Feynman integrals, quasi-periods and black holes

David Broadhurst, Open University, 10 September 2020
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1. I show that **4-loop sunrise integrals** with 5 unit internal masses give Bessel moments that evaluate **periods** and **quasi-periods** of **modular forms** with levels 6, 14 and 34 and weights up to 4.

2. When the external mass is unity, such integrals contribute to the **magnetic moment of the electron** and give the periods and quasi-periods of a modular form of **level 6** and weight 4.

3. The relevant external parameters for **levels 14 and 34** are $\sqrt{-7}$ and $\sqrt{17}-4$, where Feynman integrals determine the **areas** of the event horizons of **black holes** obtained from compactifying a 10-dimensional supergravity theory on a **Calabi-Yau threefold**.

1 Dramatis personæ

The principal characters in this narrative are **Bessel moments** of the form

$$\begin{aligned} M_{m,n}(z) &= \int_0^\infty I_0(xz)[I_0(x)]^m[K_0(x)]^{5-m}x^{2n+1}\mathrm{d}x \\ N_{m,n}(z) &= z\int_0^\infty I_1(xz)[I_0(x)]^m[K_0(x)]^{5-m}x^{2n+2}\mathrm{d}x \end{aligned}$$

with $m \in \{0,1,2\}$, integers $n \geq 0$ and real $z^2 < (5-2m)^2$. The relations

$$\theta_z M_{m,n}(z) = N_{m,n}(z), \quad \theta_z N_{m,n}(z) = z^2 M_{m,n+1}(z),$$

with $\theta_z = z(\mathrm{d}/\mathrm{d}z)$, follow from **differentiation** of Bessel functions.

Also prominent are the **integers** a_n of a sequence beginning with
`1,5,45,545,7885,127905,2241225,41467725,798562125,15855173825,`
for $n = 0\ldots 9$. It is generated by the **fifth** power of a Bessel function:

$$(I_0(x))^5 = \left(\sum_{n\geq 0}\left(\frac{x^n}{2^n n!}\right)^2\right)^5 = \sum_{n\geq 0} a_n \left(\frac{x^n}{2^n n!}\right)^2.$$

The explicit formula

$$a_n = \sum_{i+j+k+l+m=n} \left(\frac{n!}{i!j!k!l!m!} \right)^2$$

shows that a_n enumerates the **number of self-returning walks** of length $2n$ on a 4-dimensional **diamond lattice**. To compute this sequence **recursively** let

$$P_0(t) = t^4,\ \ P_1(t) = 35t^4 + 42t^2 + 3,$$
$$P_2(t) = 259t^4 + 104t^2,\ \ P_3(t) = 225(t^2 - 1)^2.$$

Then for $n > 2$ the recursion is

$$\sum_{k=0}^{3} (-1)^k P_k(2n - k) a_{n-k} = 0.$$

The **fourth** order **Calabi-Yau** differential equation for $y_0(x) = \sum_{n\geq 0} a_n x^n$ is

$$\sum_{k=0}^{3} (-x)^k P_k(2\theta_x + k) y_0(x) = 0$$

with $\theta_x = x(\mathrm{d}/\mathrm{d}x)$. For general x, there are **16 periods**, formed from 4 solutions and their first three derivatives.

The map $x \to 1/z^2$ gives 4th-order differential equations for **Feynman integrals**:

$$\sum_{k=0}^{3}(-z^2)^{1-k}P_k(\theta_z+2-k)M_{m,0}(z) = 5!\delta_{m,0}$$

for $m = 0, 1, 2$, with an **inhomogeneous** constant in the sole case $m = 0$, for the **4-loop sunrise** diagram with 5 uncut internal lines.

The moments $M_{m,n}(z)$ and $N_{m,n}(z)$ with $n > 1$ are determined by **recursion**. For example, one may determine $M_{m,2}(z)$ from

$$\begin{aligned} 16(5-z^2)M_{m,0}(z) + 32(7-z^2)N_{m,0}(z) - 4(285-168z^2+11z^4)M_{m,1}(z) \\ -4(259-70z^2+3z^4)N_{m,1}(z) + D(z)M_{m,2}(z) = 5!\delta_{m,0} \end{aligned}$$

at the seemingly daunting expense of **dividing** by

$$D(z) = (1-z)(1+z)(3-z)(3+z)(5-z)(5+z).$$

A delightful feature of the special point $z = \sqrt{-\mathbf{7}}$ is that

$$D\left(\sqrt{-7}\right) = (1+7)(9+7)(25+7) = 2^{12}.$$

An even more remarkable trick is performed by $u = \sqrt{\mathbf{17}} - 4$, a **unit** of $Q(\sqrt{17})$. The **norm** of a rational function $f(u)$ is $f(u)f(-1/u)$. Each of the 6 factors of $D(u) = (1-u)(1+u)(3-u)(3+u)(5-u)(5+u)$ has a norm of the form $\pm 2^n$. The norm of $(1-u)$ is $5^2 - 17 = 2^3$ and the norm of $D(u)$ is $-2^{25} = -33554432$, which follows from the splendid **quintic** identity

$$(1+u)(3-u)(3+u)(5-u)(5+u) = 2^{11}u.$$

It is hard to imagine better **regular points** at which to evaluate the integrals.

The special values $z = \sqrt{-7}$ and $z = \sqrt{17} - 4$ come from discoveries by Philip **Candelas**, Xenia **de la Ossa**, Mohamed **Elmi** and Duco **van Straten**.

They worked on the **Calabi-Yau** side, oblivious to Feynman integrals.

They identified $x = -1/7$ and $x = 33 + 8\sqrt{17}$ as interesting arguments for the **analytic continuation** of $y_0(x) = \sum_{n\geq 0} a_n x^n$, which converges for $|x| < 1/25$.

Hearing of this on a visit to Oxford in November 2019, I transferred **unsolved** Calabi-Yau problems to the **Feynman** side, where they are eminently **soluble**.

2 The Laporta case with $z = 1$ at level 6

The 4-loop unit-mass uncut **sunrise** diagram in two spacetime dimensions gives

$$\begin{aligned} M_{0,0}(z) &= \int_0^\infty I_0(xz)K_0^5(x)x\mathrm{d}x \\ &= \left(\prod_{n=1}^{4}\int_0^\infty \frac{\mathrm{d}x_n}{2x_n}\right)\frac{1}{\Phi(x_1,x_2,x_3,x_4,1;z^2)}, \\ \Phi(x_1,x_2,x_3,x_4,x_5;z^2) &= \left(\sum_{j=1}^{5}x_j\right)\left(\sum_{k=1}^{5}\frac{1}{x_k}\right) - z^2 \end{aligned}$$

and appears at $z = 1$ in Stefano **Laporta**'s heroic evaluation of 4-loop contributions to the **magnetic moment of the electron**, along with $M_{0,1}(1)$.

For $m = 0, 1$, the integrals $N_{m,n}(1)$ can be eliminated, using

$$\frac{M_{0,0}(1) + 3N_{0,0}(1)}{35} = \frac{\zeta(3)}{16}, \quad \frac{M_{1,0}(1) + 3N_{1,0}(1)}{3} = \frac{\zeta(2)}{4}$$

and recursions. That leaves 4 integrals at $z = 1$, with a **quadratic relation**

$$\det\begin{bmatrix} M_{0,0}(1) & M_{0,1}(1) \\ M_{1,0}(1) & M_{1,1}(1) \end{bmatrix} = \frac{\pi^4}{24^2}.$$

2.1 Modular forms for the Laporta case

Let $\eta(\tau) = q^{1/24}\prod_{k>0}(1-q^k)$ with $q = \exp(2\pi i\tau)$ and τ in the upper half plane. The **periods** are Eichler integrals of $f_{4,6}(\tau) = (\eta_1\eta_2\eta_3\eta_6)^2$ with η_n denoting $\eta(n\tau)$. Let $w(\tau) = 3(\eta_3/\eta_1)^4(\eta_2/\eta_6)^2$. The **quasi-periods** are Eichler integrals of

$$\widehat{f}_{4,6} = \mu f_{4,6}, \quad \mu = \frac{1}{32}\left(w + \frac{3}{w}\right)^4 - \frac{9}{16}\left(w + \frac{3}{w}\right)^2 = -1 + 28q + O(q^2).$$

It takes merely 3 seconds to compute 10,000 digits of the 4 **Eichler lintegrals**

$$\frac{\Omega_s}{(2\pi)^s} = \int_{1/\sqrt{3}}^{\infty} f_{4,6}\left(\frac{1+iy}{2}\right) y^{s-1} dy, \quad \frac{\widehat{\Omega}_s}{(2\pi)^s} = \int_{1/\sqrt{3}}^{\infty} \widehat{f}_{4,6}\left(\frac{1+iy}{2}\right) y^{s-1} dy,$$

with $s = 1, 2$. Then Laporta's **intersection number** is the **determinant** of

$$\mathcal{M} = \frac{24}{\pi^2}\left[\begin{array}{cc} 4M_{0,0}(1) & \frac{36}{5}\left(M_{0,0}(1) + M_{0,1}(1)\right) \\ \frac{5}{3}M_{1,0}(1) & 3\left(M_{1,0}(1) + M_{1,1}(1)\right) \end{array}\right] = 12\left[\begin{array}{cc} -\Omega_2 & \widehat{\Omega}_2 \\ -\Omega_1 & \widehat{\Omega}_1 \end{array}\right]$$

namely $\det\mathcal{M} = 12$. David **Roberts** and I conjectured explicit forms of all such **quadratic relations** for **all loops**. Javier **Fresán**, Claude **Sabbah** and Jeng-Daw **Yu** have proven our formulas up to **20 loops**.

3 The space-like case $z = \sqrt{-7}$ at level 14

A **Frobenius basis** for the Calabi-Yau equation near $x = 0$ is completed by

$$y_1 = y_0 L + f_1,\ y_2 = y_0 L^2 + 2 f_1 L + f_2,\ y_3 = y_0 L^3 + 3 f_1 L^2 + 3 f_2 L + f_3$$

where $L = \log(x)$. The series expansions of $y_0 = \sum_{n \geq 0} a_n x^n$ and

$$f_1 = 8x + O(x^2), \quad f_2 = 2x + O(x^2), \quad f_3 = -12x + O(x^2)$$

have **rational** coefficients that are readily developed by **recursion**.

These series converge for $|x| < 1/25$, with the asymptotic expansion

$$\frac{(4\pi)^2 a_n}{25^{n+1}\sqrt{5}} = \frac{1}{n^2} - \frac{1}{2n^3} + \frac{7}{16n^4} + \frac{15}{32n^5} + \frac{411}{256n^6} + \frac{443}{64n^7} + O\left(\frac{1}{n^8}\right)$$

revealing a dilogarithmic **singularity** of $y_0(x)$ at $x = 1/25$.

To reach $x = -1/7$, **outside** this domain of convergence, Candelas et al. resorted to **Runge-Kutta integration** of the fourth-order differential equation.

In **quantum field theory** we have no such problem. The **Feynman integrals** $M_{m,n}(z)$ and $N_{m,n}(z)$ are readily computable at the **space-like** point $z = \sqrt{-7}$.

3.1 Modular forms for the space-like case

Within a day of hearing of interest in $x = -1/7$, I found that

$$f_{4,14}(\tau) = \frac{(\eta_2\eta_7)^6}{(\eta_1\eta_{14})^2} - 4(\eta_1\eta_2\eta_7\eta_{14})^2 + \frac{(\eta_1\eta_{14})^6}{(\eta_2\eta_7)^2}$$

is the relevant modular form of weight 4 and **level 14**, for $z = \sqrt{-7}$. Its **periods** are **critical values** of the L-function $L(f_{4,14}, s) = ((2\pi)^s/\Gamma(s)) \int_0^\infty f_{4,14}(\mathrm{i}y)y^{s-1}\mathrm{d}y$, with

$$\begin{aligned} L(f_{4,14},3) &= M_{1,0}(\sqrt{-7}) = \int_0^\infty J_0(\sqrt{7}x)I_0(x)K_0^4(x)x\mathrm{d}x = \frac{\pi^2}{7}L(f_{4,14},1) \\ \tfrac{1}{2}L(f_{4,14},2) &= M_{2,0}(\sqrt{-7}) = \int_0^\infty J_0(\sqrt{7}x)I_0^2(x)K_0^3(x)x\mathrm{d}x. \end{aligned}$$

There is also a modular form of **weight 2** to consider, $f_{2,14}(\tau) = \eta_1\eta_2\eta_7\eta_{14}$. This provides a modular parametrization of a **quartic elliptic curve**, namely

$$\begin{aligned} y^2 &= (1+x)(1+8x)(1+5x+8x^2), \\ x &= \left(\frac{\eta_2\eta_{14}}{\eta_1\eta_7}\right)^3 = q + 3q^2 + 6q^3 + 13q^4 + O(q^5), \\ y &= \frac{q}{f_{2,14}}\frac{\mathrm{d}x}{\mathrm{d}q} = 1 + 7q + 27q^2 + 92q^3 + 259q^4 + O(q^5). \end{aligned}$$

The **periods** of $f_{2,14}$ are computable at **lightning speed** by the process of the **arithmetic-geometric mean**, yielding an **L-value** and a **j-invariant**:

$$L(f_{2,14},1)=\frac{\omega_+}{3},\quad j\left(\frac{\omega_+ + \mathrm{i}\omega_-}{2\omega_+}\right)=\left(\frac{5\times 43}{28}\right)^3,$$

$$\omega_\pm=\frac{2\pi}{\mathrm{agm}\left(\sqrt{2^{9/2}\pm 13},\,2^{11/4}\right)}.$$

These **elliptic periods** are also determined by **Feynman integrals**:

$$\begin{aligned}\frac{\omega_+}{2}&=3M_{2,0}(\sqrt{-7})+4N_{2,0}(\sqrt{-7}),\\ \frac{\pi\omega_-}{2}&=3M_{1,0}(\sqrt{-7})+4N_{1,0}(\sqrt{-7}).\end{aligned}$$

3.2 Erik Panzer's big question

When I arrived at this stage in December 2019, Erik shrewdly asked me, in Padova: "Can you really derive **all 16 Calabi-Yau periods** from Feynman integrals?"

I answer in the **affirmative**. In fact, **6 Feynman integrals** suffice at $z=\sqrt{-7}$.

3.3 Quasi-periods at level 14

The **16 Calabi-Yau periods** are analytic continuations of $\theta_x^j y_k(x)$, with $j = 0, 1, 2, 3$ and $k = 0, 1, 2, 3$, to $x = -1/7$. They are linear combinations of **8 Feynman integrals** $M_{m,n}(\sqrt{-7})$ and $N_{m,n}(\sqrt{-7})$, with $m = 1, 2$ and $n = 0, 1$.

The quasi-periods at **weight 2** are $\widehat{\omega}_\pm$, with

$$\begin{aligned} \frac{3\widehat{\omega}_+}{16} &= 7M_{2,0}(\sqrt{-7}) + 8N_{2,0}(\sqrt{-7}) + 28M_{2,1}(\sqrt{-7}), \\ \frac{3\pi\widehat{\omega}_-}{16} &= 7M_{1,0}(\sqrt{-7}) + 8N_{1,0}(\sqrt{-7}) + 28M_{1,1}(\sqrt{-7}). \end{aligned}$$

Suppressing the argument $z = \sqrt{-7}$, I obtain the quadratic relation

$$\det \begin{bmatrix} 3M_{2,0} + 4N_{2,0} & M_{2,0} + 28M_{2,1} \\ 3M_{1,0} + 4N_{1,0} & M_{1,0} + 28M_{1,1} \end{bmatrix} = -\frac{3\pi^2}{32}$$

from **Legendre's relation** for complete elliptic integrals. At **weight 4** I found

$$\det \begin{bmatrix} M_{2,0} & 39N_{2,0} - 427M_{2,1} - 112N_{2,1} \\ M_{1,0} & 39N_{1,0} - 427M_{1,1} - 112N_{1,1} \end{bmatrix} = \frac{3\pi^2}{32}$$

as the **quadratic relation** between the periods and quasi-periods of $f_{4,14}$.

Hence only **6 Feynman integrals** are algebraically independent.

4 The time-like case $z = \sqrt{17} - 4$ at level 34

The **8 Feynman integrals** at $z = u = \sqrt{17} - 4$ satisfy **2 quadratic relations**.

Let $\chi(n)$ be the **Dirichlet character** defined for prime p by $\chi(17) = 0$ and otherwise by $\chi(p) = \pm 1$ according as whether 17 is or is not a square modulo p.

There are **12 cusp forms** of **level 34 and weight 4** with this character. Feynman integrals choose a pair of **newforms** whose **Fourier coefficients**, $A_4(n)$ and $\overline{A}_4(n)$, are **Gaussian integers**, related by **complex conjugation**.

Let $L_4(s)$ be the analytic continuation of

$$L_4(s) = \sum_{n>0} \frac{A_4(n)}{n^s} = \frac{1}{1 + 2^{1-s}} \prod_{p>2} \frac{1}{1 - A_4(p)p^{-s} + \chi(p)p^{3-2s}}$$

with the choice of sign $A_4(3) = 2\mathrm{i}$. For prime p, $A_4(p)$ is **real** if $\chi(p) = +1$ and **imaginary** if $\chi(p) = -1$, while $A_4(17)/17 = 1 - 4\mathrm{i}$ is truly **complex**.

Feyman integrals determine the **critical L-values** at **weight 4**:

$$\begin{aligned} L_4(3) &= \left(\frac{13-u+(1+13u)\mathrm{i}}{17}\right) M_{1,0}(u), \\ L_4(2) &= 4\left(\frac{5-3u+(3+5u)\mathrm{i}}{17}\right) M_{2,0}(u), \\ L_4(1) &= \left(\frac{7-11u+(11+7u)\mathrm{i}}{\pi^2}\right) M_{1,0}(u). \end{aligned}$$

At **weight 2** they determine the **periods** and **quasi-periods** of the elliptic curve

$$y^2 = \left(x+\frac{5-u}{8}\right)\left(x+\frac{5+u}{8}\right)\left(x+\frac{3+u}{2}\right)$$

whose real and imaginary periods are

$$\omega_1 = \frac{4\pi}{\mathrm{agm}\left(\sqrt{4u},\sqrt{14+10u}\right)}, \quad \omega_2 = \frac{-4\pi\mathrm{i}}{\mathrm{agm}\left(\sqrt{14+6u},\sqrt{14+10u}\right)}.$$

The elliptic **periods** $\omega_{1,2}$ and **quasi-periods** $\widehat{\omega}_{1,2}$ are determined by

$$\begin{aligned}
\frac{\omega_1}{4} &= \mathcal{P}_2 = (2+3u)M_{2,0}(u) + 4(4+u)N_{2,0}(u)\\
\frac{\pi\mathrm{i}\omega_2}{4} &= \mathcal{P}_1 = (2+3u)M_{1,0}(u) + 4(4+u)N_{1,0}(u)\\
\frac{3\widehat{\omega}_1}{8(1+u)} &= \widehat{\mathcal{P}}_2 = M_{2,0}(u) + 2(5+u)N_{2,0}(u) + 2u(3+u)(4+u)M_{2,1}(u)\\
\frac{3\pi\mathrm{i}\widehat{\omega}_2}{8(1+u)} &= \widehat{\mathcal{P}}_1 = M_{1,0}(u) + 2(5+u)N_{1,0}(u) + 2u(3+u)(4+u)M_{1,1}(u)
\end{aligned}$$

with **Legendre's condition** giving $\mathcal{P}_1\widehat{\mathcal{P}}_2 - \mathcal{P}_2\widehat{\mathcal{P}}_1 = 3(\pi/4)^2/(1+u)$.

At **weight 4**, the **periods** $\mathcal{H}_m = M_{m,0}(u)$ and **quasi-periods**

$$\begin{aligned}
\widehat{\mathcal{H}}_m &= 81M_{m,0}(u) + 3(2+u)(u-6)N_{m,0}(u)\\
&+ u^2(2+u)(4+u)(96+11u)M_{m,1}(u) + 136(1-u)N_{m,1}(u)
\end{aligned}$$

yield the **intersection number** $\mathcal{H}_1\widehat{\mathcal{H}}_2 - \mathcal{H}_2\widehat{\mathcal{H}}_1 = 3(\pi/8)^2/u$.

5 Black holes and identification of constants

Feynman integrals determine the **area** of the event horizon of a **black hole** with charges specified by (k, ℓ) studied by Candelas at al. namely

$$A = \mathbf{34}\pi\left(\frac{k^2}{v} + \ell^2 v\right), \quad v = 4\pi\frac{M_{2,0}(u)}{M_{1,0}(u)} = 4\pi\frac{\int_0^\infty I_0(ux)I_0^2(x)K_0^3(x)x\mathrm{d}x}{\int_0^\infty I_0(ux)I_0(x)K_0^4(x)x\mathrm{d}x}$$

where $u = \sqrt{\mathbf{17}} - 4$ is the **external mass** in the **4-loop sunrise diagram**, with two internal propagators cut in the numerator of v and one in the denominator. Similarly, $M_{2,0}(\sqrt{-7})/M_{1,0}(\sqrt{-7})$ determines the area for their **level 14** problem.

Candelas et al. **lacked identification** of 6 constants found approximately by numerical integration of the Calabi-Yau equation from $x = 0$ to the points $x = -1/7$ and $x = 33 \pm 8\sqrt{17}$. Following a lead from Francis **Brown** and Dick **Hain**, I identified these as originating from 4 **permanents** of matrices of **Feynman integrals** whose determinants gives the **intersection numbers** at levels 14 and 34. In this I was greatly aided by my work with Kevin **Acres** on **Rademacher sums**, whose details I have omitted here for the sake of brevity.

6 Summary

1. Candelas, de la Ossa, Elmi and van Straten made the fine discovery that a Calabi-Yau equation giving periods of a level 6 modular form at a singular point also gives periods of modular forms of levels 14 and 34 at regular points.

2. Laporta and I had fully mastered the level 6 case, on the Feynman side. I was thus in the happy position of being able to bring expertise from quantum field theory to bear on unsolved Calabi-Yau problems at levels 14 and 34.

3. The 48 constants that determine the expansions of the 4 Calabi-Yau solutions in the 3 neighourhoods of the modular points $x = -1/7$ and $x = 33 \pm 8\sqrt{17}$ are determined by 16 Feynman integrals satisfying 4 quadratic relations.

4. I computed 10,000 good digits of each of the 16 Feynman integrals with ease, exploiting the intersection numbers, modularity and the pleasing inequalities

$$\exp(-\pi/\sqrt{7}) < 0.306, \quad \exp(-2\pi/\sqrt{34}) < 0.341, \quad 25(33 - 8\sqrt{17}) < 0.379.$$

5. Two ratios of these Feynman integrals determine areas of black hole horizons.