Geometry of Feynman Integrals in Twistor Space

Based on `arXiv:2005.08771` in collaboration with Cristian Vergu.

Building on work with Jacob Bourjaily, Andrew McLeod, Matt von Hippel, Matthias Wilhelm.

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Context

Feynman integrals for scattering amplitudes (e.g. $\mathcal{N} = 4$ SYM)

- "Simple" amplitudes: iterated integrals and multiple polylogarithms

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t)$$

- Algorithmic obstructions (identities, integration, …), but the functions are relatively well-understood.
- At higher loops we cannot expect such simple integrals.
- More complicated integrals involve interesting geometry (elliptic curves, K3 surfaces, Calabi-Yau manifolds).

Questions to address

- How to attach a geometry to a Feynman integral?
  (Parametric representations, momentum twistor space, ...)
- What are the properties this geometry?
  (Invariants, moduli space, ...)
- Where to go next?

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Introduction

Two ways to find a geometry

Direct integration using Feynman parameters:

$$\int \mathrm{d}^4\ell_1 \cdots \mathrm{d}^4\ell_L \times \textbf{Rational integrand} \xrightarrow{\text{integrations}} \int_\gamma \omega \times \textbf{Polylogs}$$

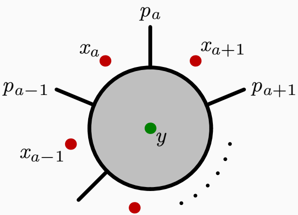
Here γ is some integration contour and ω is a holomorphic form on some variety.

Example: $\omega = \frac{\mathrm{d}x}{\sqrt{4x^3 - g_2 x - g_3}} \to$ elliptic curve $y^2 = 4x^3 - g_2 x - g_3$.

Taking residues:

- Take residues around the poles of the propagators.
- Jacobians may allow for more residues than propagators
- Residue of highest codimension: leading singularity
- Constant means the integral is polylogarithmic

Momentum twistors



Dual coordinates: $p_a = x_{a+1} - x_a$

Momentum twistors: [Hodges (2009)]

$$Z = \begin{pmatrix} \lambda^\alpha \\ x^{\alpha\dot\alpha}\tilde\lambda_{\dot\alpha} \end{pmatrix} \in \mathbb{P}^3$$

The twistor dictionary:

Dual momentum space	Momentum twistor space $\mathbb{P}^3$
Point x	Line $L_x = A_x \wedge B_x$
$(x-y)^2$	Four bracket $\langle A_x B_x A_y B_y \rangle$
$(x-y)^2 = 0$	Lines L_x and L_y intersect
Conformal transformations	$\mathrm{PGL}(4)$ transformations

The massive box integral

$$\int \frac{\mathrm{d}^4 x_\ell}{(2\pi)^4} \frac{\text{(Some normalization)}}{(x_\ell - x_1)^2 (x_\ell - x_2)^2 (x_\ell - x_3)^2 (x_\ell - x_4)^2}$$

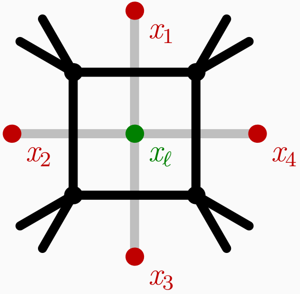


Figure 1: The box integral

Apply the twistor dictionary: [Hodges (2010)]

- External points: four skew lines L_i
- $(x_\ell - x_i)^2 = 0$: Find a fifth line transversal to L_i.
- Two solutions characterized by cross ratios κ and $\tilde{\kappa}$ on $\mathbb{P}^1$.
- Everything is manifestly conformal.

There are two configurations where all dual points are light-like separated.

Traintrack integrals

Traintrack integrals

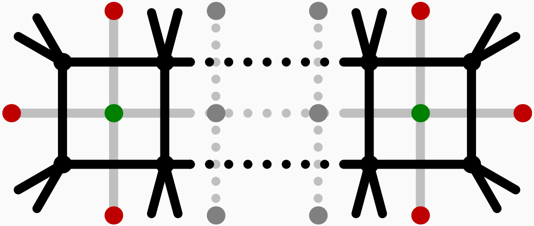


Figure 2: The traintrack integral family

Previously studied by [Bourjaily, He, McLeod, von Hippel, Wilhelm (2018)]:

- Feynman parameters, direct integration
- Hypersurfaces in weighted projective space
- Two loops: elliptic curve
- Three loops: K3 surface, four loops: Calabi-Yau threefold

Two loops: building an elliptic curve

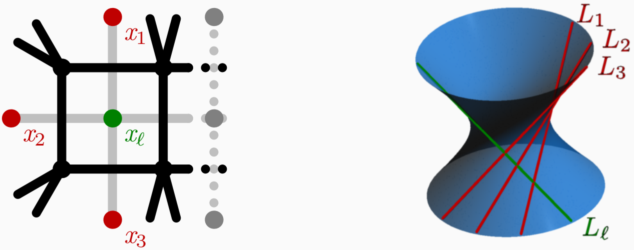


Figure 3: Relationship between the endcap of the traintrack and the quadric.

Three skew lines $L_i = A_i \wedge B_i$ determine a quadric in $\mathbb{P}^3$:

$$Q(Z) = \langle Z A_1 B_1 A_3 \rangle \langle Z A_2 B_2 B_3 \rangle - \langle Z A_1 B_1 A_3 \rangle \langle Z A_2 B_2 A_3 \rangle$$

A quadric has two rulings (families of lines):

- The lines within one ruling are skew.
- Two lines from different rulings intersect.

Two loops: building an elliptic curve

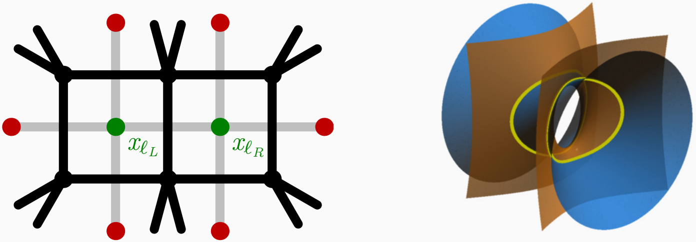


Figure 4: The double box integral

Build quadrics Q_L and Q_R for the left and the right loop respectively.

Imposing $(x_{\ell_L} - x_{\ell_R})^2 = 0$ means that the quadrics intersect.

The intersection C of two quadrics in $\mathbb{P}^3$ is an elliptic curve.

Elliptic curve: holomorphic form

Take Poincaré residues to get a holomorphic form on $C = Q_L \cap Q_R$:

$$\omega_C = \operatorname*{Res}_{Q_L, Q_R} \frac{\omega_{\mathbb{P}^3}}{Q_L Q_R}, \quad \omega_{\mathbb{P}^3} = Z_0 \, \mathrm{d}Z_1 \, \mathrm{d}Z_2 \, \mathrm{d}Z_3 \pm \text{(permutations)}$$

Check the weight under rescaling $Z \to \alpha Z$:

$$\omega_{\mathbb{P}^3} \to \alpha^4 \omega_{\mathbb{P}^3}, \quad Q_L \to \alpha^2 Q_L, \quad Q_R \to \alpha^2 Q_R.$$

Elliptic curves are characterized by one modulus, the j-invariant. Here:

$$j = 256 \frac{(z^2 - z + 1)^3}{z^2 (z-1)^2},$$

where z depends on the quadrics Q_L and Q_R.

Elliptic curve: comparison

Previous work by [Bourjaily, McLeod, Spradlin, von Hippel, Wilhelm (2017)]:

- Direct integration (Feynman parameters):

$$\int \mathrm{d}^4\ell_1 \mathrm{d}^4\ell_2 \times \textbf{Rational integrand} \xrightarrow{\text{int.}} \int_0^\infty \frac{\mathrm{d}x}{\sqrt{P_4(x)}} \times \textbf{Polylogs}$$

- Elliptic curve defined by $y^2 = P_4(x)$ with complicated j-invariant.

Here: elliptic curve as intersection of two quadrics with j-invariant

$$j = 256\frac{(z^2 - z + 1)^3}{z^2(z-1)^2}.$$

Result

The two j-invariants agree.

Three loops: K3 surface

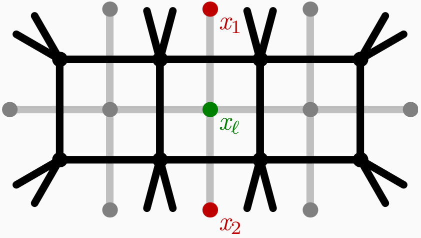


Figure 5: The three-loop traintrack integral

Geometry in twistor space:

- As before: two quadrics Q_L and Q_R
- Lines L_1 and L_2 associated to x_1 and x_2
- Line L_ℓ parameterized by two points $P_1 \in L_1$ and $P_2 \in L_2$
- Bezout's theorem: L_ℓ intersects Q_L and Q_R in two points each.

Three loops: K3 surface

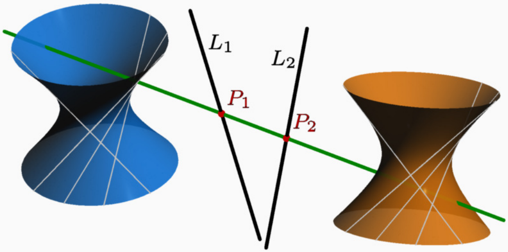


Figure 6: The geometry of the three-loop traintrack integral

Geometry in twistor space:

- As before: two quadrics Q_L and Q_R
- Lines L_1 and L_2 associated to x_1 and x_2
- Line L_ℓ parameterized by two points $P_1 \in L_1$ and $P_2 \in L_2$
- Bezout's theorem: L_ℓ intersects Q_L and Q_R in two points each.

Three loops: K3 surface

Where is the K3 surface?

- We can freely choose $P_1 \in L_1$ and $P_2 \in L_2$ while satisfying all constraints. Thus, the leading singularity is two-dimensional.
- For chosen P_1 and P_2, there are $2 \times 2 = 4$ choices for the intersection points of L_ℓ with Q_L and Q_R.
- Thus, we have a four-fold cover of $\mathbb{P}^1 \times \mathbb{P}^1$.

Aside: elliptic curve in $\mathbb{P}^2$

- Pick four points in $\mathbb{P}^1$.
- Set three of them to $\{0, 1, \infty\}$ and call the last one λ.
- Legendre form: $y^2 = x(x-1)(x-\lambda)$
- The curve is a double cover of $\mathbb{P}^1$ branched over four points.

K3 surface: branching

Branching occurs when the line L_ℓ is tangent to Q_L or Q_R.

L_ℓ is tangent if the following equations are fulfilled:

$$\Delta_L \equiv Q_L(P_1, P_2)^2 - Q_L(P_1, P_1)Q_L(P_2, P_2) = 0$$
$$\Delta_R \equiv Q_R(P_1, P_2)^2 - Q_R(P_1, P_1)Q_R(P_2, P_2) = 0$$

We get two curves Δ_L and Δ_R of bi-degree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$:

- They are themselves elliptic curves.
- Two branches of the surface over each curve
- Bezout: eight intersection points (only one branch)

K3 surface: characteristics

Points in $\mathbb{P}^1 \times \mathbb{P}^1$	Branches
$\mathbb{P}^1 \times \mathbb{P}^1 - \Delta_L \cup \Delta_R$	4
$\Delta_L \cup \Delta_R - \Delta_L \cap \Delta_R$	2
$\Delta_L \cap \Delta_R$	1

We compute the Euler characteristic using surgery:

$$\begin{aligned}\chi = 4 \times &\left[\chi(\mathbb{P}^1 \times \mathbb{P}^1) - \chi(\Delta_L \cup \Delta_R)\right] \\ &+ 2 \times [\chi(\Delta_L \cup \Delta_R) - \chi(\Delta_L \cap \Delta_R)] \\ &+ 1 \times \chi(\Delta_L \cap \Delta_R)\end{aligned}$$

Using for example $\chi(\Delta_L) = \chi(\Delta_R) = 0$ we get $\chi = 24$ as required.

K3 surface: characteristics

Dimension of the moduli space: 11 (Picard rank $\rho = 9$)

Holomorphic form:

$$\omega_{K3} = \frac{\omega_{\mathbb{P}^1}\omega_{\mathbb{P}^1}}{\sqrt{\Delta_L}\sqrt{\Delta_R}}$$

Nikulin involutions and automorphisms: [Nikulin (1979)]

- Number of fixpoints gives bounds on Picard rank
- In this case: $\rho \geq 9$

Open questions: How to compare to the hypersurface in weighted projective space? What are the invariants?

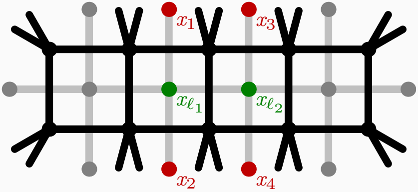


Figure 7: The four-loop traintrack integral

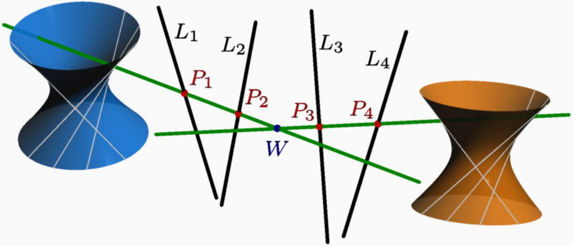


Figure 8: The geometry of the four-loop traintrack integral

Four and more loops

We can build a Calabi-Yau as a complete intersection in a toric variety:

- Use combinatorial description to compute Hodge numbers [Batyrev, Borisov (1994); ...]
- Three-fold at four loops: $h^{11} = 12$, $h^{12} = 28$ and $\chi = -32$ (computed with PALP [Kreuzer, Skarke (2004); ...])
- Problem: codimension in the embedding space grows with the number of loops

General traintrack integral with L loops

Calabi-Yau $(L-1)$-fold in a toric variety of dimension $2(L-1)$.

Conclusion and further directions

Summary

Leading singularity of the traintrack integrals:

- Two loops: elliptic curve as the intersection of two quadrics in $\mathbb{P}^3$
- Three loops: four-fold cover of $\mathbb{P}^1 \times \mathbb{P}^1$
- More loops: Calabi-Yau complete intersection

Good properties of momentum twistor space:

- Intersections of lines are easier than quadratic equations.
- No extra (unphysical) parameters
- Dual-conformal symmetry manifest

Further directions

Supersymmetrization:

- Amplitudes in $\mathcal{N} = 4$ SYM are superconformal
- Formulate the intersections in terms of δ-functions (invariant under $\mathrm{PSL}(4)$)
- Replace $\mathbb{P}^3$ by $\mathbb{P}^{3|4}$ and the δ-functions by supersymmetric versions.

More complicated diagrams:

- Fishnet-type $N \times M$ box graphs: also Calabi-Yau
- Massive internal propagators

Even further directions

How does the geometry degenerate?

- Polylogarithmic limits, e.g. ladder diagrams [Ussyukina, Davydychev (1993); Broadhurst (1993)]
- Basso-Dixon integrals [Basso, Dixon (2017)]

How to deal with non-planar integrals?

The role of different parametrizations?

- Feynman parameters, Baikov representation, …
- Isogenies, e.g. [Bogner, Müller-Stach, Weinzierl (2019)]
- Which changes of variables are allowed?

Calabi-Yau mirror symmetry (for example as in [Bloch, Kerr, Vanhove (2016); Bönisch, Fischbach, Klemm, Nega, Safari (2020); …])?

Thank you.

Elliptic curve: j-invariant

Construct an invariant in four steps:

1. Pencil of quadrics: $\lambda_L Q_L + \lambda_R Q_R$ with $[\lambda_L : \lambda_R] \in \mathbb{P}^1$.
2. Think of Q_L and Q_R as 4×4 symmetric matrices and compute

$$\det(\lambda_L Q_L + \lambda_R Q_R) \sim \# \lambda_L^4 + \tilde{\#} \lambda_L^3 \lambda_R + \cdots .$$

3. Compute the cross ratio of the four roots $\lambda^i = [\lambda_L^i : \lambda_R^i]$:

$$z = \frac{\langle 12 \rangle \langle 34 \rangle}{\langle 13 \rangle \langle 24 \rangle}, \quad \langle ij \rangle = \det(\lambda^i, \lambda^j).$$

   Permutations of the λ^i send $z \to z' \in \{\frac{1}{z}, 1 - z, \ldots\}$.

4. The j-invariant is the permutation-invariant combination

$$j = 256 \frac{(z^2 - z + 1)^3}{z^2 (z-1)^2}.$$