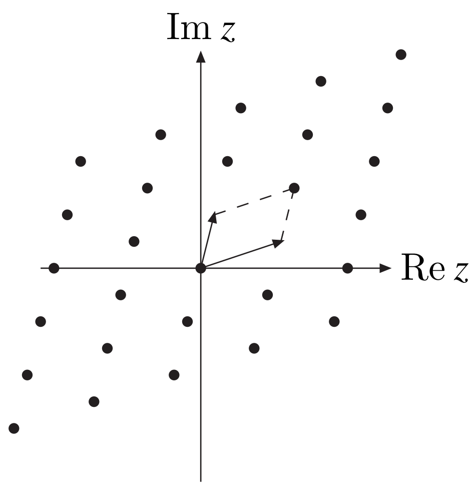
Let ψ_1 and ψ_2 be two complex numbers with $\mathrm{Im}(\psi_2/\psi_1) \neq 0$.

Elliptic curve represented by



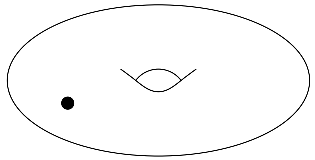
Weierstrass normal form

$y^2 = 4x^3 - g_2 x - g_3$



Torus

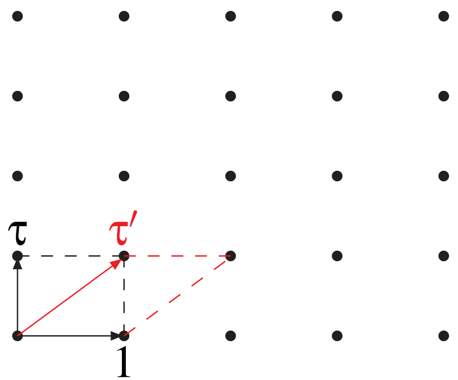
$\mathbb{C}/\Lambda$



Real Riemann surface of genus one with one marked point

Bases of lattices

The periods ψ_1 and ψ_2 generate a lattice. Any other basis as good as (ψ_2, ψ_1).
Convention: Normalise $(\psi_2, \psi_1) \to (\tau, 1)$ where $\tau = \psi_2/\psi_1$.



Change of basis: $$\begin{pmatrix} \psi_2' \\ \psi_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix},$$

Transformation should be invertible: $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

In terms of τ and τ': $$\tau' = \frac{a\tau + b}{c\tau + d}$$

Modular forms

Denote by $\mathbb{H}$ the complex upper half plane. A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of modular weight k for $\mathrm{SL}_2(\mathbb{Z})$ if

(i) f transforms under Möbius transformations as

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \cdot f(\tau) \qquad \text{for } \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \mathrm{SL}_2(\mathbb{Z})$$

(ii) f is holomorphic on $\mathbb{H}$,

(iii) f is holomorphic at ∞.

Congruence subgroups

Apart from $\mathrm{SL}_2(\mathbb{Z})$ we may also look at congruence subgroups, for example

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \bmod N,\ c \equiv 0 \bmod N \right\}$$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \bmod N,\ b, c \equiv 0 \bmod N \right\}$$

Modular forms for congruence subgroups: Require "nice" transformation properties only for subgroup Γ (plus holomorphicity on $\mathbb{H}$ and at the cusps).

Dirichlet character

Let N be a positive integer. A function $\chi : \mathbb{Z} \to \mathbb{C}$ is called a Dirichlet character modulo N, if

(i) $\chi(n) = \chi(n+N) \qquad \forall n \in \mathbb{Z}$,

(ii) $\chi(n) = 0$ if $\gcd(n,N) > 1$ and $\chi(n) \neq 0$ if $\gcd(n,N) = 1$,

(iii) $\chi(nm) = \chi(n)\chi(m) \qquad \forall\, n,m \in \mathbb{Z}$.

The conductor of χ is the smallest positive divisor $d|N$ such that there is a character χ' modulo d with

$$\chi(n) = \chi'(n) \qquad \forall\, n \in \mathbb{Z} \qquad \text{with } \gcd(n,N) = 1.$$

The Kronecker symbol

Let a be an integer and n a non-zero integer with prime factorisation $n = up_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $u \in \{1, -1\}$ is a unit. The Kronecker symbol is defined by

$$\left(\frac{a}{n}\right) = \left(\frac{a}{u}\right) \left(\frac{a}{p_1}\right)^{\alpha_1} \left(\frac{a}{p_2}\right)^{\alpha_2} \cdots \left(\frac{a}{p_k}\right)^{\alpha_k}.$$

If a is the discriminant of a quadratic field, then it is a primitive Dirichlet character with conductor $|a|$.

Including the trivial character (for which $a = 1$) the possible values for a with smallest absolute value are

$$1, -3, -4, 5, -7, 8, -8, -11, 12, \ldots$$

Modular forms with character

We may relax the transformation law:

Let N be a positive integer and let χ be a Dirichlet character modulo N. A function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k for $\Gamma_0(N)$ with character χ if

(i) f is holomorphic on $\mathbb{H}$,

(ii) f is holomorphic at the cusps of $\Gamma_1(N)$,

(iii) $f\left(\dfrac{a\tau+b}{c\tau+d}\right) = \chi(d)(c\tau+d)^k f(\tau)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

The space of modular forms

- The modular forms for a given congruence subgroup form a vectorspace.
- This vectorspace is finite dimensional.
- It decomposes into a subspace of cusp forms and the Eisenstein subspace.
- We have

$$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_{\chi} \mathcal{M}_k(N,\chi)$$

  and similar for the subspace of cusp forms and the Eisenstein subspace.
- Basis of Eisenstein subspace $\mathcal{E}_k(N,\chi)$ given in terms of generalised Eisenstein series.

Generalised Eisenstein series

Let a be the discriminant of a quadratic field. The Kronecker symbol defines then a primitive Dirichlet character

$$\chi_a(n) = \left(\frac{a}{n}\right)$$

Set

$$E_{k,a,b}(\tau) = a_0 + \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi_a(n/d) \cdot \chi_b(d) \cdot d^{k-1} \right) \bar{q}^n, \qquad \bar{q} = e^{2\pi i \tau}$$

Generalised Eisenstein series:

$$E_{k,a,b,K}(\tau) = \begin{cases} E_{k,a,b}(K\tau), & (k,a,b) \neq (2,1,1), \quad K \geq 1, \\ E_{2,1,1}(\tau) - K E_{2,1,1}(K\tau), & (k,a,b) = (2,1,1), \quad K > 1. \end{cases}$$

Iterated integrals of modular forms

Iterated integrals of modular forms:

$$I\left(f_1, f_2, \ldots, f_n; q\right) = \left(2\pi i\right)^n \int\limits_{\tau_0}^{\tau} d\tau_1 f_1\left(\tau_1\right) \int\limits_{\tau_0}^{\tau_1} d\tau_2 f_2\left(\tau_2\right) \ldots \int\limits_{\tau_0}^{\tau_{n-1}} d\tau_n f_n\left(\tau_n\right)$$

An integral over a modular form is in general not a modular form.

Analogy: An integral over a rational function is in general not a rational function.

The Kronecker function

The first Jacobi theta function $\theta_1(z,q)$:

$$\theta_1(z,q) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{\left(n+\frac{1}{2}\right)^2} e^{i(2n+1)z}, \qquad q = e^{i\pi\tau}$$

The Kronecker function $F(z,\alpha,\tau)$:

$$F(z,\alpha,\tau) = \pi\theta_1'(0,q) \frac{\theta_1(\pi(z+\alpha),q)}{\theta_1(\pi z,q)\,\theta_1(\pi\alpha,q)} = \frac{1}{\alpha} \sum_{k=0}^{\infty} g^{(k)}(z,\tau)\alpha^k,$$

We are mainly interested in the coefficients $g^{(k)}(z,\tau)$ of the Kronecker function.

The coefficients $g^{(k)}(z,\tau)$ of the Kronecker function

Properties of $g^{(k)}(z,\tau)$:

- only simple poles as a function of z
- quasi-periodic as a function of z: Periodic by 1, quasi-periodic by τ.
- almost modular: Nice modular transformation properties only spoiled by divergent Eisenstein series $E_1(z,\tau)$.

Brown, Levin, '11,

Broedel, Duhr, Dulat, Penante, Tancredi, '18

Can we have the full cake and eat it?

No, it is not possible to have integration kernels, which are

- double periodic
- meromorphic
- only simple poles

Elliptic polylogarithms ($\widetilde{\Gamma}$-version)

Elliptic polylogarithms (meromorphic version, simple poles, but not double periodic)

$$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z; \tau\right) \quad = \quad \int_0^z dz' \, g^{(n_1)}(z' - z_1, \tau) \, \widetilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_k \\ z_2 & \dots & z_k \end{smallmatrix}; z'; \tau\right)$$

Broedel, Duhr, Dulat, Tancredi, ’17

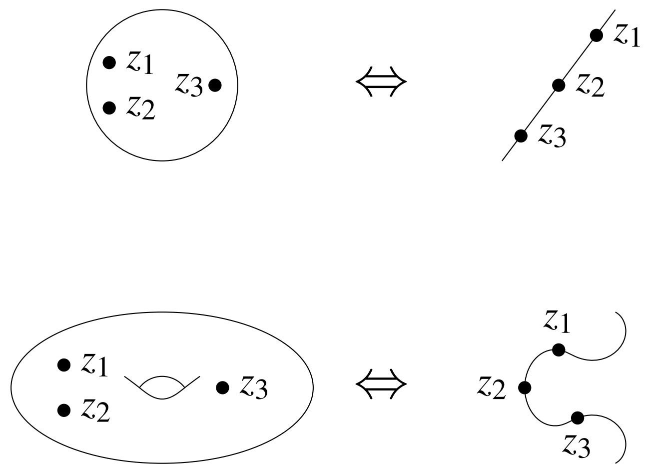
Note: $\tau = \mathrm{const}$

Moduli spaces

$\mathcal{M}_{g,n}$: Space of isomorphism classes of smooth (complex, algebraic) curves of genus g with n marked points.

Recall:

real surface



complex curve

Coordinates

Genus 0: $\dim \mathcal{M}_{0,n} = n-3.$

Sphere has a unique shape

Use Möbius transformation to fix $z_{n-2} = 1, \quad z_{n-1} = \infty, \quad z_n = 0$

Coordinates are $(z_1, \dots, z_{n-3})$

Genus 1: $\dim \mathcal{M}_{1,n} = n.$

One coordinate describes the shape of the torus

Use translation to fix $z_n = 0$

Coordinates are $(\tau, z_1, \dots, z_{n-1})$

In particular:

$\dim \mathcal{M}_{1,1} = 1$	with coordinate τ,	(equal mass sunrise)
$\dim \mathcal{M}_{1,3} = 3$	with coordinates τ, z_1, z_2,	(unequal mass sunrise).

The ELi-functions

Recall the definition of the classical polylogarithms:

$$\mathrm{Li}_n(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^n}.$$

Generalisation, the two sums are coupled through the variable q:

$$\mathrm{ELi}_{n;m}(x;y;q) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^j}{j^n} \frac{y^k}{k^m} q^{jk}.$$

Define the linear combinations

$$\overline{\mathrm{E}}_{n;m}(x;y;q) = \mathrm{ELi}_{n;m}(x;y;q) - (-1)^{n+m} \mathrm{ELi}_{n;m}\left(x^{-1};y^{-1};q\right).$$

Expansion of the coefficients of the Kronecker function

The functions $\overline{\mathrm{E}}_{n;m}$ are helpful for the $\bar{q}$-expansion of the functions $g^{(n)}(z,\tau)$. Explicitly one has with $\bar{q} = \exp(2\pi i \tau)$ and $\bar{w} = \exp(2\pi i z)$

$$\begin{aligned}
g^{(0)}(z,\tau) &= 1, \\
g^{(1)}(z,\tau) &= -2\pi i \left[\frac{1+\bar{w}}{2(1-\bar{w})} + \overline{\mathrm{E}}_{0,0}(\bar{w};1;\bar{q}) \right], \\
g^{(k)}(z,\tau) &= -\frac{(2\pi i)^k}{(k-1)!} \left[-\frac{B_k}{k} + \overline{\mathrm{E}}_{0,1-k}(\bar{w};1;\bar{q}) \right], \qquad k > 1,
\end{aligned}$$

where B_k denote the k-th Bernoulli number.

More variants of elliptic polylogarithms

Some authors consider iterated integrals of the form

$$\int_0^{y_0} dy f(y) G(z_1, \dots, z_{k-1}; y),$$

where only the outermost integration is non-polylogarithmic, for example

$$f(y) = \frac{1}{\sqrt{(y - z_1)(y - z_2)(y - z_3)(y - z_4)}}$$

Remiddi, Tancredi, '17;

Bourjaily, McLeod, Spradlin, von Hippel, Wilhelm, '17;

Hidding, Moriello, '17;

Part III

Evaluation of elliptic multiple polylogarithms

Iterated integrals

For ω_1, ..., ω_k differential 1-forms on a manifold M and $\gamma : [0,1] \to M$ a path, write for the pull-back of ω_j to the interval $[0,1]$

$$f_j(\lambda)\, d\lambda \quad = \quad \gamma^* \omega_j.$$

The iterated integral is defined by (Chen '77)

$$I_\gamma(\omega_1, ..., \omega_k; \lambda) \quad = \quad \int_0^{\lambda} d\lambda_1 f_1(\lambda_1) \int_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \ldots \int_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k).$$

Shuffle product and trailing zeros

Let $\lambda_0 \in \mathbb{R}_{>0}$ and denote by U the domain $U = \{z \in \mathbb{C} | |z| \leq \lambda_0\}$. Let us assume that all ω_j are holomorphic in $U \backslash \{0\}$ and have at most a simple pole at $z = 0$. In other words

$$\omega_j = f_j(z)\, dz = \sum_{n=0}^{\infty} c_{j,n}\, z^{n-1} dz, \qquad c_{j,n} \in \mathbb{C}.$$

We say that ω_j has a trailing zero, if $c_{j,0} \neq 0$.

We may use shuffle product regularisation / tangential base point as in the case of multiple polylogarithms.

Integration kernels

- Kernels related to $g^{(k)}(z, \tau)$:

```
Kronecker_dtau_kernel(k, z_j); Kronecker_dz_kernel(k, z_j, tau);
```

- Kernels related to modular forms:

```
Eisenstein_kernel(k, a, b, K); modular_form_kernel(k, P, qbar);
```

- Kernels related to ELi-functions:

```
ELi_kernel(n, m, x, y); Ebar_kernel(n, m, x, y);
```

- User-defined kernels:

```
user_defined_kernel(f, y);
```

Example

```
Digits = 50;

ex tau = 10*I;
ex qbar = evalf(exp(2*Pi*I*tau));

ex z = 0.9;

ex g_2 = Kronecker_dtau_kernel(2,z);
ex g_3 = Kronecker_dtau_kernel(3,z);

std::cout << iterated_integral(lst{g_3,g_2},qbar).evalf() << std::endl;

3.2253571394850843286565907071596312651610339124775388346726E-27
-4.370890856300573854123107098377978052316020623730771456153E-85*I
```

Conclusions

- At the end we want a number!
- Numerical evaluations with arbitrary precision helpful for PSLQ algorithm.
- Implementation of functions related to elliptic Feynman integrals in GiNaC.