Series expansions methods for Feynman integrals, the DiffExp Mathematica package, and various applications

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Based on arXiv:2006.05510

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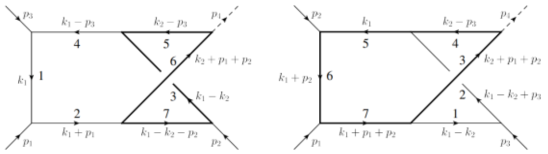
Outline of the talk

Introduction

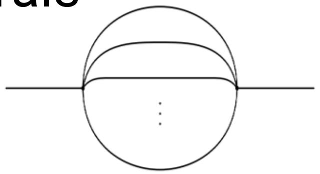
- The method of differential equations
  - Canonical form
- Series solutions along 1-dim. contours
  - Finding series solutions
  - Line segmentation & analytic continuation
- The DiffExp Mathematica package
  - Usage & boundary conditions

Applications

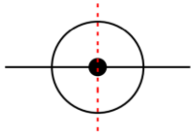


[Bonciani, Del Duca, Frellesvig, Henn, Hidding, Maestri, Moriello, Salvatori, Smirnov] [1907.13156, 1911.06308]

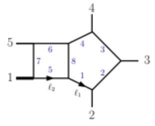
- Higgs plus jet integrals
- Banana graphs



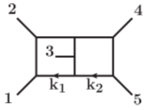
  - 3-Loop equal/unequal mass
  - 4-Loop equal mass



- Computation of cut integrals [Abreu, Britto, Duhr, Gardi, Gonzo, Hidding, upcoming work]
- Examples from the literature



[Abreu et al, 2005.04195]



[Chicherin et al, 1812.11160]

- Final remarks
  - Future prospects

Analytic methods

- Express Feynman integrals in terms of classes of iterated integrals

  (MPLs, eMPLs, “and beyond”)

- Strengths:
  - Algebraic properties are manifest (Hopf algebra, symbol map, …)
  - Good numerical performance (see also prev. talks.)
- Drawbacks:
  - Analytic continuation can be quite complicated
  - Not in general known how to obtain analytic solutions

Numerical methods

- Use numerical algorithms or integrators to evaluate Feynman integrals (sector decomposition, numerical integration of diff. eqns, numerical Mellin-Barnes, …)
- Benefits:
  - Typically fully algorithmic and general purpose
- Drawbacks:
  - Numbers might not expose symmetries and/or structures underlying amplitudes / integrals
  - Performance can greatly lack behind analytic methods

Series expansions of differential equations

- An “intermediate” strategy, is to derive the differential equations and simplify them analytically, and then to solve the differential equations numerically.
- In particular, we will solve the differential equations of Feynman integrals in terms of series expansions, following the general strategy of F. Moriello’s paper arXiv:1907.13234.
  - The same strategy was applied further for non-planar H+j integrals in arXiv:1907.13156, arXiv:1911.06308, and to five-point one-mass functions in arXiv:2005.04195.
- Main steps:
  - Write down a sequence of line segments to a kinematic point.
  - Series expand the differential equations along each segment
  - Solve the differential equations in terms of series expansions, along each path, and use the result to fix the boundary conditions for the next path.

Series expansions

- Note also the range of previous literature on series expansions. For single scale problems, see e.g.:

S. Pozzorini and E. Remiddi, *Precise numerical evaluation of the two loop sunrise graph master integrals in the equal mass case*, *Comput. Phys. Commun.* **175** (2006) 381–387, [hep-ph/0505041].

U. Aglietti, R. Bonciani, L. Grassi, and E. Remiddi, *The Two loop crossed ladder vertex diagram with two massive exchanges*, *Nucl. Phys.* **B789** (2008) 45–83, [arXiv:0705.2616].

R. Mueller and D. G. Öztürk, *On the computation of finite bottom-quark mass effects in Higgs boson production*, *JHEP* **08** (2016) 055, [arXiv:1512.08570].

B. Mistlberger, *Higgs boson production at hadron colliders at N^3LO in QCD*, *JHEP* **05** (2018) 028, [arXiv:1802.00833].

R. N. Lee, A. V. Smirnov, and V. A. Smirnov, *Solving differential equations for Feynman integrals by expansions near singular points*, *JHEP* **03** (2018) 008, [arXiv:1709.07525].

R. N. Lee, A. V. Smirnov, and V. A. Smirnov, *Evaluating elliptic master integrals at special kinematic values: using differential equations and their solutions via expansions near singular points*, *JHEP* **07** (2018) 102, [arXiv:1805.00227].

R. Bonciani, G. Degrassi, P. P. Giardino, and R. Gröber, *A Numerical Routine for the Crossed Vertex Diagram with a Massive-Particle Loop*, *Comput. Phys. Commun.* **241** (2019) 122–131, [arXiv:1812.02698].

- For multi-scale problems, series expansions have been considered before in special kinematic limits. See e.g.:

K. Melnikov, L. Tancredi, and C. Wever, *Two-loop $gg \to Hg$ amplitude mediated by a nearly massless quark*, *JHEP* **11** (2016) 104, [arXiv:1610.03747].

K. Melnikov, L. Tancredi, and C. Wever, *Two-loop amplitudes for $qg \to Hq$ and $q\bar{q} \to Hg$ mediated by a nearly massless quark*, *Phys. Rev.* **D95** (2017), no. 5 054012, [arXiv:1702.00426].

R. Bonciani, G. Degrassi, P. P. Giardino, and R. Grober, *Analytical Method for Next-to-Leading-Order QCD Corrections to Double-Higgs Production*, *Phys. Rev. Lett.* **121** (2018), no. 16 162003, [arXiv:1806.11564].

R. Bruser, S. Caron-Huot, and J. M. Henn, *Subleading Regge limit from a soft anomalous dimension*, *JHEP* **04** (2018) 047, [arXiv:1802.02524].

J. Davies, G. Mishima, M. Steinhauser, and D. Wellmann, *Double-Higgs boson production in the high-energy limit: planar master integrals*, *JHEP* **03** (2018) 048, [arXiv:1801.09696].

J. Davies, G. Mishima, M. Steinhauser, and D. Wellmann, *Double Higgs boson production at NLO in the high-energy limit: complete analytic results*, *JHEP* **01** (2019) 176, [arXiv:1811.05489].

B. Mistlberger, *Higgs boson production at hadron colliders at N^3LO in QCD*, *JHEP* **05** (2018) 028 [1802.00833].

DiffExp

- In this talk we focus in particular on my public Mathematica package DiffExp, introduced with the associated paper arXiv:2006.05510
- Capable of computing "coupled" systems of more than two integrals
- Takes in (any) system of differential equations of the form

$$\frac{\partial}{\partial s}\vec{f}(\{S\},\epsilon) = \mathbf{A}_s \vec{f}(\{S\},\epsilon) \qquad \mathbf{A}_x(x,\epsilon) = \sum_{k=0}^{\infty} \mathbf{A}_x^{(k)}(x)\epsilon^k$$

- Uses: compute Feynman integrals numerically at high precision. Analytically continue results across thresholds. Transporting boundary conditions from one special point to another.

The method of differential equations

Differential equations

- Start from a family of scalar Feynman integrals:

$$I_{a_1,\ldots,a_{n+m}} = \int \left(\prod_{i=1}^{l} d^d k_i \right) \frac{\prod_{i=n+1}^{n+m} N_i^{-a_i}}{\prod_{i=1}^{n} D_i^{a_i}}, D_i = -q_i^2 + m_i^2 - i\delta$$

- Derivatives of Feynman integrals can be expressed in the same family.

- By IBP-reduction we may thus obtain a closed system of the form:

$$\boxed{d\vec{f} = \sum_{s\in S} \mathbf{M}_s \vec{f} ds}$$

[Kotikov, 1991], [Remiddi, 1997]
[Gehrmann, Remiddi, 2000]

- For some vector of master integrals $\vec{f}$

Canonical basis

- We may simplify the differential equations by a change of basis

- Let $\vec{B} = \boldsymbol{T}\vec{f}$, then we have: $\dfrac{\partial}{\partial s_i}\vec{B} = \left[(\partial_{s_i}\mathbf{T})\,\mathbf{T}^{-1} + \mathbf{T}\mathbf{M}_{s_i}\mathbf{T}^{-1}\right]\vec{B}\,.$

- The canonical basis conjecture claims that $\exists\mathbf{T}:\ \boxed{d\vec{B} = \epsilon d\tilde{\mathbf{A}}\vec{B}}$ [Henn, 2013]

- Furthermore, if the integrals are polylogarithmic, we then have:

$$d\tilde{\mathbf{A}} = \sum_{i\in\mathcal{A}} \mathbf{C}_i d\log(l_i)$$

Canonical basis

- The formal solution can be given in terms of Chen's iterated integrals: [Chen, 1977]

$$d\vec{B} = \epsilon \left(d\tilde{\mathbf{A}} \right) \vec{B} \quad \Rightarrow \quad \vec{B} = \mathbb{P}\exp\left[\epsilon \int_\gamma d\tilde{\mathbf{A}} \right] \vec{B}_{\text{boundary}} \qquad \gamma(x) : [0,1] \to \mathbb{C}^p$$

$$\vec{B} = \vec{B}^{(0)}(\gamma(0)) + \sum_{k\geq 1} \epsilon^k \sum_{j=1}^{k} \int_0^1 \gamma^*(d\tilde{\mathbf{A}})\,(x_1) \int_0^{x_1} \gamma^*(d\tilde{\mathbf{A}})\,(x_2) \times \ldots \times \int_0^{x_{j-1}} \gamma^*(d\tilde{\mathbf{A}})\,(x_j)\, \vec{B}^{(k-j)}(\gamma(0))$$

- Or recursively in ϵ we have simply: $\vec{B} = \sum_{i\geq 0} \vec{B}^{(i)} \varepsilon^i$

$$\mathbf{A}_x = \sum_{s\in S} \frac{\partial \tilde{\mathbf{A}}}{\partial s}(\gamma(x)) \frac{\partial \gamma_s(x)}{\partial x} dx$$

$$\boxed{\vec{B}^{(i)}(\gamma(1)) = \int_0^1 \mathbf{A}_x \vec{B}^{(i-1)} dx + \vec{B}^{(i)}(\gamma(0))}$$

set of masses and kinematic invariants

Canonical basis

$$\vec{B}^{(i)}(\gamma(1)) = \int_0^1 \mathbf{A}_x \vec{B}^{(i-1)} dx + \vec{B}^{(i)}(\gamma(0))$$

- When $d\tilde{\mathbf{A}} = \sum_{i\in\mathcal{A}} \mathbf{A}_i d\log(l_i)$ and when $\mathcal{A}$ contains only (simult.) rat. alg. functions the results are expressible in terms of MPLs:

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t)\,, \quad G(;z) \equiv 1$$

- Allows for a fully analytic solution of the differential equations, which is also efficient to evaluate.

Series solutions to differential equations

Series solutions of a canonical basis

$$\vec{B}^{(i)}(\gamma(1)) = \int_0^1 \mathbf{A}_x \vec{B}^{(i-1)} dx + \vec{B}^{(i)}(\gamma(0))$$

- Let us expand along the contour:

$$\tilde{\mathbf{A}}_x = x^r \left[\sum_{p=0}^{k} \mathbf{C}_p x^p + \mathcal{O}\left(x^{k+1}\right) \right] ,$$

  where r is a rational number.

- Then all integrations can be performed straightforwardly:

$$\int x^w \log(x)^n = \boldsymbol{H}_{ijk}\, c_i x^{w_j} \log(x)^{n_k}$$

- E.g: $\int x^{-3/5} \log^2(x) dx = \frac{5}{4} x^{2/5} \left(2\log^2(x) - 10\log(x) + 25\right)$

Canonical basis

- In the presence of non-rationalizable roots, the results may not be expressible in terms of MPLs at all orders in ϵ. [Brown, Duhr, arXiv:2006.09413]
- Furthermore, in general $d\widetilde{\boldsymbol{A}}$ is not a sum of logarithms:
  - More complicated kernels appear (modular forms)
- In general we might not be able / know how to obtain a canonical form at all
- In such cases, series expansion methods can help us out

Solving non-canonical systems

- Start from: $$\frac{\partial}{\partial s}\vec{f}(\{S\},\epsilon) = \mathbf{A}_s \vec{f}(\{S\},\epsilon)$$
- Consider the contour: $\gamma(x) : [0,1] \to \mathbb{C}^p$
- Then: $$\frac{\partial}{\partial x}\vec{f}(x) = \mathbf{A}_x \vec{f}(x,\epsilon) \quad \mathbf{A}_x = \sum_{s\in S} \mathbf{A}_s(\gamma(x))\frac{\partial \gamma_s(x)}{\partial x}$$
- Expand in ϵ: $$\mathbf{A}_x(x,\epsilon) = \sum_{k=0}^{\infty} \mathbf{A}_x^{(k)}(x)\epsilon^k \quad \vec{f}(x,\epsilon) = \sum_{k=0}^{\infty} \vec{f}^{(k)}(x)\epsilon^k$$

Assume no poles

- Then: $$\boxed{\partial_x \vec{f}^{(k)} = \mathbf{A}_x^{(0)} \vec{f}^{(k)} + \sum_{j=0}^{k-1} \mathbf{A}_x^{(k-j)} f^{(j)}}$$

Integration sequence:

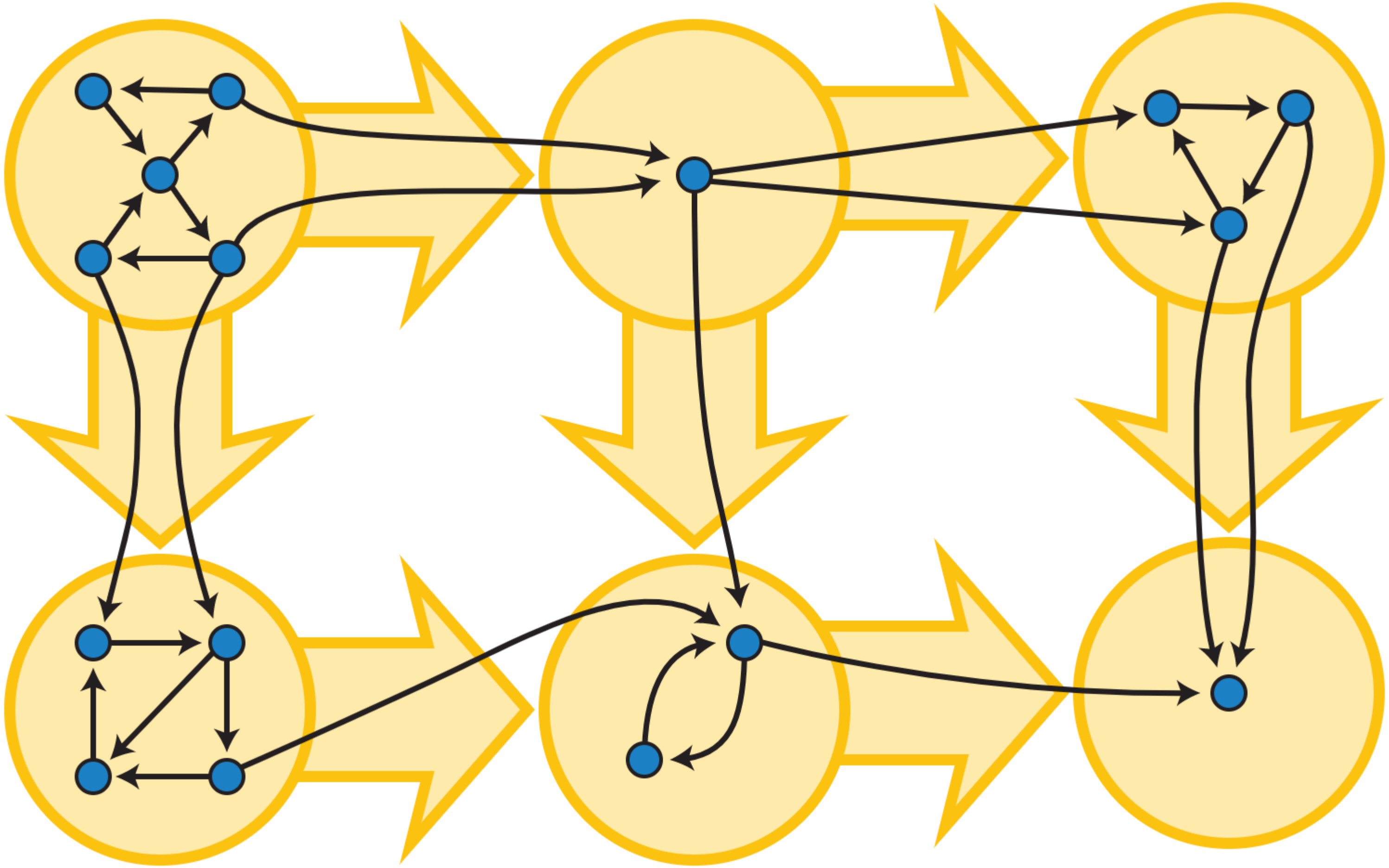
- Lower orders in ϵ integrated first
- Subtopologies integrated first
- Some integrals are coupled

Integration sequence

- Read off integration sequence & coupled integrals from $\boldsymbol{A}_x^{(0)}$:
- Let $\boldsymbol{C}_{ij} = \begin{cases} 0 \text{ if } \left(\boldsymbol{A}_x^0\right)_{ji} = 0 \\ 1 \text{ if } \left(\boldsymbol{A}_x^0\right)_{ji} = 1 \end{cases}$, be the adjacency matrix of a directed graph G
- Then $f_j \to f_i$ is an edge of G if the derivative of f_i includes a contribution from f_j
- Next, determine the strongly connected components of G
  - Sets of vertices for which there is a directed path between every pair of vertices.
  - Every vertex is connected to itself by the trivial path

Integration sequence

- Example of condensation of a directed cyclic graph



Picture taken from https://commons.wikimedia.org/wiki/File:Graph_Condensation.svg

Integration sequence

- Consider the condensation $\tilde{G}$ of G:
  - The vertices of $\tilde{G}$ are the strongly connected components of G
  - If c_1, c_2 are vertices in $\tilde{G}$, and if $\exists f_i \in c_1, f_j \in c_2$ such that $f_i \to f_j$ is an edge in G then $c_1 \to c_2$ is an edge in $\tilde{G}$.
  - $\tilde{G}$ is a directed acyclic graph
- The integration sequence is then determined by topologically sorting the vertices of $\tilde{G}$:
  - If $\exists$ path $c_i \to c_j$, then $c_i < c_j$.
  - We determine a sequence $(c_{\rho_1}, c_{\rho_2}, \dots)$ compatible with the ordering

Solving non-canonical systems

$$\partial_x \vec{f}^{(k)} = \mathbf{A}_x^{(0)} \vec{f}^{(k)} + \sum_{j=0}^{k-1} \mathbf{A}_x^{(k-j)} f^{(j)}$$

- Let $\{f_{\sigma_1}, \ldots, f_{\sigma_p}\}$ be a set of coupled integrals, and let $f_{\sigma_1} \to g_1, f_{\sigma_2} \to g_2, \ldots$.
- Then $\boxed{\partial_x \vec{g}^{(k)} = \mathbf{M}\vec{g}^{(k)} + \vec{b}^{(k)}}$

Where: $\underbrace{\mathbf{M}_{ij} = \left(\mathbf{A}_x^{(0)}\right)_{\sigma_i,\sigma_j}}_{\text{Homogeneous part: the same at all orders in } \epsilon}, \quad \vec{b}_i^{(k)} = \sum_{j \notin \Sigma} \underbrace{\left[\left(\mathbf{A}_x^{(0)}\right)_{\sigma_i j} f_j^{(k)} + \sum_{l=0}^{k-1} \left(\mathbf{A}_x^{(k-l)}\right)_{\sigma_i j} f_j^{(l)}\right]}_{\text{Inhomogeneous part: subtopology terms \& lower orders in } \epsilon}$

Homogeneous part: the same at all orders in ϵ

Inhomogeneous part: subtopology terms & lower orders in ϵ

Homogeneous differential equations

$$\partial_x \vec{g} = \mathbf{M}\vec{g} + \vec{b}$$

- First, we solve the homogeneous part: $\partial_x \vec{g} = \mathbf{M}\vec{g}$.
- Strategy:
  1. Combine the system into a p-th order differential equation for g_i
  2. Find p (homogeneous) solutions for g_i using the Frobenius method and reduction of order
  3. Solve for the remaining g_j in terms of g_i
- Detailed steps can be found in [MH, 2006.05510].

Homogeneous differential equations

$$\partial_x \vec{g} = \mathbf{M}\vec{g}$$

- Let: $\vec{g}^{\partial} = \left(g_1, \partial g_1, \ldots, \partial^{p-1} g_1\right)$
- Then we can find a matrix $\tilde{\mathbf{M}}$ such that $\vec{g}^{\partial} = \tilde{\mathbf{M}}\vec{g}$
- By Frobenius method & reduction of order: $\mathbf{W} = \begin{vmatrix} h_1 & \cdots & h_p \\ \partial h_1 & \cdots & \partial h_p \\ \vdots & \ddots & \vdots \\ \partial^{p-1} h_1 & \cdots & \partial^{p-1} h_p \end{vmatrix}$

Wronskian matrix of homogeneous solutions for homogeneous equation of g_1.

- Then let: $\boxed{\mathbf{F} = \tilde{\mathbf{M}}^{-1}\mathbf{W},}$ $\quad \partial\mathbf{F} = \mathbf{MF}$

Invert by solving associated linear system.
(Note: we assume it is invertible. Non-invertible case requires some more work but can be computed as well.)

Inhomogeneous differential equations

$$\partial_x \vec{g} = \mathbf{M}\vec{g} + \vec{b}$$

- Consider the matrix: $\mathbf{B} = \frac{1}{p}(\vec{b}, \ldots, \vec{b})$
- Consider a matrix $\mathbf{G}$ satisfying: $\partial\mathbf{G} = \mathbf{M}\mathbf{G} + \mathbf{B}$
- If we write, $\mathbf{G} = \mathbf{F}\mathbf{H}$, then:

Constant matrix

$$\mathbf{F}\partial\mathbf{H} = \mathbf{B} \Rightarrow \mathbf{H} = \int \mathbf{F}^{-1}\mathbf{B} + \mathbf{E}$$

Choose

- Thus:

k-th column

$$\boxed{\vec{g} = \sum_{\mathrm{j}=1}^{\mathrm{p}} \vec{\mathbf{G}}_{\mathrm{j}}, \mathbf{G} = \mathbf{F}\left(\int \mathbf{F}^{-1}\mathbf{B} + \mathbf{E}\right)}$$

$$\mathbf{E} = \mathrm{diag}\,(e_1, \ldots, e_p)$$

Inhomogeneous differential equations

$$\vec{g} = \sum_{\mathrm{j}=1}^{\mathrm{p}} \vec{\mathbf{G}}_{\mathrm{j}}, \mathbf{G} = \mathbf{F}\left(\int \mathbf{F}^{-1}\mathbf{B} + \mathbf{E}\right)$$

- How to compute $\mathbf{F}^{-1} = \mathbf{W}^{-1}\widetilde{\mathbf{M}}$?
- If $\mathbf{F}$ contains no logarithms in the series, we can invert it using Mathematica's routines. If $\mathbf{F}$ contains logarithms, then Mathematica will greatly struggle with the inversion.
- Solution: If $\partial\mathbf{W} = \mathbf{N}\mathbf{W}$ then $\partial(\mathbf{W}^{-1})^{\top} = -\mathbf{N}^{\top}(\mathbf{W}^{-1})^{\top}$
- We can solve for $\mathbf{W}^{-1}$ using the Frobenius method

Inhomogeneous differential equations

$$\vec{g} = \sum_{\mathrm{j}=1}^{\mathrm{p}} \vec{\mathbf{G}}_{\mathrm{j}}, \mathbf{G} = \mathbf{F}\left(\int \mathbf{F}^{-1}\mathbf{B} + \mathbf{E}\right)$$

- If we reintroduce a superscript for the order in ϵ we have that:

$$\vec{g}^{(k)} = \sum_{\mathrm{j}=1}^{\mathrm{p}} \vec{\mathbf{G}}_{\mathrm{j}}^{(k)}, \mathbf{G}^{(\mathrm{k})} = \mathbf{F}\left(\int \mathbf{F}^{-1}\mathbf{B}^{(\mathrm{k})} + \mathbf{E}^{(k)}\right)$$

- Thus, we only compute $\mathbf{F}$ and $\mathbf{F}^{-1}$ once at order ϵ^0.
- Higher orders in ϵ are computed by two matrix multiplications, and a single integration (which is implemented using an efficient replacement rule.)
- This compares favorably to a “straightforward” variation of parameters implementation, which involves computing p determinants of size $(p-1)\times(p-1)$ for each order in ϵ.

Line segmentation

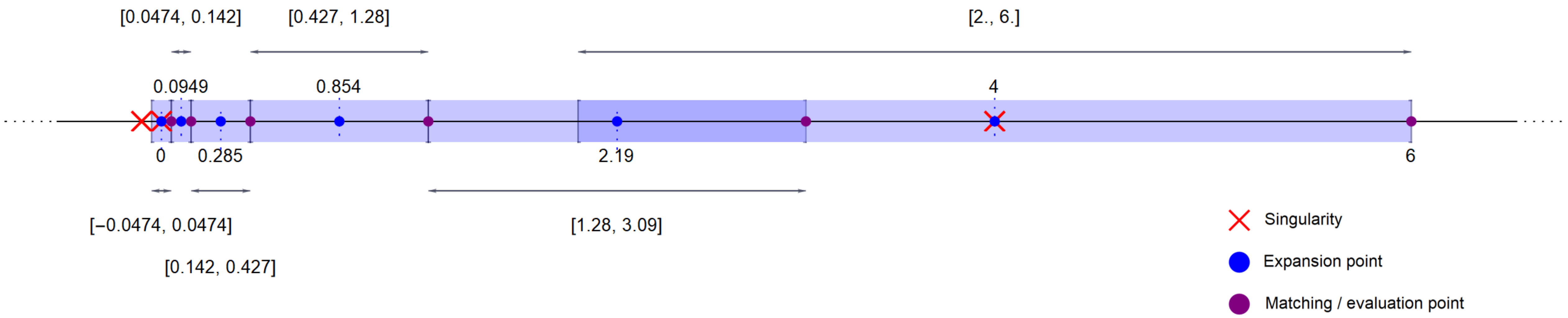
- The series solutions have a finite radius of convergence.
- By concatenating solutions centered at different line segments we can reach any point in phase-space.
- We may choose the line segments such that each expansion is evaluated at most $1/k$ the distance to the nearest singularity, where $k > 1$.
- To cross singularities, we center an expansion at the singularity

Line segmentation

- For example, suppose: $X_{\text{sing}} = (\dots, -0.095, 0, 4, 16, \dots)$

$$x_{\text{start}} = 0, x_{\text{end}} = 6$$

- Then we may pick the following partitioning into 6 line segments, such that each evaluation happens at most ½ the distance to the nearest singularity:



Analytic continuation

- The series solutions centered at singularities may contain logarithms and square roots.
  - Logarithms appear by integration of poles $1/x$.
  - Square roots can arise from the homogeneous solutions (when the indicial equation has a half-integer root), or from the basis definition.
- By transferring an $i\delta$-prescription to the line parameter, we can perform the analytic continuation of these functions. In particular we can let:

$$\log(x + i\delta) = \log(x), \qquad \sqrt{x + i\delta} = \sqrt{x}$$

$$\log(x - i\delta) = \log(x) - 2\pi i \theta_m, \qquad \sqrt{x - i\delta} = (\theta_p - \theta_m)\sqrt{x}$$

Analytic continuation

- We don't like to carry theta functions around in the series expansions (for performance reasons), so we may instead use replacement rules.
- For example, if x carries $-i\delta$, and we evaluate at a point $x < 0$, we let:

$$\log(x) \to \log(x) - 2\pi i, \quad \sqrt{x} \to -\sqrt{x}$$

- Additional comments:
  - The $i\delta$-prescriptions can be determined from the Feynman prescription
  - Typically we should avoid crossing two singular regions at the same time

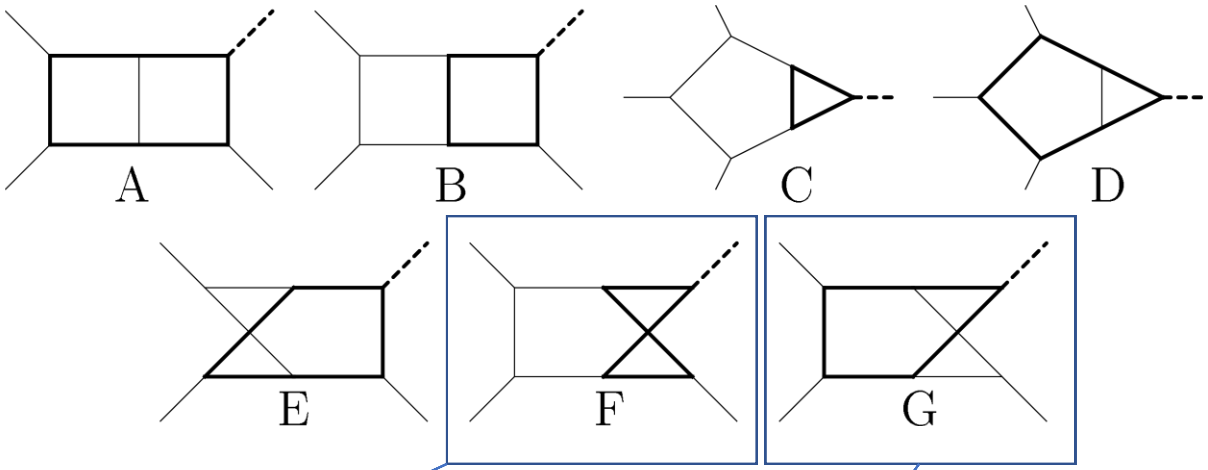
Examples

Higgs + jet integrals

R. Bonciani, V. Del Duca, H. Frellesvig, J. M. Henn, MH, L. Maestri, F. Moriello, G. Salvatori, V. A. Smirnov

[Bonciani et al, 1609.06685]

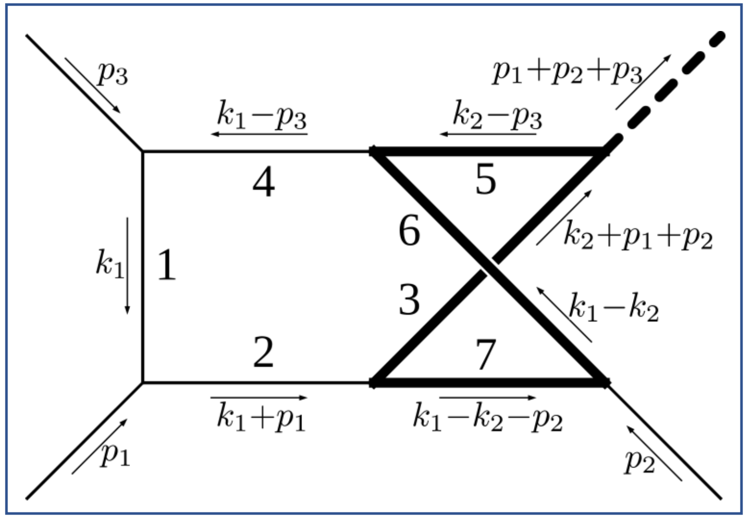
- Integrals relevant for H+j production at NLO with full heavy quark mass dependence
- Dependence on three scales (after normalizing out mass dependence)
- Families A, F, and G contain elliptic sectors



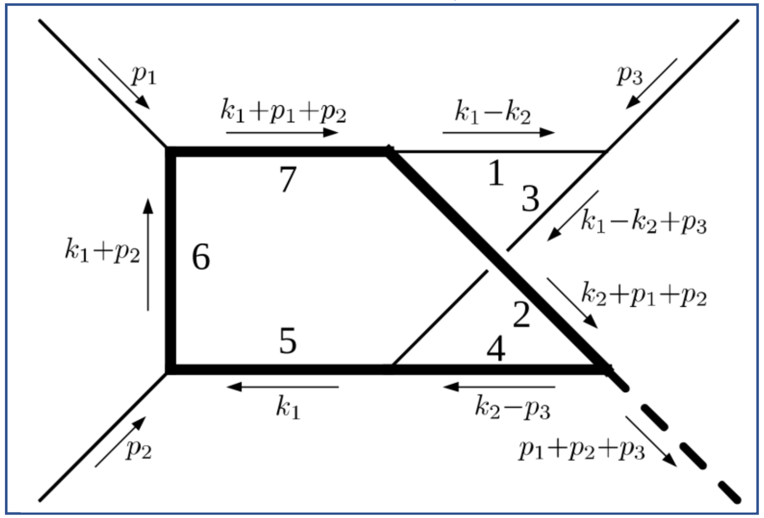
$$s = (p_1 + p_2)^2\,, \quad t = (p_1 + p_3)^2$$

$$p_4^2 = (p_1 + p_2 + p_3)^2 = s + t + u$$

$$m_q$$



[Bonciani et al, 1907.13156]



[Frellesvig et al, 1911.06308]

Family F

Master integrals

- IBP-reduction:
  - 73 master integrals
  - Default FIRE basis: $\mathcal{O}$(1 GB)
  - More suitable (pre-canonical) basis: $\mathcal{O}$(100 MB)
  - Possible using either FIRE or KIRA
- Differential eqns: $\mathcal{O}$(10 MB)

  [Bonciani et al, 1907.13156]

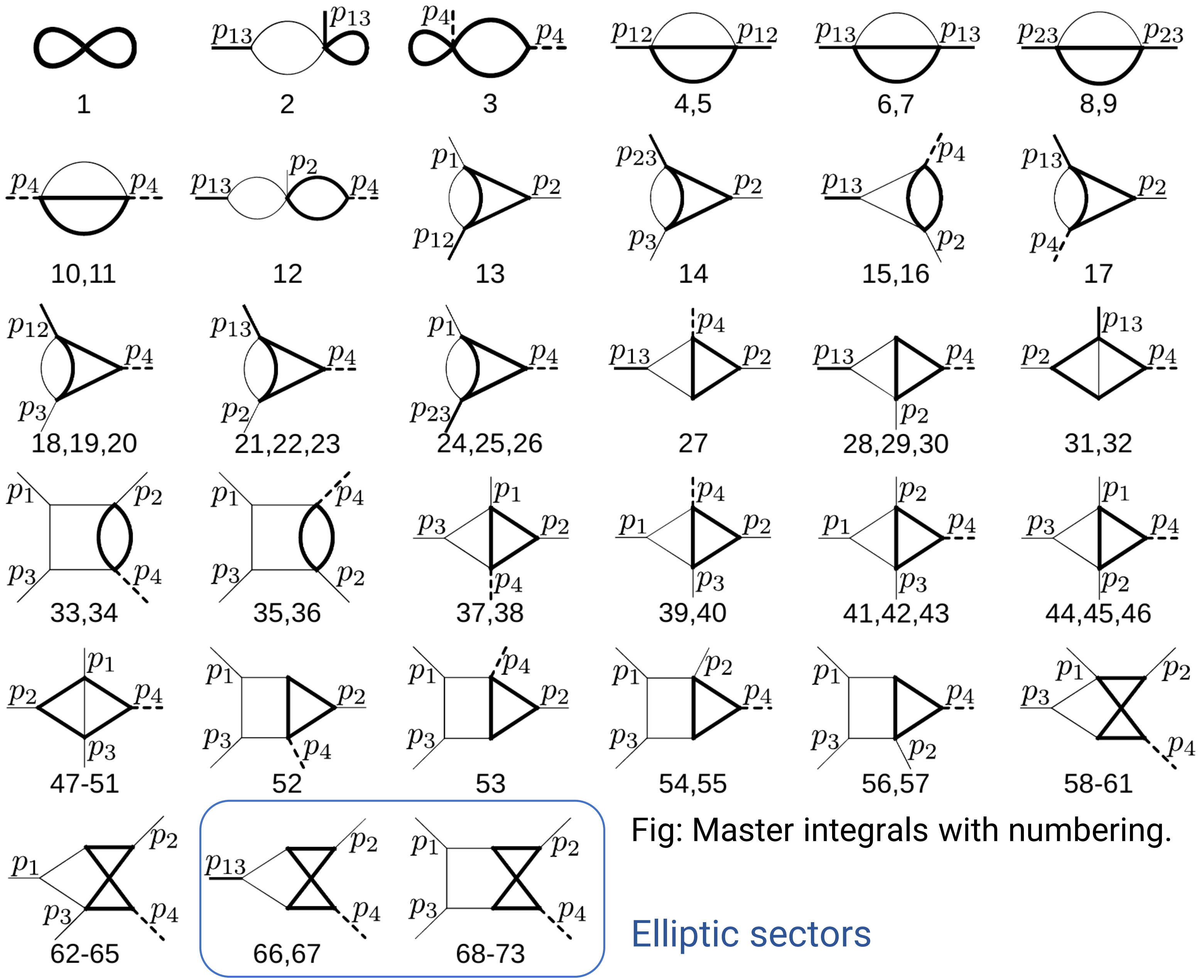


Fig: Master integrals with numbering.

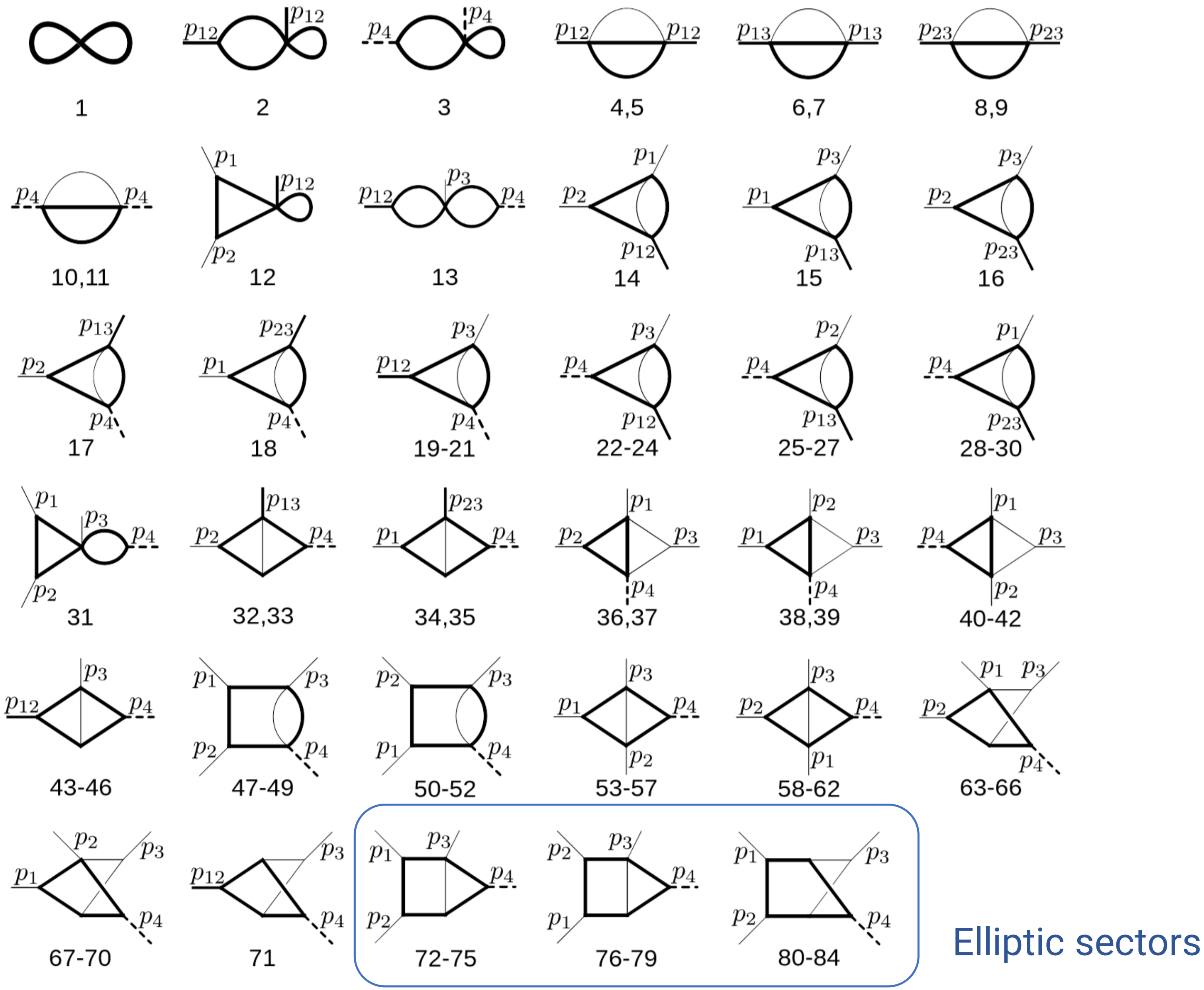
Elliptic sectors

Family G

Master integrals

- IBP-reduction:
  - 84 master integrals
  - Default FIRE basis: $\mathcal{O}$(1 GB)
  - More suitable (pre-canonical) basis: $\mathcal{O}$(100 MB)
  - Possible using either FIRE or KIRA
- Differential eqns: $\mathcal{O}$(10 MB)

  [Bonciani et al, 1911.06308]



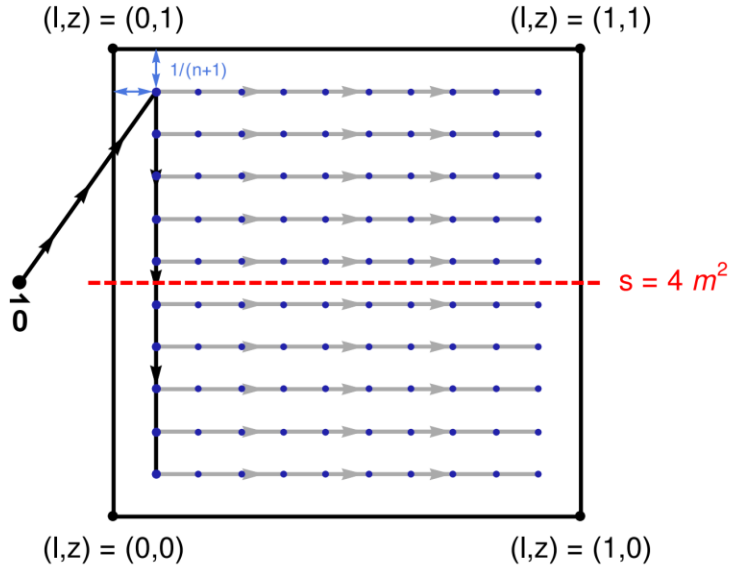
Higgs + jet integrals

[1907.13156, 1911.06308]

- We can obtain 3-dimensional plots, if we sample enough points. Consider the parametrization:

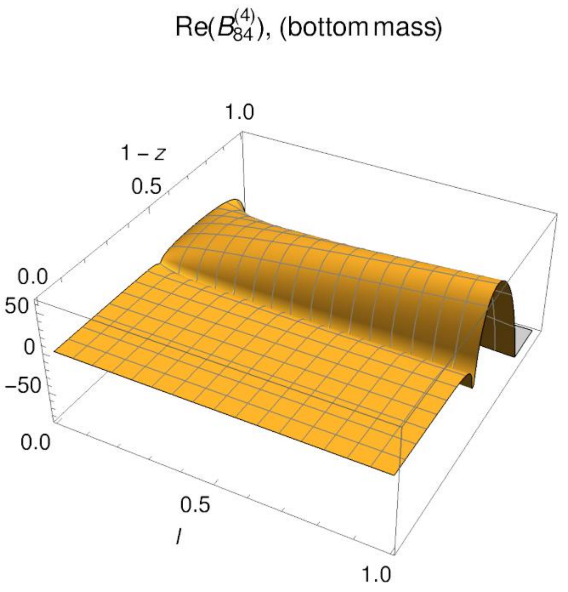
$$\text{top} \quad (l,z)_t: \quad s = \frac{87-74z}{25z}, \qquad t = \frac{87\,l\,(z-1)}{25z}, \qquad p_4^2 = \frac{13}{25},$$
$$\text{bottom} \quad (l,z)_b: \quad s = \frac{323761}{361z}, \qquad t = \frac{323761\,l\,(z-1)}{361z}, \qquad p_4^2 = \frac{323761}{361}.$$

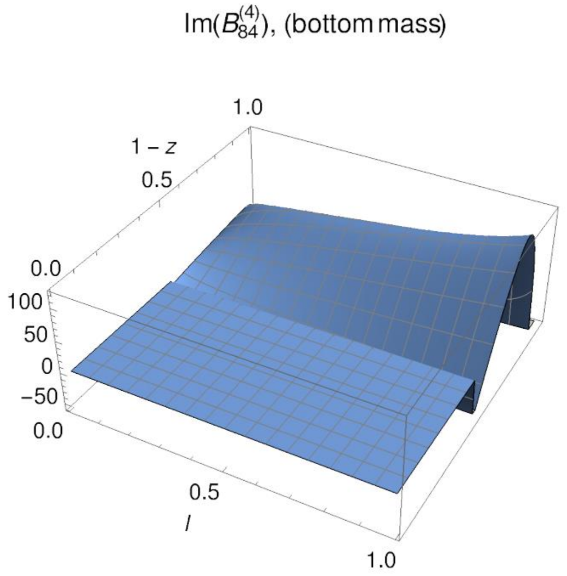
- Which maps the physical regions of the top quark and bottom quark contributions to the unit square:

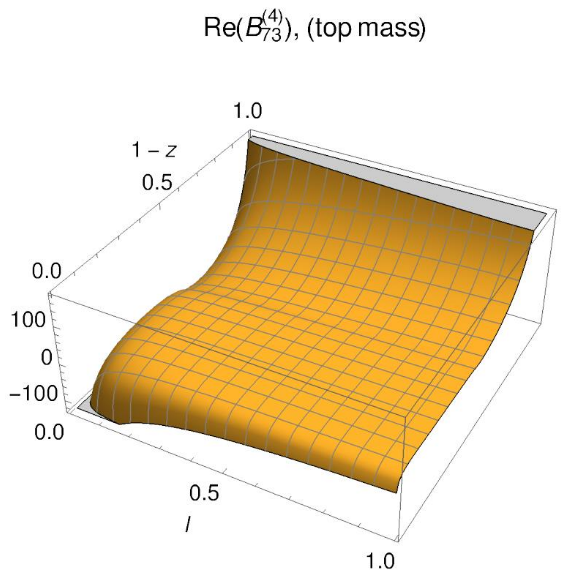


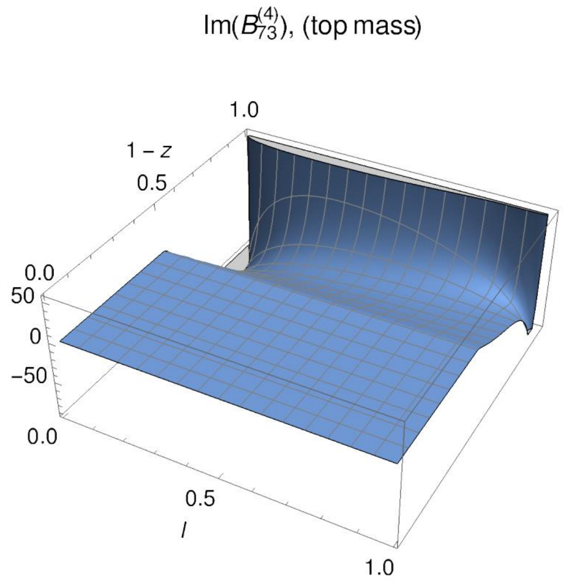
Higgs + jet integrals

Plots sampled from 10000 points on an evenly spaced grid.









Family G

Family F

3-loop banana graph

- First, we consider the equal-mass case:

$$I^{\text{banana}}_{a_1a_2a_3a_4} = \left(\frac{e^{\gamma_E \epsilon}}{i\pi^{d/2}}\right)^3 (m^2)^{a-\frac{3}{2}(2-2\epsilon)} \left(\prod_{i=1}^{4} \int d^d k_i\right) D_1^{-a_1} D_2^{-a_2} D_3^{-a_3} D_4^{-a_4}$$

$$D_1 = -k_1^2 + m^2\,, \quad D_2 = -k_2^2 + m^2\,, \quad D_3 = -k_3^2 + m^2\,, \quad D_4 = -(k_1+k_2+k_3+p_1)^2 + m^2$$

- The differential equations are given by:

$$\vec{B}^{\text{banana}} = (\epsilon I^{\text{banana}}_{2211}\,, \epsilon(1+3\epsilon) I^{\text{banana}}_{2111}\,, \epsilon(1+3\epsilon)(1+4\epsilon) I^{\text{banana}}_{1111}\,, \epsilon^3 I^{\text{banana}}_{1110}\,)$$

$$\partial_t \vec{B}^{\text{banana}} = \begin{pmatrix} -\frac{64-2t+t^2+(8+t)^2\epsilon}{t(t-16)(t-4)} & \frac{2(t+20)(2\epsilon+1)}{t(t-16)(t-4)} & -\frac{6(2\epsilon+1)}{t(t-16)(t-4)} & -\frac{2\epsilon}{t(t-16)} \\ \frac{3t(3\epsilon+1)}{t(t-4)} & -\frac{2(t+8)\epsilon+t+4}{t(t-4)} & \frac{3\epsilon+1}{t(t-4)} & 0 \\ 0 & \frac{4(4\epsilon+1)}{t} & \frac{-3\epsilon-1}{t} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{B}^{\text{banana}}$$

- With $t = p_1^2/m^2$

3-loop banana graph

- We need to obtain boundary conditions
- The Feynman parametrization is given by:

$$I_{1111}^{\text{banana}} = ie^{3\gamma\epsilon}\Gamma(3\epsilon+1)\left(m^2\right)^{-3\epsilon-1} x^{3\epsilon+1}\int_\Delta d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 \left(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_4\alpha_3+\alpha_2\alpha_4\alpha_3+\right.$$
$$\left.\alpha_1\alpha_2\alpha_4\right)^{4\epsilon}\left(\alpha_2\alpha_3\alpha_1^2 x+\alpha_2\alpha_4\alpha_1^2 x+\alpha_3\alpha_4\alpha_1^2 x+\alpha_2\alpha_3^2\alpha_1 x+\alpha_2\alpha_4^2\alpha_1 x+\alpha_3\alpha_4^2\alpha_1 x+\right.$$
$$\alpha_2^2\alpha_3\alpha_1 x+\alpha_2^2\alpha_4\alpha_1 x+\alpha_3^2\alpha_4\alpha_1 x+4\alpha_2\alpha_3\alpha_4\alpha_1 x+\alpha_2\alpha_3\alpha_4^2 x+\alpha_2\alpha_3^2\alpha_4 x+\alpha_2^2\alpha_3\alpha_4 x+$$
$$\left.+\alpha_2\alpha_3\alpha_4\alpha_1\right)^{-3\epsilon-1}$$

- Where we let $t = -1/x$.
- We will compute boundary conditions in the limit $x \to 0$, which is equivalent to the limit where the mass vanishes.

3-loop banana graph

- We use the method of expansions by regions and the program `asy` to obtain the regions.
- The regions are given by:

$$\begin{aligned} &R_1 = \{0,-1,-1,-1\}\,, && R_2 = \{0,-1,-1,0\}\,, && R_3 = \{0,0,0,0\}\,,\\ &R_4 = \{0,0,0,-1\}\,, && R_5 = \{0,1,1,0\}\,, && R_6 = \{0,0,1,0\}\,,\\ &R_7 = \{0,-1,0,-1\}\,, && R_8 = \{0,-1,0,0\}\,, && R_9 = \{0,0,0,1\}\,,\\ &R_{10} = \{0,1,1,1\}\,, && R_{11} = \{0,0,1,1\}\,, && R_{12} = \{0,1,0,0\}\,,\\ &R_{13} = \{0,0,-1,-1\}\,, && R_{14} = \{0,1,0,1\}\,, && R_{15} = \{0,0,-1,0\}\,. \end{aligned}$$

- Their contributions work out to:

$$\begin{aligned} &\mathrm{I}^{R_1}_{1111} \sim x e^{3\gamma\epsilon}\Gamma(\epsilon)^3\,, && \mathrm{I}^{R_2}_{1111} \sim \frac{e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)}\,, && \mathrm{I}^{R_3}_{1111} \sim \frac{3e^{3\gamma\epsilon}\epsilon x^{3\epsilon+1}\Gamma(-\epsilon)^4\Gamma(3\epsilon)}{\Gamma(-4\epsilon)}\,,\\ &\mathrm{I}^{R_4}_{1111} \sim \frac{2e^{3\gamma\epsilon}\epsilon x^{2\epsilon+1}\Gamma(-\epsilon)^3\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(-3\epsilon)}\,, && \mathrm{I}^{R_5}_{1111} \sim \frac{e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)}\,, && \mathrm{I}^{R_6}_{1111} \sim x e^{3\gamma\epsilon}\Gamma(\epsilon)^3\,,\\ &\mathrm{I}^{R_7}_{1111} \sim \frac{e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)}\,, && \mathrm{I}^{R_8}_{1111} \sim \frac{2e^{3\gamma\epsilon}\epsilon x^{2\epsilon+1}\Gamma(-\epsilon)^3\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(-3\epsilon)}\,, && \mathrm{I}^{R_9}_{1111} \sim x e^{3\gamma\epsilon}\Gamma(\epsilon)^3\,,\\ &\mathrm{I}^{R_{10}}_{1111} \sim \frac{2e^{3\gamma\epsilon}\epsilon x^{2\epsilon+1}\Gamma(-\epsilon)^3\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(-3\epsilon)}\,, && \mathrm{I}^{R_{11}}_{1111} \sim \frac{e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)}\,, && \mathrm{I}^{R_{12}}_{1111} \sim x e^{3\gamma\epsilon}\Gamma(\epsilon)^3\,,\\ &\mathrm{I}^{R_{13}}_{1111} \sim \frac{e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)}\,, && \mathrm{I}^{R_{14}}_{1111} \sim \frac{e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)}\,, && \mathrm{I}^{R_{15}}_{1111} \sim \frac{2e^{3\gamma\epsilon}\epsilon x^{2\epsilon+1}\Gamma(-\epsilon)^3\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(-3\epsilon)} \end{aligned}$$

3-loop banana graph

- Summing over all contributions, we obtain the following result:

$$I_{1111}^{\text{banana}} \overset{x\downarrow 0}{\sim} \frac{6e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)} + \frac{8e^{3\gamma\epsilon}\epsilon x^{2\epsilon+1}\Gamma(-\epsilon)^3\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(-3\epsilon)} + \frac{3e^{3\gamma\epsilon}\epsilon x^{3\epsilon+1}\Gamma(-\epsilon)^4\Gamma(3\epsilon)}{\Gamma(-4\epsilon)} + 4xe^{3\gamma\epsilon}\Gamma(\epsilon)^3 + \mathcal{O}(x^2)\,.$$

$$I_{1110}^{\text{banana}} = e^{3\gamma\epsilon}\Gamma(\epsilon)^3$$

- Next, we show how to obtain results for any values of p^2 using DiffExp

DiffExp

- Typical usage of the package:
  - Set configuration options using the method `LoadConfiguration[opts_]`
  - Prepare a list of boundary conditions using `PrepareBoundaryConditions[bcs_, line_]`
  - Then we can find series solutions along a line using the function:

    ```
    IntegrateSystem[bcsprepared_, line_]
    ```

  - Or one can transport the boundary conditions to a new point using:

    ```
    TransportTo[bcsprepared_, point_]
    ```

3-loop banana graph

- Load DiffExp:

```
Get[FileNameJoin[{NotebookDirectory[], "..", "DiffExp.m"}]];
```

```
Loading DiffExp version 1.0.2
Author: Martijn Hidding. Email: hiddingm@tcd.ie.
```

- Set the configuration options and load the matrices

```
EqualMassConfiguration = {
   DeltaPrescriptions → {t - 16 + I δ},
   MatrixDirectory → NotebookDirectory[] <> "Banana_EqualMass_Matrices/",
   UseMobius → True, UsePade → True
  };
```

```
LoadConfiguration[EqualMassConfiguration];
```

```
DiffExp: Loading matrices.
DiffExp: Found files: {dt_0.m, dt_1.m, dt_2.m, dt_3.m, dt_4.m}
DiffExp: Kinematic invariants and masses: {t}
DiffExp: Getting irreducible factors..
DiffExp: Configuration updated.
```

3-loop banana graph

- Prepare the boundary conditions along an asymptotic limit:

```
EqualMassBoundaryConditions = {
    "?",
    "?",
    ϵ (1 + 3 ϵ) (1 + 4 ϵ) (-(4 ⅇ^(3 EulerGamma ϵ) Gamma[ϵ]^3)/t + (6 ⅇ^(3 EulerGamma ϵ) (-1/t)^(1+ϵ) ϵ Gamma[-ϵ]^2 Gamma[ϵ]^3)/Gamma[-2 ϵ] +
      (8 ⅇ^(3 EulerGamma ϵ) (-1/t)^(1+2 ϵ) ϵ Gamma[-ϵ]^3 Gamma[ϵ] Gamma[2 ϵ])/Gamma[-3 ϵ] + (3 ⅇ^(3 EulerGamma ϵ) (-1/t)^(1+3 ϵ) ϵ Gamma[-ϵ]^4 Gamma[3 ϵ])/Gamma[-4 ϵ]),
    ⅇ^(3 EulerGamma ϵ) ϵ^3 Gamma[ϵ]^3
  } // PrepareBoundaryConditions[#, <|t → -1/x|>] &;
```

```
DiffExp: Integral 1: Ignoring boundary conditions.
DiffExp: Integral 2: Ignoring boundary conditions.
DiffExp: Assuming that integral 3 is exactly zero at epsilon order 0.
DiffExp: Prepared boundary conditions in asymptotic limit, of the form:
```

	?	?	?	?	?
	?	?	?	?	?
DiffExp:	$O[x]^{51}$	$(\ldots)\, x + O[x]^{3/2}$	$(\ldots)\, x + O[x]^{3/2}$	$(\ldots)\, x + O[x]^{3/2}$	$(\ldots)\, x + O[x]^{3/2}$
	$(\ldots) + \sqrt{O[x]}$	$\sqrt{O[x]}$	$(\ldots) + \sqrt{O[x]}$	$(\ldots) + \sqrt{O[x]}$	$(\ldots) + \sqrt{O[x]}$

3-loop banana graph

- Next, we transport the boundary conditions:

```
Transport1 = TransportTo[EqualMassBoundaryConditions, <|t → -1|>];
Transport2 = TransportTo[Transport1, <|t → x|>, 32, True];
```

DiffExp: Transporting boundary conditions along $\langle\left| t \to -\frac{1.}{x} \right|\rangle$ from x = 0. to x = 1.

DiffExp: Preparing partial derivative matrices along current line..

DiffExp: Determining positions of singularities and branch-cuts.

DiffExp: Possible singularities along line at positions {0.}.

DiffExp: Analyzing integration segments.

DiffExp: Segments to integrate: 3.

DiffExp: Integrating segment: $\langle\left| t \to \frac{8.\,(-1. + 1.\,x)}{x} \right|\rangle$.

DiffExp: Integrated segment 1 out of 3 in 20.8565 seconds.

DiffExp: Evaluating at x = 0.0625

DiffExp: Current segment error estimate: 5.14483×10^{-31}

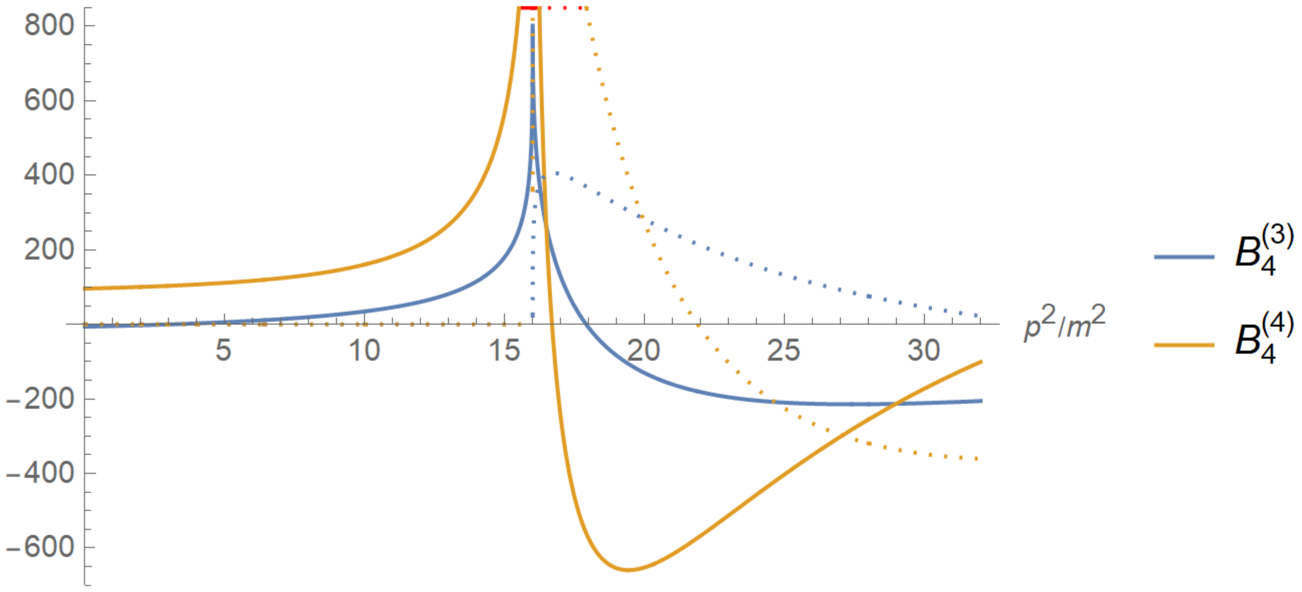
DiffExp: Total error estimate: 5.14483×10^{-31}

DiffExp: Integrating segment: $\langle\left| t \to \frac{-1. + 1.\,x}{\ldots} \right|\rangle$

3-loop banana graph

- Lastly, we plot the result:

```
ResultsForPlotting = ToPiecewise[Transport2];
Quiet[ReImPlot[{ResultsForPlotting[[3, 4]][x], ResultsForPlotting[[3, 5]][x]}, {x, 0, 32},
  ClippingStyle → Red, PlotLegends → {"B4^(3)", "B4^(4)"}, AxesLabel → {"p^2/m^2"}, PlotRange → {-700, 850},
  MaxRecursion → 15, WorkingPrecision → 100]]
```



3-loop banana graph

- Timing:
  - Moving from $p^2 = -\infty$ to $p^2 = 30$ at a precision of 25 digits takes about 90s, where we computed the top sector integrals up and including order ϵ^3.
  - Moving from $p^2 = -\infty$ to $p^2 = 30$ at a precision of 100 digits takes a bit under 20m, where we computed the top sector integrals up and including order ϵ^3.
  - Obtaining 100+ digits at $p^2 = -100$ up to and including order ϵ^0 takes about 50s.
  - Obtaining 100+ digits at $p^2 = -100$ up to and including order ϵ^3 takes about 2.5 min.

- $B_3^{(k)}$:
```
0
4.08241320270405960780199146104509733985501253774222434496563798314848283907330199489603248642178129
-0.7713150915227857546258559692543676298350939151980774607908277236769934490973612004866036340787026038
-15.52268532416518855576696548019433617730937578226039207428302008586262767404183548619606743796239099
78.12509728148001692986790482079302619114776011817121195506011258285334682242128391076363566162968586
```

3-Loop banana graph

- In the unequal mass case we choose the basis:

$$\vec{B}^{\text{banana}} = \left\{ \begin{array}{l} \epsilon I^{\text{banana}}_{1122},\ \epsilon I^{\text{banana}}_{1212},\ \epsilon I^{\text{banana}}_{1221},\ \epsilon I^{\text{banana}}_{2112},\ \epsilon I^{\text{banana}}_{2121},\ \epsilon I^{\text{banana}}_{2211}, \\ \epsilon(1+3\epsilon) I^{\text{banana}}_{1112},\ \epsilon(1+3\epsilon) I^{\text{banana}}_{1121},\ \epsilon(1+3\epsilon) I^{\text{banana}}_{1211}, \\ \epsilon(1+3\epsilon) I^{\text{banana}}_{2111},\ \epsilon(1+3\epsilon)(1+4\epsilon) I^{\text{banana}}_{1111}, \\ \epsilon^3 I^{\text{banana}}_{0111},\ \epsilon^3 I^{\text{banana}}_{1011},\ \epsilon^3 I^{\text{banana}}_{1101},\ \epsilon^3 I^{\text{banana}}_{1110} \end{array} \right\}$$

- The unequal mass case is significantly more difficult to compute for DiffExp, due to the presence of 11 coupled master integrals.
- The series expansions grow wildly at intermediate stages of the calculations, which puts the linear algebra routines off track.
- Therefore we must work at a high working precision (1000+)

3-Loop banana graph

- We provide 55 digits of basis integral B_{11} below, in the point

$$(p^2 = 50, m_1^2 = 2, m_2^2 = 3/2, m_3^2 = 4/3, m_4^2 = 1)$$

$$\begin{aligned}
B_{11}^{(0)} &= 0 \\
B_{11}^{(1)} &= 5.1972521136965043170129578538563652405618939122389078645 \\
&\quad + i\, 6.8755169535390207501370685645538902299559024551830956594 \\
B_{11}^{(2)} &= -17.958010811209406089952336169892847894878068705389907573 3 \\
&\quad + i\, 31.743670363369309090840293229901197191350895064949423104 7 \\
B_{11}^{(3)} &= -121.51011520681775652033928075412160849628807729083063706 68 \\
&\quad - i\, 40.769076236020276645377599991717222653742825852914575474 6 \\
B_{11}^{(4)} &= 125.61133880236055347455937640047989582321186326812570739 23 \\
&\quad - i\, 229.92002571723885899520627575712151768344717834951127550 27
\end{aligned}$$

- This point can be obtained in about 23 min.

4-Loop banana graph

- We can also compute higher loop banana graphs.

$$I^{\text{banana}}_{a_1 a_2 a_3 a_4} = \left(\frac{e^{\gamma_E \epsilon}}{i\pi^{d/2}}\right)^4 \left(m^2\right)^{a-2(2-2\epsilon)} \left(\prod_{i=1}^{5} \int d^d k_i\right) D_1^{-a_1} D_2^{-a_2} D_3^{-a_3} D_4^{-a_4} D_5^{-a_5}$$

$$D_1 = -k_1^2 + m^2, \quad D_2 = -k_2^2 + m^2, \quad D_3 = -k_3^2 + m^2$$
$$D_4 = -k_4^2 + m^2, \quad D_5 = -\left(k_1 + k_2 + k_3 + k_4 + p_1\right)^2 + m^2$$

- We consider the following basis of masters:

$$B_1 = \epsilon \mathrm{I}^{\text{banana}}_{11222}$$
$$B_2 = \epsilon(2\epsilon + 1)\mathrm{I}^{\text{banana}}_{11122}$$
$$B_3 = \epsilon(2\epsilon + 1)(4\epsilon + 1)\mathrm{I}^{\text{banana}}_{11112}$$
$$B_4 = \epsilon(2\epsilon + 1)(4\epsilon + 1)(5\epsilon + 1)\mathrm{I}^{\text{banana}}_{11111}$$
$$B_5 = \epsilon^4 \mathrm{I}^{\text{banana}}_{11110}$$

4-Loop banana graph

- We may find boundary conditions by imposing the vanishing of non-physical singularities.
- This allows a determination of the integrals completely from the differential equations, without any need for asymptotic expansions.
- This follows the approach of
  - [Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser, 1809.06240]
  - [Abreu, Ita, Page, Tschernow, Zeng, 2005.04195]
- First, we need to provide an overall normalization for the basis. This is provided by the tadpole integral which is equal to: $I_{11110}^{\text{banana}} = e^{4\gamma\epsilon}\epsilon^4\Gamma(\epsilon)^4$

4-Loop banana graph

- Then we do the following:
  - We compute the general solution of the top sector integrals at $t = 0$.
  - The expansions contain powers of logarithms, we set their coefficients to zero, which solves some of the indeterminate constants.
  - Next, we transport and center an expansion at $t = 1$. There are again logarithms in the expansions, and we set their coefficients to zero.
  - We repeat this a final time and get rid of a non-physical singularity at $t = 9$.

- Only the physical singularity at $t = 25$ remains at the end and all coefficients are fixed

4-Loop banana graph

- Our original expansion was centered at $t = 0$, where we now find the results:

0	0.5626161626035411	0.3475481638835365	1.911555944481455	0.2718352134528369
0	1.923605373745244	0.6752648394943755	6.876325052991839	−1.339280364786555
0	7.989117602399249	10.23095239518146	7.206853721986161	86.05832181407076
0	39.94558801199625	91.10034998790354	−72.59332146214688	900.8362379685953
1.000000000000000	0	3.289868133696453	−1.602742537546126	6.493939402266829

- Or, moving to the point $t = 50$, we have:

0	−0.127301395 + 0.060055594 i	0.44510796 + 0.05196919 i	−3.6356361 + 2.3160967 i	−3.337360 − 16.053530 i
0	−0.6629555 + 1.5090835 i	−4.6446551 − 6.2746996 i	24.255982 − 2.263457 i	−9.971355 + 57.386664 i
0	4.6398127 + 9.9513277 i	−41.998524 + 31.366660 i	−130.68313 − 120.45461 i	327.35602 − 310.64441 i
0	50.934939 + 28.927728 i	77.63780 + 297.73512 i	−952.72123 + 736.44670 i	−1977.1722 − 1153.1457 i
1.0000000	0	3.2898681	−1.6027425	6.4939394

Application to cut integrals

- As a last example we briefly consider the computation of cut integrals
- Cut integrals are typically very difficult to compute.
- But luckily, families of cut Feynman integrals satisfy the same differential equations as uncut integrals!
- Thus we may compute cuts directly from differential equations, as long as we have suitable boundary points.

Conclusion

- Series expansion methods provide an efficient way to evaluate Feynman integrals
- Allows for obtaining high-precision numerical results for beyond elliptic type integrals
- Public Mathematica package DiffExp can be used to apply these methods

Thank you for listening!