UNIVERSITY OF OXFORD
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Motivic coaction and cohomology with twisted coefficients

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Oxford Mathematics

Feynman integrals in parametric representation are integrals of the form:

$$I_G = \int_{\sigma_G} \frac{1}{\Psi_G^{d/2}} \left(\frac{\Psi_G}{\Xi_G}\right)^{N_G - h_G d/2} \Omega_G$$

where:

- $G = (V_G, E_G, E_G^{ext})$ is a Feynman graph.
- d is the number of space-time dimensions, $N_G = |E_G|$ and h_G is the number of loops of G.
- To each edge E_G associate a variable α_e, and define
- $\Psi_G = \sum_{T \subset G} \prod_{e \notin T} \alpha_e$ 1st Symanzik and
- $\Xi_G = \sum_{T_1 \cup T_2 \subset G} (q^{T_1})^2 \prod_{e \notin T_1 \cup T_2} \alpha_e + \left(\sum_{e \in E_G} m_e^2 \alpha_e \right) \Psi_G$ 2nd Symanzik.
- $\Omega_G = \sum_{i=1}^{N_G} (-1)^i \alpha_i d\alpha_1 \wedge \ldots \wedge \widehat{d\alpha_i} \wedge \ldots \wedge d\alpha_{N_G}$.
- Domain of integration $\sigma = \{[\alpha_1 : \ldots : \alpha_{N_G}] : \alpha_i \geq 0\} \subset \mathbb{P}^{N_G - 1}(\mathbb{R})$.
- Only consider convergent integrals (for now).

We can set up a motivic Galois theory of families of periods by considering the category $\mathcal{H}(S)$ of triples $(\mathbb{V}_{\mathrm{B}}, (\mathcal{V}_{\mathrm{dR}}, \nabla), c)$ such that:

- $\mathbb{V}_{\mathrm{B}}$ is a local system of finite dimensional $\mathbb{Q}$-vector spaces on $S(\mathbb{C})$.
- $(\mathcal{V}_{\mathrm{dR}}, \nabla)$ is an algebraic vector bundle on S with a flat connection with regular singularities at infinity.
- $c : (\mathcal{V}_{\mathrm{dR}}, \nabla)^{an} \xrightarrow{\sim} \mathbb{V}_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathcal{O}_{S^{an}}$ is an isomorphism of analytic vector bundles with connection.
- $\mathbb{V}_{\mathrm{B}}$ underlies a variation of mixed Hodge structures (Hodge and weight filtrations).

$\mathcal{H}(S)$ is a Tannakian category with fiber functors:

$$\begin{aligned} \omega_{\mathrm{dR}}^{gen} :& \mathcal{H}(S) \to \mathrm{Vec}_{k_S} \\ & (\mathbb{V}_{\mathrm{B}}, (\mathcal{V}_{\mathrm{dR}}, \nabla), c) \mapsto \Gamma(\mathrm{Spec}(k_S), \mathcal{V}_{\mathrm{dR}}) \end{aligned} \tag{1}$$

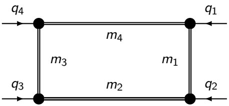
$$\begin{aligned} \omega_{B}^{Z} :& \mathcal{H}(S) \to \mathrm{Vec}_{\mathbb{Q}} \\ & (\mathbb{V}_{\mathrm{B}}, (\mathcal{V}_{\mathrm{dR}}, \nabla), c) \mapsto \Gamma(Z, \mathbb{V}_{\mathrm{B}}), \\ & \text{where } Z \subset S(\mathbb{C}) \text{ simply connected.} \end{aligned} \tag{2}$$

We can now define:

- The ring of motivic periods $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}(S)} = \mathcal{O}(\mathrm{Isom}^{\otimes}_{\mathcal{H}(S)}(\omega^{gen}_{\mathrm{dR}}, \omega^{Z}_{B}))$.
- Elements of $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}(S)}$ are triples $[V, \sigma, \omega]$, where $V \in \mathrm{Ob}(\mathcal{H}(S))$, σ is a section of $\mathbb{V}_{\mathrm{B}}$ (cycle of integration) and ω is a section of $\mathcal{V}_{\mathrm{dR}}$ (differential form).
- There is a *period morphism* $\mathrm{per} : \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}(S)} \to M_Z(S(\mathbb{C}))$ – multivalued meromorphic functions on $S(\mathbb{C})$ with a prescribed branch along Z.
- The (de Rham) motivic Galois group $G^{\mathfrak{dr},gen}_{\mathcal{H}(S)} = \mathrm{Aut}^{\otimes}_{\mathcal{H}(S)}(\omega^{gen}_{\mathrm{dR}})$.
- We have the Galois action $G^{\mathfrak{dr},gen}_{\mathcal{H}(S)} \times \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}(S)} \to \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}(S)}$, or dually
- Coaction

$$\Delta : \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}(S)} \to \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}(S)} \otimes_{k_S} \mathcal{O}(G^{\mathfrak{dr},gen}_{\mathcal{H}(S)})$$

- Brown (2015) associates to any Feynman graph with generic kinematics a *graph motive* $mot_G \in \mathrm{ob}(\mathcal{H}(S))$, and an element $I^{\mathfrak{m}}_G \in \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}(S)}$, such that $\mathrm{per}(I^{\mathfrak{m}}_G) = I_G$.
- $I^{\mathfrak{m}}_G$ is the motivic lift of the Feynman integral that Brown calls the motivic Feynman amplitude.

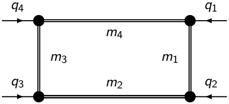
With this in mind, consider  where m_i, q_i generic, $d = 4$.

Theorem (T.)

Let G be as in the figure and $I_G^{\mathfrak{m}}$ its motivic Feynman amplitude. Then:

$$\Delta I_G^{\mathfrak{m}} = \sum_{1 \le j < k \le 4} I^{\mathfrak{m}, d=2}_{G/\{e_j, e_k\}} \otimes P_{j,k} \log^{\mathfrak{dr}}(f_{j,k}) \mathbb{L}^{\mathfrak{dr}} + \\ + I_G^{\mathfrak{m}} \otimes \left(\mathbb{L}^{\mathfrak{dr}}\right)^2 + 1 \otimes I_G^{\mathfrak{dr}}. \tag{3}$$

where $f_{j,k} = \frac{\sqrt{(U)_{j,k}^2 - (U)_{j,j}(U)_{k,k}} - (U)_{j,k}}{\sqrt{(U)_{j,k}^2 - (U)_{j,j}(U)_{k,k}} + (U)_{j,k}}$, *and $C, D_{j,k}$ are the matrices of the quadratic forms Ξ_G, $\Xi_{G/\{e_j, e_k\}}$ respectively, $U = C^{-1}$, and* $P_{j,k} = \frac{\sqrt{|\det D_{j,k}|}}{8\sqrt{|\det C|}}$.

With this in mind, consider  where m_i, q_i generic,

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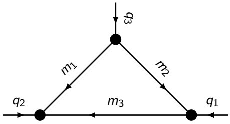
- Galois conjugates given by motivic periods associated to sub-quotient graphs!

- We can then reasonably ask when a statement like this for the Galois conjugates holds true?
- Is it always true that the conjugates of a Feynman integral are motivic periods associated to motives of sub-quotient graphs, with weight of the conjugates bounded by the number of edges of the sub-quotient graphs?
- It relates the integral to the integrals associated to sub-quotient graphs, thereby providing information and constraints on the original integral.

Consider 

where $m_i = 0$ and q_i generic, $d = 4$.

Theorem (T.)

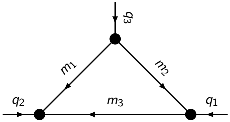
Let G be the Feynman graph in the figure. Then:

$$\Delta I_G^{\mathfrak{m}} = \left(a_1 \log^{\mathfrak{m}}\left(\frac{q_2^2}{q_3^2}\right) + a_2 \log^{\mathfrak{m}}\left(\frac{q_1^2}{q_3^2}\right)\right) \otimes \left(\log^{\mathfrak{dr}}(f_1) + \log^{\mathfrak{dr}}(f_2)\right)\mathbb{L}^{\mathfrak{dr}} +$$

$$+ I_G^{\mathfrak{m}} \otimes \left(\mathbb{L}^{\mathfrak{dr}}\right)^2 + 1 \otimes I_G^{\mathfrak{dr}}, \text{ where}$$

$$f_1 = \frac{(q_1^2+q_2^2-q_3^2+\sqrt{q_1^4+q_2^4+q_3^4-2q_1^2q_3^2-2q_2^2q_3^2})^2}{4q_1^2q_2^2}, \; f_2 = f_1 \frac{q_1^2+q_3^2-q_2^2-\sqrt{q_1^4+q_2^4+q_3^4-2q_1^2q_3^2-2q_2^2q_3^2}}{q_1^2+q_3^2-q_2^2+\sqrt{q_1^4+q_2^4+q_3^4-2q_1^2q_3^2-2q_2^2q_3^2}}$$

and $a_i \in k_S$.

Consider 

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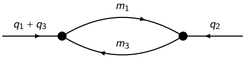
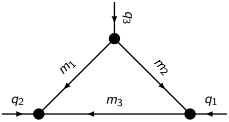
and $a_i \in k_S$.

- Not motivic periods associated to quotient graphs.
- Quotient graphs, e.g. 

, do not converge when $m_i = 0$.

Dimensional regularization

Set $4-2\epsilon$ dimensions:

$G =$ 

$\rightsquigarrow \quad I_G := \int_{\sigma_G} \left(\frac{\Psi_G^2}{\Xi_G}\right)^{\epsilon} \frac{\Omega_G}{\Psi_G^2}.$

Expanding in ϵ we get:

$$(I_G)_L = \sum_{k\geq 0} \epsilon^k \int_{\sigma_G} \frac{1}{k!} \log^k \left(\frac{\Psi_G^2}{\Xi_G}\right) \frac{\Omega_G}{\Psi_G^2}$$

Applying the coaction *term by term* in ϵ, apriori get:

$$\Delta((I_G)_L^{\mathfrak{m}}) \in (\mathcal{P}^{\mathfrak{m}} \otimes \mathcal{O}(G_{\mathcal{H}(S)}^{\mathfrak{dr},gen}))[[\epsilon]]$$

However, claim:

$$\Delta((I_G)_L^{\mathfrak{m}}) \in \mathcal{P}^{\mathfrak{m}}[[\epsilon]] \otimes \mathcal{O}(G_{\mathcal{H}(S)}^{\mathfrak{dr},gen})[[\epsilon]]$$

and even more sharply (Abreu, Britto, Duhr and Gardi):

$$\Delta((I_G)_L^{\mathfrak{m}}) = \sum_{i=1}^{3} (I_{G/e_i})_L^{\mathfrak{m}} \otimes (b_i)_L^{\mathfrak{dr}} + (I_G)_L^{\mathfrak{m}} \otimes (b_0)_L^{\mathfrak{dr}}$$

A first approximation: fix $\epsilon \notin \frac{1}{2}\mathbb{Z}$

- Want to realize the integral $I_G := \int_{\sigma_G} \left(\frac{\Psi_G^2}{\Xi_G}\right)^{\epsilon} \frac{\Omega_G}{\Psi_G^2}$ as a 'period'.
- Let $Q : \Xi_G = 0$, and $L : \Psi_G = 0$ be defined over $k \subset \mathbb{C}$ (fix kinematics)
- Consider the vector bundle $\mathcal{O}_X$ on $X = \mathbb{P}^2 \setminus Q \cup L$ with connection:

$$\nabla_\epsilon : d + \epsilon d \log\left(\frac{\Psi_G^2}{\Xi_G}\right)$$

- Let $\mathcal{L}_\epsilon = \mathbb{Q}(e^{2\pi i \epsilon})\left(\frac{\Psi_G^2}{\Xi_G}\right)^{-\epsilon}$. Then $\mathcal{L}_\epsilon \otimes_{\mathbb{Q}(e^{2\pi i\epsilon})} \mathbb{C} \cong (\mathcal{O}_X^{an})^{\nabla_\epsilon}$.
- There is a comparison isomorphism:

$$\mathrm{comp}_{\mathrm{B,dR}}(\epsilon) : H^2_{\mathrm{dR}}(X, D; \nabla_\epsilon) \otimes_{k(\epsilon)} \mathbb{C} \cong H^2_{\mathrm{B}}(X, D; \mathcal{L}_\epsilon) \otimes_{\mathbb{Q}(\epsilon)} \mathbb{C}$$

  where $D = \bigcup_{i=1}^3 D_i \setminus D_i \cap (X \cup L)$ and $D_i : \alpha_i = 0$.
- $[\frac{\Omega_G}{\Psi_G^2}] \in H^2_{\mathrm{dR}}(X, D; \nabla_\epsilon)$ and $\left[\sigma_G \otimes \left(\frac{\Psi_G^2}{\Xi_G}\right)^{\epsilon}\right] \in H_2^{lf}(X, D; \mathcal{L}_\epsilon^\vee)$
- Therefore the integral I_G is a coefficient of $\mathrm{comp}_{\mathrm{B,dR}}(\epsilon)$.

The 'motive' $H^2(X, D; \mathcal{L}_\epsilon)$

Claim: it sits in a short exact sequence

$$0 \to \bigoplus_{i=1}^{3} H^1(D_i \smallsetminus D_i \cap (Q \cup L); \mathcal{L}_\epsilon|_{D_i}) \to H^2(X, D; \mathcal{L}_\epsilon) \to H^2(\mathbb{P}^2 \smallsetminus Q \cup L; \mathcal{L}_\epsilon) \longrightarrow 0$$

Lemma

The cohomology of $X = \mathbb{P}^2 \smallsetminus Q \cup L$ with coefficients in $\mathcal{L}_\epsilon$ is concentrated in cohomological degree 2 and has rank 1.

- The sequence is the long exact sequence in relative cohomology, where...
- the lemma implies the zero on the left, i.e. $H^1(X; \mathcal{L}_\epsilon) = 0$, and
- the zero on the right, i.e. $H^2(D_i \smallsetminus D_i \cap (Q \cup L); \mathcal{L}_\epsilon|_{D_i}) = 0$, follows from Artin vanishing beacuse the space $D_i \smallsetminus D_i \cap (Q \cup L)$ is affine.

Lemma

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Sketch:

- Let $j : X \hookrightarrow \mathbb{P}^2$ denote the natural open immersion.
- If $\epsilon \notin \frac{1}{2}\mathbb{Z}$ we have $j_! \mathcal{L}_\epsilon \xrightarrow{\sim} Rj_* \mathcal{L}_\epsilon$ in $D^b(\mathbb{P}^2)$. (True whenever monodromy of the local system doesn't have 1 as an eigenvalue.)
- Since $H_c^\bullet(\mathbb{P}^2, j_! \mathcal{L}_\epsilon) \cong H_c^\bullet(X, \mathcal{L}_\epsilon)$, and $H^\bullet(\mathbb{P}^2, Rj_* \mathcal{L}_\epsilon) \cong H^\bullet(X, \mathcal{L}_\epsilon)$, and $\mathbb{P}^2$ is compact, we get $H^\bullet(X, \mathcal{L}_\epsilon) \cong H_c^\bullet(X, \mathcal{L}_\epsilon)$.
- We have Poincare duality $H_c^k(X, \mathcal{L}_\epsilon)^\vee \cong H^{2n-k}(X, \mathcal{L}_\epsilon^\vee)$, and Artin vanishing: X is affine $\implies$ $H^k(X, \mathcal{L}_\epsilon) = 0$ for $k > 2$, which implies the first statement.
- Also, we have generally that $\chi(X, \mathcal{L}_\epsilon) = \mathrm{rk}(\mathcal{L}_\epsilon)\chi(X)$ $= 1$ in this case.
- From point 2 it also follows that $H_\bullet(X, \mathcal{L}_\epsilon) \cong H_\bullet^{lf}(X, \mathcal{L}_\epsilon)$, intersection pairings etc.

The 'motive' $H^2(X, D; \mathcal{L}_\epsilon)$

$$0 \to \bigoplus_{i=1}^{3} H^1(D_i \smallsetminus D_i \cap (Q \cup L); \mathcal{L}_\epsilon) \to H^2(X, D; \mathcal{L}_\epsilon) \to H^2(\mathbb{P}^2 \smallsetminus Q \cup L; \mathcal{L}_\epsilon) \longrightarrow 0\,.$$

- The boundary motives are

$$H^1(D_i \smallsetminus D_i \cap (Q \cup L); \mathcal{L}_\epsilon|_{D_i}) \cong H^1(\mathbb{P}^1 \smallsetminus \{0, -1, \infty\}; \mathcal{L}_\epsilon|_{D_i})$$

- This is a 'beta motive', up to a twist. One can check, by integrating directly, that the associated 'period' is $(q_i^2)^{-\epsilon}\beta(-\epsilon, -\epsilon)$.
- See Brown-Dupont (2019) for generalized Beta integrals in this setting.

On the right hand side of the s.e.s. we have

Lemma

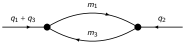
$H^2(\mathbb{P}^2 \smallsetminus Q \cup L; \mathcal{L}_\epsilon)$ is a c^ϵ twist, for a constant c, of a "pure Artin–Tate motive of weight 2" i.e., is isomorphic to $\mathbb{Q}(-1)c^\epsilon$ after a finite extension of the field of coefficients.

- The period matrix of the motive $H^2(X, D; \mathcal{L}_\epsilon)$ is then of the form:

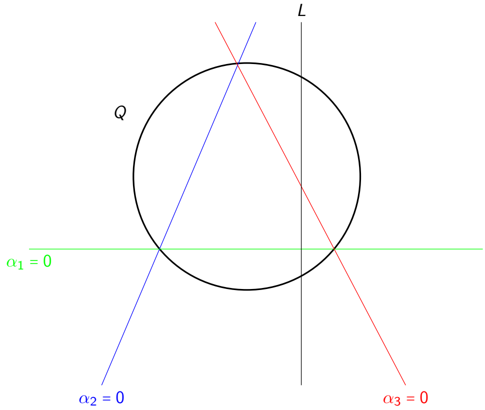
$$\begin{bmatrix} (q_1^2)^{-\epsilon}\beta(-\epsilon,-\epsilon) & 0 & 0 & I_1 \\ 0 & (q_2^2)^{-\epsilon}\beta(-\epsilon,-\epsilon) & 0 & I_2 \\ 0 & 0 & (q_3^2)^{-\epsilon}\beta(-\epsilon,-\epsilon) & I_3 \\ 0 & 0 & 0 & c^\epsilon 2\pi i \end{bmatrix}$$

- The Feynman integral I_G is a linear combination of I_1, I_2, I_3.
- We didn't define a Galois group and action in this setting, but if we had (see Brown-Dupont), the conjugates of I_G would be precisely $(q_i^2)^{-\epsilon}\beta(-\epsilon,-\epsilon)$, and the trivial term corresponding to $c^\epsilon 2\pi i$.
- The first few terms of the expansion in ϵ of this are

$$(1 - \epsilon\log(q_i^2) + \ldots)(-\frac{2}{\epsilon} - \epsilon^2\zeta(2) + \ldots) = -\frac{2}{\epsilon} + 2\log(q_i^2) + \ldots$$

- This was observed by Abreu, Britto, Duhr and Gardi, to correspond to the first few terms in expansion of the bubble integral 

with trivial internal masses in dimensional regularization.


L
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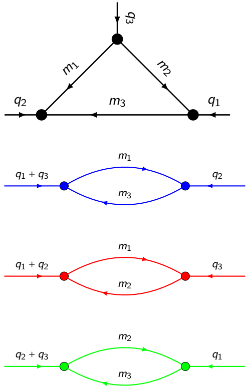
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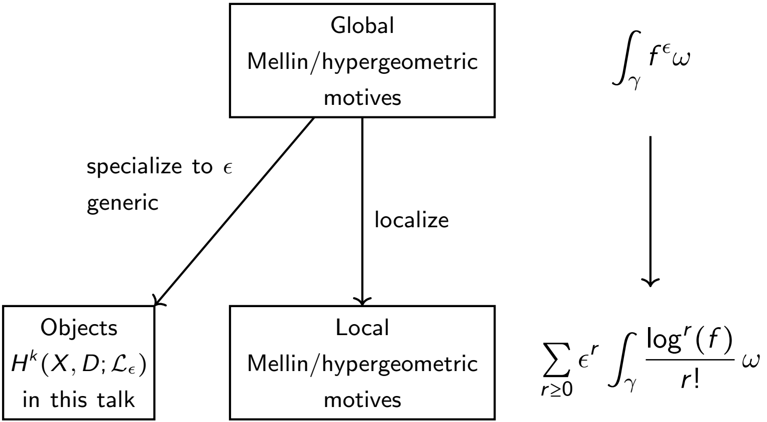

- Global and local parts of the picture should correspond to Tannakian categories, each with its own 'Galois group', with a functor 'localize' enabling us to compare the coaction before and after expanding in ϵ.
- Topic of upcoming joint work with Brown, Dupont, Fresán.
- No reason for it to be limited to 1-loop graphs.

Another example

- $I(a;s) = \frac{1}{s}\left({}_2F_1(s,1;1+s;a) - 1\right) = \int_0^1 x^s \frac{a\,dx}{1-ax} = \sum_{n\geq 0}(-s)^n \operatorname{Li}_{n+1}(a)$, where $a \in \mathbb{C} \setminus \{0,1\}$
- Period of $M = H^1(\mathbb{A}^1 \setminus \{a^{-1}\}, \{0,1\}; \mathcal{L}_s)$, where $\mathcal{L}_s$ corresponds to the vector bundle $(\mathcal{O}_X, \nabla = d + s\frac{dx}{x}))$ on $X = \mathbb{A}^1 \setminus \{a^{-1}\}$.
- Sits in a short exact sequence:

$$0 \to H^1(\mathbb{A}^1, \{0,1\}; \mathcal{L}_s) \to M \to H^0(\{a^{-1}\}; (\mathcal{L}_s)_{a^{-1}})(-1) \to 0$$

- The period matrix of M is

$$\begin{pmatrix} 1 & I(a;s) \\ 0 & 2\pi i\, a^{-s} \end{pmatrix} .$$

Another example

- Let us lift the local expansion to periods term by term, and define

$$I_{\mathrm{L}}^{\mathfrak{m}/\mathfrak{dr}}(a;s) = \sum_{n\geq 0}(-s)^n \mathrm{Li}_{n+1}^{\mathfrak{m}/\mathfrak{dr}}(a) \ ,$$

- If we coact on s by

$$\Delta(s) = s(1 \otimes (\mathbb{L}^{\mathfrak{dr}})^{-1})$$

where $\mathbb{L}^{\mathfrak{dr}} \in \mathcal{O}(G_{\mathcal{H}}^{\mathfrak{dr}})$, we get

$$\Delta(I_{\mathrm{L}}^{\mathfrak{m}}(a;s)) = \sum_{n\geq 0}(-s)^n \Big(1\otimes(\mathbb{L}^{\mathfrak{dr}})^{-n}\mathrm{Li}_{n+1}^{\mathfrak{dr}} + \sum_{k=0}^{n} \mathrm{Li}_{n+1-k}^{\mathfrak{m}}(a) \otimes \frac{(\log^{\mathfrak{dr}}(a))^k}{k!}(\mathbb{L}^{\mathfrak{dr}})^{1-k}\Big) \ ,$$

- which can be rewritten as:

$$\Delta(I_{\mathrm{L}}^{\mathfrak{m}}(a;s)) = 1 \otimes I_{\mathrm{L}}^{\mathfrak{dr}}(a; s(\mathbb{L}^{\mathfrak{dr}})^{-1}) + I_{\mathrm{L}}^{\mathfrak{m}}(a;s) \otimes \mathbb{L}^{\mathfrak{dr}} \left(a^{-s(\mathbb{L}^{\mathfrak{dr}})^{-1}}\right)_{\mathrm{L}}^{\mathfrak{dr}} \ ,$$

where we have defined

$$(a^s)_{\mathrm{L}}^{\mathfrak{dr}} = \sum_{n\geq 0} \frac{(\log^{\mathfrak{dr}}(a))^n}{n!} s^n \ .$$