Vector Fields and Integrals

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Motivation

The complexity of amplitude computations arising from the number of integral topologies often requires automated approaches.

Perturbative amplitudes and their integrals naturally setup in algebraic geometry.

In recent years we have developed algebraic toolkit which handles algebra well, however, multi-dimensional geometry is a hard-to-crack bottleneck

We will discuss a few geometric properties and how they can be obtained from algebraic ones. The vehicle/language are vector fields. No direct relevant to elliptic integrals but potentially useful tools.

Numerical Amplitude Methods

- Parametrisation for amplitude integrand in terms of surface terms (= total derivatives) and master integrals.
- Generalised unitarity & numerical evaluation in finite fields to determine coefficients.
- Numerical formulation of dimensional regularisation (treatment of fermions)
- Reconstruction of dimensional regulator D=4-2ε
- Replace integrals by function basis (ε-expansion)
- Extract finite remainder
- Analytic reconstruction in Mandelstam variables (in finite field)
- Simplify coefficients unique partial-fraction decomposition (Leinartes-like algorithm) and reconstruct rationals from finite field

many recent contributions [Abreu, Febres Cordero, Dormans, Ita, Page, Sotnikov, Ruf, Klinkert, Zeng; Badger, Hartanto, Bronnum-Hansen, Peraro; Larsen, Zhang; Mastrolia, Mirabela, Ossola]

Requires vector fields for surface terms.

Aided by good function-basis.

Cut surface

We consider Feynman integrals in momentum space.

Feynman integrals modulo pinches lead to algebraic varieties,

$$I = \int d^{nD}\ell \, \frac{t(\ell, k)}{\rho_1 \cdots \rho_m} \quad \text{mod} \quad \text{pinches} \quad \to \quad t(\ell, k) \quad \text{mod} \quad \langle \rho_1, \ldots, \rho_n \rangle$$

defined by propagator ideal. Insertions modulo pinches means polynomials modulo the propagator ideal.

The propagator ideal defines an algebraic variety which corresponds geometrically to setting propagators to zero as in generalised cuts, i.e. the maximal cut surface.

The insertions modulo propagators are the interesting functions on the cut surface.

Natural coordinates

Natural coordinates for integrals lead to the Baikov/Cutkosky parametrisation in terms of propagators and irreducible scalar products (ISP),

$$\{\ell_i^\mu\} \to \{\rho_j, \alpha_k\}, \qquad I \sim \int \prod_i d\rho_i \prod_j d\alpha_j \frac{t(\alpha,\rho) B^{d(D)}(\alpha,\rho)}{\rho_1 \cdots \rho_m} .$$

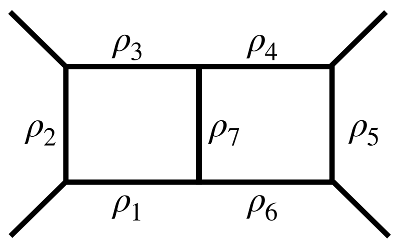
The measure factor, the Baikov polynomial is a Gram determinant. It signals the loss of dimension of the loop momentum,

$$B = \det G(\ell_1, \ell_2, \ldots, k_1, \ldots).$$

Measure factor is volume of internal space that is fibered over the affine space of the ISP variables and propagators. Surfaces without boundary are then open surfaces that end on the zero-set of the Baikov polynomial.

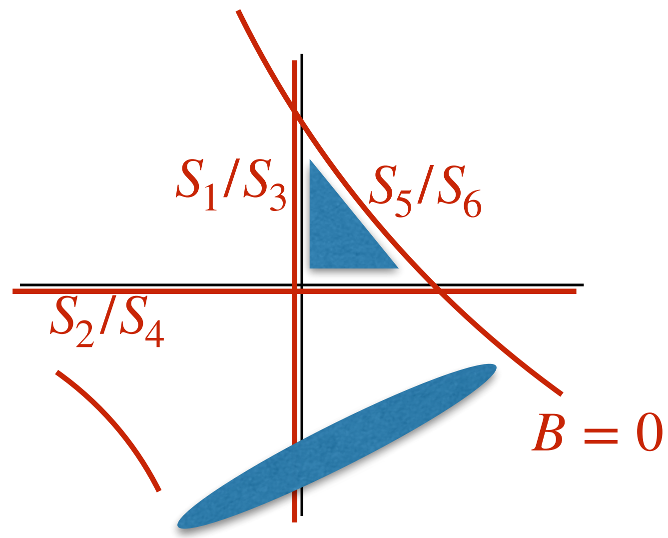
Cut integrals appear naturally after taking residues at $\rho_i = 0$ and include the measure factor $B^{d(D)}(\alpha,0)$.

Example:



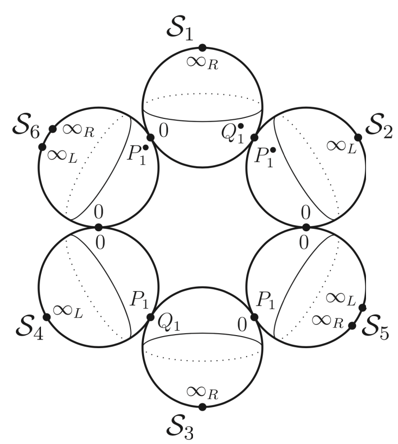
two ISPs: $\{\alpha_1, \alpha_2\}$

cut surface: α_1, α_2 -plane



[Tancredi Primo `16; Bosma, Sogaard, Zhang `17;]

4-dim slice: $B = 0$



[Huot, Larsen `12]

Cut surface is double-cover of the affine space of the ISP variables, i.e. the loop momenta are quadratically related to ISP variables,

$$0 = \rho_1 = (\ell_{4d})^2 + (\ell_\epsilon)^2 = (\ell \cdot p_i)(\ell \cdot p_j)(p_i \cdot p_j)^{-1} + (\ell_\epsilon)^2 \ .$$

Properties of cut surfaces

Comparison of exact versus closed polynomials give master integrals.

Dimension foliations: minors of Gram matrix $G(\ell_1, \ell_2, \ldots, k_1, \ldots)$ define distinguished surfaces. These corresponds to cuts in various dimensions when the loop momenta ℓ_i are lined up with external momenta k_i.

Regular versus singular Baikov surface,

$$0 = \partial_1 B = \ldots = \partial_n B = B.$$

For maximal co-dimension singularities see below.

Experience: the more singularities the simpler, and also 'less elliptic'.
Mechanism for simplicity (e.g. in IBP reduction) unclear.

Maximal Singularities

Singularities determined by loss of rank of the differential of defining equations:

$$\text{rank}(\partial_1 B \ldots \partial_n B)\,|_{B=0} = 0 \quad \rightarrow \quad I_1 = \langle \partial_1 B, \ldots, \partial_n B, B \rangle$$

These singularities are common being the four-dimensional foliation:

$$B = \det(\mathrm{p_i} \cdot \mathrm{p_j})(\mu_{11}\mu_{22} - \mu_{12}^2) \quad \rightarrow \quad \partial_i B \sim \mu_{jk}$$

Considers the singularities of the singularity varieties. Recursive construction: vanishing minors of differential of vanishing minors [see books by Arnold, Gusein-Zade, Varchenko]

$$\text{rank}(\partial_i \partial_1 B \ldots \partial_i \partial_n B)\,|_{\partial_i B = B = 0} = \text{minimal} \quad \rightarrow \quad I_2 = \langle \text{minor}_{ik}(\partial_l \partial_m B), \partial_i B, B \rangle$$

where index ‘i’ labels minors and ‘k’ the type of minor. One chooses the minimal ‘k’ such that the minors still have a common zero.

Motivation: vector fields

1) Vector fields appear naturally in derivations of IBP relations and play a central role in integral parametrisation,

$$0 = \int d^{nD}\ell \sum_{i=1}^{n} \frac{\partial}{\partial \ell_i^\mu} \frac{v_i^\mu}{\rho_1 \cdots \rho_m}, \quad v_i^\mu = \text{polynomial}$$

2) Vector fields trivially give rise to dlog forms,

$$\int d^{nD}\ell \prod_{i,\mu} dlog(v_i^\mu)$$

which count multiplicities of zeros of the vectors.

3) Critical points (=zeros) of vector fields are related to topological information of underlying surfaces. Connection between vanishing cycles (=change of topology) and new vectors (and IBP relations).

Properties of vector fields

The polynomial vector fields are naturally tangent to surfaces of reduced dimension: The expressions,

$$v_i^\mu = p_i(\ell, k)\ell_i^\mu + q_i(\ell, k)k_i^\mu$$

imply where ℓ_i^μ points into D dimensions, v_i^μ will point into D dimensions.

Algebraic formulations [Larsen, Zhang `15, Abreu et al `17]:

$$v_i^\mu \frac{\partial}{\partial \ell_i^\mu} B \sim B\,, \quad v_i^\nu \frac{\partial}{\partial \ell_i^\nu} \mu_{ij} \sim \text{linear combination of } \mu_{ij}$$

Statement follows also from integral measures in Baikov coordinates and measures including $d\mu_{ij}$.

Natural vectors fields

Introduced as vector fields that do not double propagators in a given topology [Gluza, Kajda, Kosower `11],

$$v_i^\mu(\ell, k)\frac{\partial}{\partial \ell_i^\mu}\rho_j(\ell, k) = f_j(\ell, k)\rho_j(\ell, k), \quad \rho_j \in \text{topology}$$

Geometric meaning of equations is that the vectors are tangent to cut surface of topology and all its pinches. (Note correlation of propagator of rhs and lhs.)

Equivalently in natural coordinates: give up dimensionality condition and impose propagator condition by hand [Larsen, Zhang; Ita `15],

$$\{\ell_i^\mu\} \to \{\rho_m, \alpha_n\} \ : \qquad \sum_{m\in props\cup ISP} v^m(\rho,\alpha)\frac{\partial}{\partial \rho_m}\rho_j = f_j(\rho,\alpha)\rho_j \to v^m(\rho,\alpha) = f_m(\rho,\alpha)\rho_m$$

$$\sum_{m\in props} f_m(\rho,\alpha)\rho_m \frac{\partial}{\partial \rho_m}B(\alpha,\rho) + \sum_{n\in ISP} v^n(\rho,\alpha)\frac{\partial}{\partial \alpha_n}B(\alpha,\rho) = f_B(\rho,\alpha)B(\alpha,\rho)$$

for propagators ρ_i and irreducible scalar products α_i .

For on-shell form of vectors ρ_i-terms are dropped,

$$\sum_{n\in ISP} v^n(\alpha)\frac{\partial}{\partial \alpha_n}B(\alpha,0) = f_B(\alpha)B(\alpha,0)\,, \quad v^m = f_m\rho_m = 0\,.$$

These off-shell and on-shell 'syzygy equations' can be solved by computational algebraic geometry (e.g. Singular program). The fewer variables the better.

Types of vector fields

Direct impact of topology change and appearance of singularities:

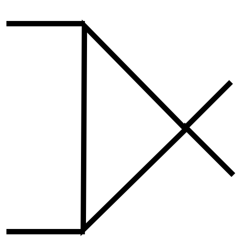
1) Generic vector fields from crossing:

$$v_g = p(\alpha)(\partial_2 B, -\partial_1 B, 0, \ldots)$$

$$v_b = p(\alpha)(B, 0, \ldots, \partial_1 B)$$

2) Non-trivial syzygy relations if zero set of Baikov is singular [see e.g. lecture notes of Y. Zhang `16]

Examples:



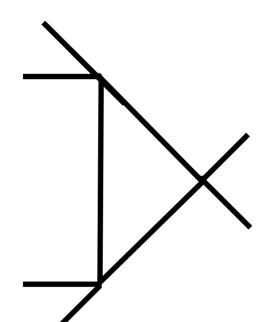
singular
no master

$B = (\alpha_1)^2 + (\alpha_2)^2$

generic: $v_g = (\alpha_2, -\alpha_1, 0, \ldots)$

$v_b = [(\alpha_1)^2 + (\alpha_2)^2, 0, \alpha_1]$

scaling: $v_s = (\alpha_1, \alpha_2, 2)$



non-singular
one master

$B = (\alpha_1)^2 + (\alpha_2)^2 + c(s_{ij})$

generic: $v_g = (\alpha_2, -\alpha_1, 0, \ldots)$

$v_b = [(\alpha_1)^2 + (\alpha_2)^2, 0, \alpha_1]$

Vector components

Two types of components in syzygy, the true vector components v_i and the coefficient of the B-term f_B.

Properties following from defining equation:

$$v^1\partial_1 B + v^2\partial_2 B + \ldots - f_B B = 0$$

a) @ extrema of function B, $\partial_i B = 0$ we find that $f_B = 0$.

b) @ singularities of zero set of $B = 0$ the $B = \partial_i B = 0$ we have $v^i = 0$ and $f_B \neq 0$.

Observations:

a) Taking a complete set of vectors and counting the zeros of $0 = f_B^1 = f_B^2 = \ldots$ matches the number of master integrals on the cut. (Compare to [Lee, Pomeransky '13].)

b) The solutions to $0 = v_1^i = v_2^i = \ldots$ give maximal co-dimension singularities of $B = 0$ surface.

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Surface terms

Surface terms from divergence of vectors with respect of proper measure,

$$m_g = \mathrm{div}(v_g) = \partial_2 B \partial_1 p - \partial_1 B \partial_2 p$$

$$m_b = \mathrm{div}(v_b) = \partial_1 p B + \big(d(D) + 1\big)\, p \partial_1 B$$

$$m_s = \mathrm{div}(p v_s) = v_s^i \partial_i p + f_{sB} d(D) p + p \partial_i v_s^i$$

Observations:

m_g and m_b vanish at singular surface $B = \partial_i B = 0$

m_g vanishes at extrema

all surface terms vanish at extrema, if polynomial p does.

IBP reduction

Reduction modulo pinches amounts to solving the linear system,

$$t(\alpha) = \sum_{i \in \text{surface terms}} c_i(s_{kl}, D)\, m^i(\alpha) + \sum_{i \in \text{masters}} c_i(s_{kl}, D) m^i(\alpha)$$

for the coefficients c_i , $i \in$ masters.

System can be solved numerically by evaluating on random values of the ISPs $\{\alpha_i\}$. Often analytic expressions are obtained from reconstruction [Peraro `16; Abreu et al. `17; Maitre, Laurentis `19; Klappert, Lange `19].

The number of coefficients determines size of system. Choice of masters and relations can impact analytic form of solution

Given vanishing properties of surface terms it is natural to sample specific points:

- singular surfaces or maximum co-dimension singularities
- sub surfaces of reduced dimensions

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Algebraic analog to evaluating on particular surfaces is to expand functions in powers of the vanishing ideal. These lead to generalisations of Taylor expansions called Weierstrass expansions,

$\langle f_1, f_2, \dots \rangle$ = ideal associated to maximal co-dimension singularities

$\{e_1, e_2, \dots\}$ = finite set of polynomials modulo the ideal $\langle f_1, f_2, \dots \rangle$

$$t(\alpha) = \sum_i h_i(s_{kl}, D)\, e^i(\alpha) + \sum_k g_k(\alpha) f_k(\alpha) \quad \text{(follows from polynomial division)}$$

Recursive application of formula (to the $g_k(\alpha)$) gives formal Taylor series of $t(\alpha)$ in terms of $f_i(\alpha)$.

Vector components are in ideal $\langle f_1, f_2, \dots \rangle$. Surface terms can be expanded in f_i and their effect can be analysed in a transparent way.

It will be interesting to see if these structures will be strong enough to turn IBP system to (block) triangular form.

Linking singularities of IR-properties suggests that functions vanishing on singularities are candidates for finite integrals. This implies that the insertion's expansion starts at subheading order.

Conclusion

Vector fields are a natural structure associated to Feynman integrals, so we should learn to use them.

Vanishing properties of vector fields impact IBP-reduction.

Interesting properties of integrals can be extracted from vectors:

- Maximal co-dimension singularities. Can we automate the computation of leading singularities?

- Count master integrals in momentum space without reduction. Can we extract further topological information about cut spaces?

Geometric understanding of vectors can help with computing them, when hard to obtain. We learned that their components live in the space of functions that vanish at singularities.