Physics and graph theory of Feynman periods

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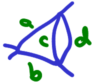
Feynman periods

Let G be a graph, perhaps a scalar Feynman diagram.

variable a_e for each edge e of graph G

$$\Psi_G = \sum_{\substack{T \\ \text{spanning tree of } G}} \prod_{e \notin T} a_e$$



$$\Psi = cd + bd + bc + ac + ad$$

$$P_G = \int_{a_e \geq 0} \frac{da_2 \cdots da_{|E(G)|}}{\Psi_G^2}\Bigg|_{a_1 = 1}$$

Eg

$$P = 6\zeta(3)$$

P_G is a period in the sense of Kontsevich and Zagier

In fact you know these already

The Feynman periods are part of the story of Feynman integrals.

In textbooks.

Folks like David Broadhurst calculated them decades ago.

Contribution to β-function.

Simplified sufficiently that higher loops is what is interesting.

In various forms

As with the Feynman integral can write them in position space, momentum space, or parametric space.
Can write them projectively or affinely.

$$\int_{a_e \geq 1} \frac{da_2 \cdots da_{|E(G)|}}{\Psi^2}\Big|_{a_1=1} = \int_{a_e \geq 1} \frac{\Omega(a)}{\Psi^2}$$

Ω = projective volume form

$$\int \frac{\prod_\ell dp_\ell}{\prod_e (p_e^2)}\Big|_{p_1=1} = \int \frac{\Omega(p)}{\prod_e (p_e^2)}$$

$$p_e = \sum_\ell \pm p_\ell$$

$$\int \frac{dx_2 \cdots}{\prod (x_i - x_j)^2}\Big|_{\substack{x_0=0 \\ x_1=1}} = \int \frac{\Omega(x)}{\prod (x_i - x_j)^2}$$

Why?

Why should you care?

As a physicist

- It is a part of what you need.
- It is relatively computable.
- Simplifications like this one seem to postpone, not remove, interesting complexity, while making the structure clearer.

As a mathematician

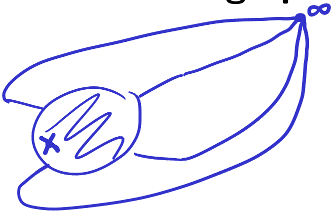
- It looks like algebraic geometry.
- It looks like algebraic graph theory.
- It asks interesting questions.

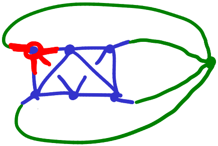
Graph structure

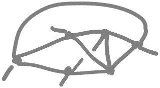
You already know this too – since Feynman diagrams began, you, as a physicist, used the graph structure to help you find the best way to integrate a given graph, to simplify integrals, or to find connections between integrals.

Feynman periods are good for computing edge-by-edge with graphical functions (Schnetz) or hyperlogarithms (Panzer, . . .).

Period invariant graph operations – completion







Period invariant graph operations – dual

The Fourier transform symmetry between position and momentm space in 4-dimensions is the planar dual of graphs.
(or more generally a matroidal dual)

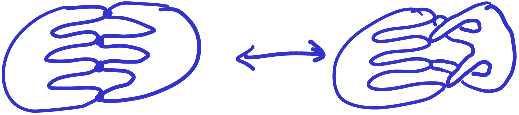
whole story works for regular matroids
graphic matroids
cographic matroids (duals of graphs)
a bit more R_{10}
and glue together
3 join

edges
vertices

Period invariant graph operations – twist

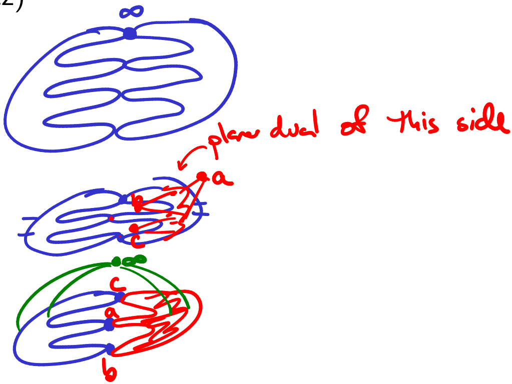
(due to Schnetz)

To state it nicely work at the level of the completed graph

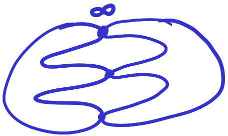


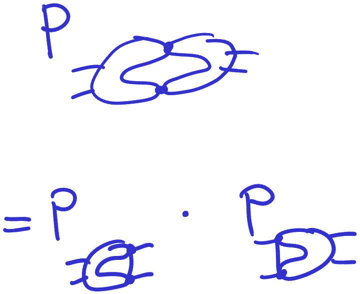
Period invariant graph operations – Fourier split

(from Hu, Schnetz, Shaw, Y, based on an observation of Golz, Panzer, Schnetz)

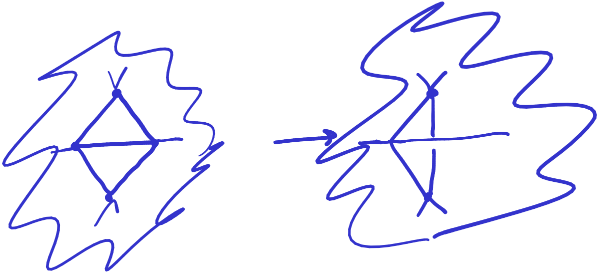


3-join





Double triangle



This is a graph reduction that appears to preserve transcendental co-weight

Of particular interest, what is the structure of all graphs that reduce to K_5?

The c_2 invariant

Recall Ψ_G.

point count over $\mathbb{F}_p$

$$c_2^{(p)}(G) = \frac{[\Psi_G]_p}{p^2} \mod p$$

$c_2^{(2^2)}(G)$

$$G \longrightarrow \left(c_2^{(2)}(G),\ c_2^{(3)}(G),\ c_2^{(5)}(G),\ \ldots\right)$$

Why?

if all MZV then should be some good mathematical reason

FALSE

so should be Mixed Tate

so $[\Psi]_p$ should be polynomial in p

in that case $\boxed{c_2^{(p)}(G)}$ is the quadratic coeff of the polynomial (lower coeffs must be 0) $\rightarrow$ indep of p

Even still $c_2^{(p)}$ is an interesting measure of what sort of thing should be going on in P_G and its geometry

Same or compatible graph symmetries?

The c_2 invariant either has or is conjectured to have

- completion invariance (conjectured)
- duality invariance (proven in planar case and more, conjectured in general)
  Doryn
- twist invariance (conjectured)
- Fourier split invariance (conjectured)
- 3-join gives 0 (proven)
- Double triangle invariance (proven)

Conjectured: [redacted]

If 2 graphs have same period then have same c_2

Other graph invariants with the symmetries

Iain Crump's permanent invariant, made from stacking signed incidence matrices. E with one row removed (full rank version)

$$E = \begin{array}{c} \text{edges} \\ \begin{bmatrix} +1 \\ -1 \end{bmatrix}^{\text{vertices}} \end{array}$$

$$\mathrm{Perm}\begin{bmatrix} E \\ E \end{bmatrix} \bmod 3$$

Erik Panzer's Hepp invariant made by taking only the dominant term of Ψ in each sector.

$$\int \frac{1}{\Psi^2} = \int \frac{1}{\text{max monomial}(\Psi)^2}$$

Why Hepp?

Conjecturally two graphs have the same Hepp bound **if and only if** they have the same period. ← for 4-regular case (disproved more generally by Schnetz)

Empirically, when suitably scaled it matches the value of the period very well
(from Panzer, arXiv:1908.09820)

← period

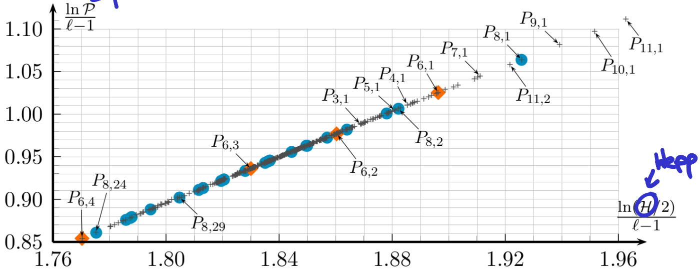


Figure 23: ϕ^4 periods from [90] and their Hepp bounds (products not included). Graphs with 6 and 8 loops are highlighted with orange diamonds and blue circles.

Elliptics and beyond

Back to c_2, what sequences do we see? (Brown and Schnetz)

- constants 0 and -1
- constant except for $p = 2$ or requiring a root of unity
- modular

a few proven

most empirical observation

weight	2	3	4	5	6	7	8
level	$\boxed{11}^{\phi>4}$	$\boxed{7}^{8}_{P}$	$\boxed{5}^{8}$	$\boxed{4}^{9}$	$\boxed{3}^{8}$	$\boxed{3}^{9}$	$\boxed{2}^{10}$
	$\boxed{14}^{\phi>4}$	$\boxed{8}^{8}_{P}$	$\boxed{6}^{9}$	7	$\boxed{4}^{9}$	7	3
	$\boxed{15}^{\phi>4}$	$\boxed{11}^{\phi>4}$	$\boxed{7}^{10}$	8	5	8	$\boxed{5}^{10}$
	17	$\boxed{12}^{9}_{P}$	$\boxed{8}^{11}$	11	6	11	6
	19	15	9	12	$\boxed{7}^{9}_{P}$	15	7
	20	15	$\boxed{10}^{11}$	15	8	15	8
	21	16	12	15	9	16	8
	24	19	$\boxed{13}^{9}_{P}$	19	$\boxed{10}^{10}$	19	9
	26	⋮	⋮	20	10	20	10
	26	$\boxed{24}^{11}$	$\boxed{17}^{10}$	20	10	20	12

TABLE 2. The weight and level of modular graphs for 11 loops and below. All modular forms are newforms. A box indicates that a modular graph of this weight and level was found. The $\phi^{>4}$ superscript indicates that this modular form appears in non-ϕ^4 theory, i.e. it comes from a graph with valence greater than 4. The superscript number indicates which loop order it was first found. The subscript P indicates that a modular graph was found and proved to be modular for all p in [7] or [19].

sticking with vertex degree ≤ 4, get NO elliptic curve.

- unknown, eg (i_{53}) 0,1,1,4,6,6,14,5,15,11,21,23,1,33,33,...

Brown Schnetz modular forms in QFT

A c_2 result

with Simone Hu.

The completion conjecture holds for c_2 when $p = 2$.

That is, for a 4-regular graph K, and v, w two vertices of K

$$c_2^{(2)}(K - v) = c_2^{(2)}(K - w)$$

idea of proof: reduce calculating c_2 to counting $(p=2)$ edge bipartitions of graph with certain properties

p: $p-1$ copies of edges $2(p-1)$-partition

Need to know same parity of counts

Find a fixed point free involution

I did this for some but...

Simone finished it.