THE CALABI-YAU GEOMETRY OF FEYNMAN INTEGRALS

BASED ON JOINT WORK WITH ANDREY NOVOSELTSEV AND PIERRE VANHOVE

$$
\begin{tikzcd}
\text{FEYNMAN GRAPH} \arrow[rr] \arrow[d] & & \text{FEYNMAN INTEGRAL} \\
\text{FEYNMAN GRAPH HYPERSURFACES} \arrow[urr, dashed, "{[BEK]}"] \arrow[dr, "A"'] & & \text{PICARD-FUCHS DIFFERENTIAL EQUATIONS} \arrow[u, "C"'] \\
& \text{MOTIVIC CALABI-YAU MANIFOLDS} \arrow[ur, "B"'] &
\end{tikzcd}
$$

TODAY: TAKE THREE INFINITE FAMILIES OF FEYNMAN GRAPHS WHERE WE ([DNV]) UNDERSTAND A B C COMPLETELY

FOCUS TODAY IS ON GEOMETRY

[FOR GEOMETRY → PICARD FUCHS EQNS.
PLEASE SEE MY MSRI MINI-COURSE
STARTING ON OCTOBER 27.]

BIG QUESTIONS:

① HOW DOES THE CORRESPONDENCE

FEYNMAN GRAPHS ⟷ MOTIVIC CALABI-YAU MANIFOLDS

WORK?

② HOW ARE KEY FEATURES OF CY GEOMETRY AND MODULI REFLECTED IN RELATIONSHIPS AMONG FEYNMAN INTEGRALS?

START WITH PRIMER ON CALABI-YAU MANIFOLDS, THEN WE'LL LOOK AT THE THREE FAMILIES ("TOWERS") OF GRAPHS

PRIMER ON CALABI-YAU MANIFOLDS AND THEIR MODULI

ELLIPTIC CURVES

- NORMAL FORMS
  - HESSE:
    $$x^3+y^3+z^3+\mu xyz=0$$
  - LEGENDRE:
    $$y^2z-x(x-z)(x-tz)=0$$
  - WEIERSTRASS:
    $$y^2z-4x^3+g_2xz^2+g_3z^3=0$$
- MODULI SPACE = "MODULAR CURVES"

K3 SURFACES

- NORMAL FORMS
  - QUARTICS $\subset \mathbb{P}^3$ $\langle 4 \rangle$
  - DOUBLE COVERS OF $\mathbb{P}^2$ BRANCHED OVER A SEXTIC CURVE $\langle 2 \rangle$
  - INTERNAL ELLIPTIC FIBRATION WITH SECT. $H=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- MODULI SPACES OF LATTICE-POLARIZED K3 SURFACES

K3 LATTICE: $\Lambda_{K3} = H^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$

$L \subset \Lambda_{K3}$ signature $(1, \rho - 1)$

$\dim_{\mathbb{C}} (\mathcal{M}_{L\text{-pol}}) = 20 - \rho$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

NEW PHENOMENON: "RANK JUMPS"

K3 SURFACES ACQUIRE ADE-SINGS

ALONG SUBLOCI IN MODULI

$\equiv$ MODULI SPACES OF NEW K3 SURFACES WITH HIGHER RANK LATTICE POLARIZATION

CALABI-YAU THREEFOLDS

- NORMAL FORMS
  - QUINTICS $\subset \mathbb{P}^4$
  - DOUBLE COVERS OF $\mathbb{P}^3$ BRANCHED OVER AN OCTIC SURFACE

INTERNAL FIBRATION BY K3 SURFACES AND/OR ELLIPTIC CURVES

CALABI-YAU THREEFOLD MODULI:

$$\dim_{\mathbb{C}}\left(\mathcal{M}_{\mathbb{C}\text{-str}}\right) = h^{2,1}$$

$$\dim_{\mathbb{C}}\left(\mathcal{M}_{\mathbb{K}\text{-str}}\right) = h^{1,1}$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,2} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

NEW PHENOMENON:

"TOPOLOGY CHANGING TRANSITIONS" AMONG MODULI SPACES

- DEGENERATE & RESOLVE SINGULARITIES

  $h^{2,1} \downarrow \quad , \quad h^{1,1} \uparrow$

- BIRATIONAL CONTRACTION & SMOOTHING

  $h^{2,1} \uparrow \quad , \quad h^{1,1} \downarrow$

- EXAMPLE: GENERIC QUINTICS AND DOUBLE OCTICS ARE RELATED IN THIS WAY

TORIC CONSTRUCTIONS

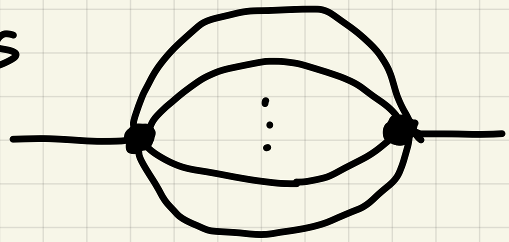
- CALABI-YAU HYPERSURFACES AND COMPLETE INTERSECTIONS IN TORIC VARIETIES
- BRANCHED DOUBLE COVERS OF TORIC VARIETIES
- CALABI-YAU FIBRATIONS OVER A TORIC BASE VARIETY

KEY: GENERICITY REDUCES MUCH OF THE GEOMETRY TO COMBINATORICS

SO, THE "EASY CASE" WOULD BE TO FIND THAT OUR MOTIVIC CALABI-YAU MANIFOLDS ARISE FROM SUCH GENERIC TORIC CONSTRUCTIONS

MULTI-LOOP SUNSET FEYNMAN GRAPHS

n masses



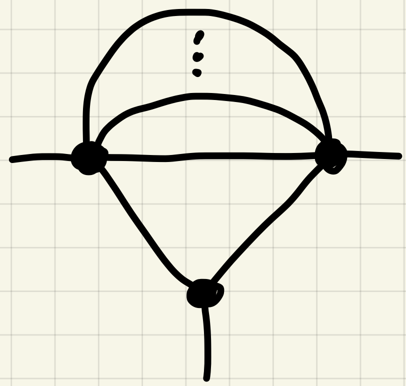
GRAPH HYPERSURFACE IS SINGULAR CY

THM ([DNV])

1. FOR GENERAL MASS PARAMETERS, THE MOTIVIC CALABI-YAU MANIFOLD $X^{(n-2)}$ OF THE $(n-1)$ LOOP SUNSET GRAPHS ARE GENERIC, SMOOTH, BIPARTITE COMPLETE INTERSECTIONS IN TORIC n-FOLDS
2. THE HODGE NUMBERS OF $X^{(n-2)}$ COUNTING COMPLEX AND KÄHLER MODULI ARE:
$$h^{n-3,1} = n \, , \quad h^{1,1} = 2^{n+1} - 3n - 4 \quad (n \geq 5)$$
3. THE $X^{(k)}$ ARE FIBERED BY $X^{(k-1)}$ OVER $\mathbb{P}^1$, INDUCING AN ITERATED CALABI-YAU FIBRATION IN ALL DIMENSIONS
4. "THRESHOLDS" CORRESPOND TO ORDINARY DOUBLE POINT (CONIFOLD) SINGULARITIES COLLISION LOCI OF THE $X^{(n-2)}$

THE BEST OF ALL POSSIBLE WORLDS!

ICE CREAM CONE FEYNMAN GRAPHS



THM ([DNV])

THE MOTIVIC CALABI-YAU MANIFOLDS $Y^{(n-2)}$ OF THE $(n-1)$ LOOP ($(n-2)$ SCOOP) I.C. CONE GRAPHS ARE ISOMORPHIC TO THOSE OF THE $(n-1)$ LOOP SUNSET GRAPHS.

COR ([DNV])

THE MAP

$$\{\text{FEYNMAN GRAPHS}\} \longrightarrow \{\text{CY MODULI SPACES}\}$$

IS NOT INJECTIVE FOR CALABI-YAU MANIFOLDS OF ANY DIMENSION.

IS THERE STILL A NATURAL GEOMETRIC WAY TO DISTINGUISH THESE CASES?

MIRROR SYMMETRY

- CY-CY MIRROR SYMMETRY

THM ([DNV])

BOTH THE SUNSET AND ICE CREAM CONE MOTIVIC CALABI-YAU MANIFOLDS ARE MIRROR TO THE COMPLETE INTERSECTION OF A PAIR OF $(1,1,\ldots,1)$-HYPERSURFACES IN $(\mathbb{P}^1)^n$

WE NEED A VERSION THAT IS SENSITIVE TO FAMILIES OF MOTIVIC CALABI-YAUS ...

- LG-FANO MIRROR SYMMETRY

LANDAU-GINZBURG MODEL WITH SUPERPOTENTIAL $\longleftrightarrow$ QUASI-FANO VARIETY

THM ([DNV])

1. THE SUNSET MOTIVIC CALABI-YAU MANIFOLDS DEFINE LANDAU-GINZBURG MODELS WITH

   SUPERPOTENTIAL = EXTERNAL MOMENTUM (p^2)

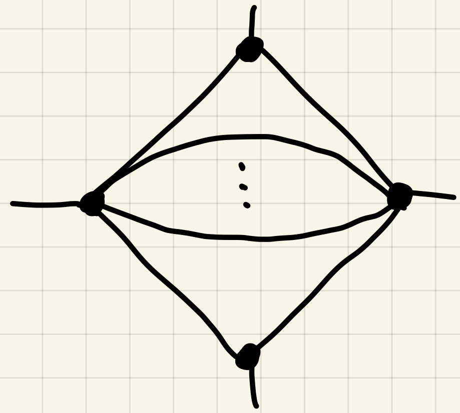
   $\mathbb{C}$-STRUCTURE DEFORMATIONS = MASS PARAMETERS

2. LG-MODEL WITH ALL MASSES EQUAL $\longleftrightarrow$ $(1,1,\ldots,1) \subset (\mathbb{P}^1)^n$ FANO MANIFOLD

   LG-MODELS WITH UNEQUAL MASSES $\longleftrightarrow$ WEAK (SINGULAR) FANO VARIANTS

3. WITH ONE OF THE EXTERNAL MOMENTA AS SUPERPOTENTIAL, THE ICE CREAM CONE MOTIVIC CALABI-YAUS DEFINE LG-MODELS MIRROR TO THE QUASI-FANO BLOW UP OF $(1,1,\ldots,1) \subset (\mathbb{P}^1)^n$ ALONG A CODIMENSION-TWO CALABI-YAU SUBMANIFOLD

KITE FEYNMAN GRAPHS



THM ([DNV])

1. THE MOTIVIC CALABI-YAU $Z^{(n-2)}$ OF THE $(n-1)$ LOOP KITE FEYNMAN GRAPH IS FIBERED IN CODIMENSION-TWO BY SUNSET MOTIVIC CALABI-YAU MANIFOLDS $X^{(n-4)}$ OVER A NON-TORIC RATIONAL SURFACE $S^{(2)}$.

2. THERE ARE EXPLICIT TOPOLOGY CHANGING TRANSITIONS BETWEEN THE KITE AND THE ICE CREAM CONE MOTIVIC CALABI-YAU MANIFOLDS DERIVED BY MATCHING FIBRATIONS.

LESSONS LEARNED:

1. THE GEOMETRY AND MODULI OF MOTIVIC CALABI-YAU MANIFOLDS CONNECT DISTINCT FEYNMAN GRAPHS AND THEIR INTEGRALS

   "ROLLING AMONG FEYNMAN INTEGRALS"

2. KEY ROLES ARE PLAYED BY
   - CALABI-YAU ITERATED FIBRATIONS
   - LG-FANO MIRROR SYMMETRY

   THE LINK BETWEEN THESE TWO IS GIVEN BY THE DHT "GLUING/SPLITTING" MIRROR CONJECTURE

3. READY TO APPLY TOOLS OF LIMITING MIXED HODGE STRUCTURES, HIGHER NORMAL FUNCTIONS, REGULATOR PERIODS, ETC. (JOINT WITH BLOCH, KERR, VANHOVE) TO STUDY OF AMPLITUDES