Analytic Structure of all Loop Banana Amplitudes

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based on [1]=arXiv:1912.06201v2 and [2]=arXiv:2008.10574v1

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Introduction:

Consider the Feynman representation of the graph contribution in terms of an integral I over the graph polynomials $\mathcal{U}(\underline{x})$ and $\mathcal{G}(\underline{x};\underline{K},\underline{M})$, $\underline{K}$ independent momenta, $\underline{M}$ masses:

$$\mathcal{F}_{\sigma_{n-1}}(\underline{K},\underline{M}) = \int_{\sigma_{n-1}} \mu_{n-1} \frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{G}^{\omega}} \prod_i x_i^{\nu_i - 1}$$

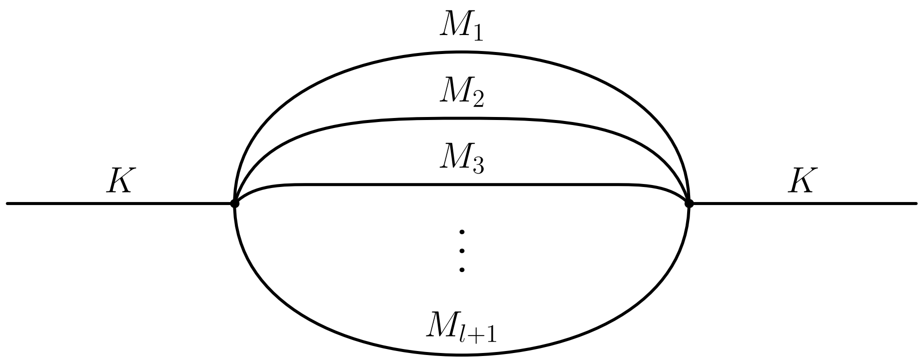
- D dimension, n # of edges, ν_i their multiplicity,
- $\omega = \sum_{i=1}^{n} \nu_i - lD/2$, l # of loops, $v = n - l + \chi - 1$,

- $\sigma_{n-1} = \{[x_1 : \ldots : x_n] \in \mathbb{P}^{n-1} | x_i \in \mathbb{R}_{\geq 0} \forall 1 \leq i \leq n\}$ an open domain,
- $\mu_{n-1} = \sum_{k=1}^{n} (-1)^{k+1} x_k \mathrm{d}x_1 \wedge \ldots \wedge \widehat{\mathrm{d}x_k} \wedge \ldots \wedge \mathrm{d}x_n$ measure on $\mathbb{P}^{n-1}$.

Generally, one needs dimensional regularization: Therefore, one evaluates in $D = 4 - 2\epsilon$ dimensions and seeks for a Laurent expansion $I = \frac{I_{-1}}{\epsilon} + I_0 + \ldots$. The Laurent coefficients are known to be (twisted) periods [Bogner, Weinzierl], potentially general Calabi-Yau motives realized in minimally $l - 1$ dimensions.

Banana Amplitudes:

A very simple series of such Feynman amplitudes with increasing loop order are the Banana diagrams in $D = 2$:



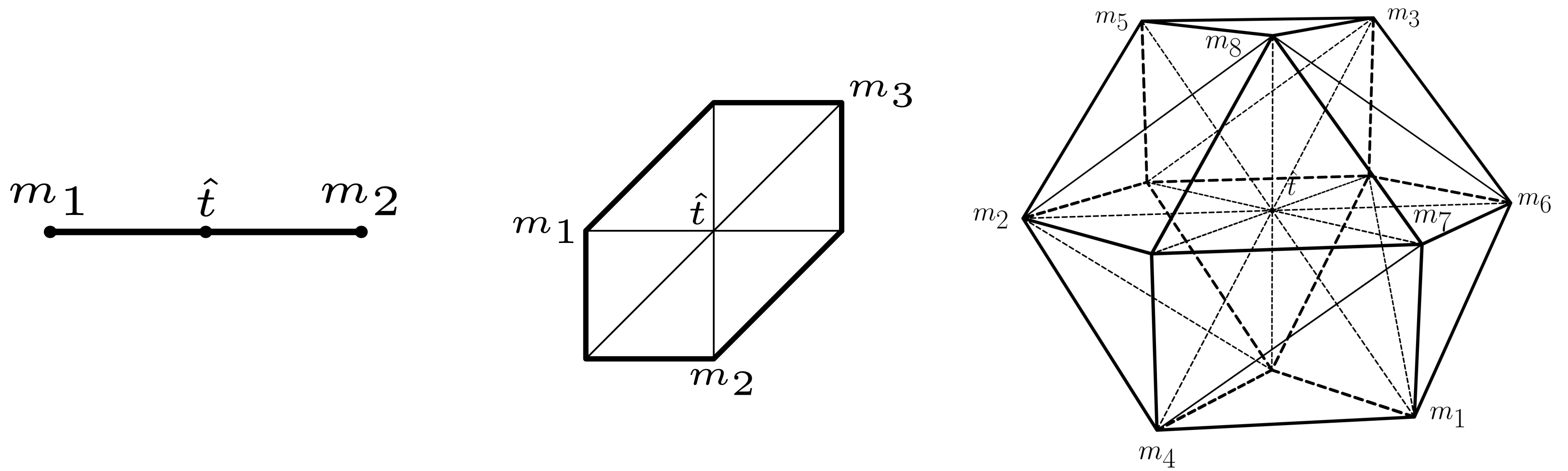
The l-loop banana diagram with external momentum K and internal masses M_i.

Since $\omega = 1$ $(\chi = 2)$ the exponent of $\mathcal{U}$ vanishes and the exponent of $\mathcal{F}$ becomes one.

With $n = l + 1$ this topology leads in dimensionless variables $t = \frac{K^2}{\mu^2}$, $\xi_i = \frac{M_i}{\mu}$ to the (relative-) period integral

$$\begin{aligned}\mathcal{F}_{\sigma_l}(t,\xi_i) &= \int_{\sigma_l} \frac{\mu_l}{P_l(t,\xi_i;x)} \\ &= \int_{\sigma_l} \frac{\mu_l}{\left(t - \left(\sum_{i=1}^{l+1} \xi_i^2 x_i\right)\left(\sum_{i=1}^{l+1} x_i^{-1}\right)\right)\prod_{i=1}^{l+1} x_i} .\end{aligned}$$

The Newton polytope Δ_l of $P_{\Delta_l} = P_l / \prod_{i=1}^{l+1} x_i$ in the canonical rank l-lattice is reflexive. For example the Δ_l for $l = 1, 2, 3$ ($t = \hat{t} + \sum_i \xi_i^2$, $M_k(m_k)$) look like:



The reflexive pair of polytopes (Δ_l, Δ_l^*) gives rise to a mirror pair of Calabi-Yau manifolds (M, W) as sections of the anti–canonical bundle in the toric projective varieties $(\mathbb{P}_{\Delta_l^*}, \mathbb{P}_{\Delta_l})$:

$$M_{l-1} = \{P_{\Delta_l} \subset \mathbb{P}_{\Delta_l^*}\} \overset{\text{Mirror Sym.}}{\longleftrightarrow} \{P_{\Delta_l^*} \subset \mathbb{P}_{\Delta}\} = W_{l-1} \; .$$

The Feynman integral $\mathcal{F}_{\sigma_l}$ is a (relative) period over a real $(l-1)$-cycle in $M \subset \mathbb{P}_{\Delta_l^*}$. Given that fact, there are in principle straightforward steps to calculate it as performed for $l \leq 3$ in [1] and $l = 4$ in [2]:

- Determine the flat Gauss-Manin connection on M_{l-1} and solve for the periods $\Pi_k = \int_{\Gamma_k} \Omega(t, \underline{\xi})$ with Γ_k a basis of $H_{l-1}(M_{l-1}, \mathbb{Z})$, $\Omega \in H^{l-1,0}(M)$ the unique Calabi-Yau form.

- This first task is best done by starting with the Gelfand-Kapranov-Zelevinskĭ (GKZ) differential system of $\mathbb{P}_{\Delta_l}$ and deriving the Picard-Fuchs differential D-module $\{\widehat{\mathcal{D}}\}$, i.e. a complete set of linear differential operators with $\widehat{\mathcal{D}}_m \Pi_k = 0$, $k = 1, \ldots, h_{l-1}$, $m = 1, \ldots, |\{\widehat{\mathcal{D}}\}|$ and no further solutions, from it.

- Typically this leads first to a local Frobenius basis $\widehat{\varpi}_k$ for $k = 1, \ldots, h_{l-1}$ of solutions and one has to determine the linear combinations which correspond to integrals over actual geometrical cycles. This is achieved using the $\widehat{\Gamma}(TM_{l-1})$-class at the MUM point.

- By commuting the deformation modulus of the inner point through the ideal $\{\widehat{\mathcal{D}}\}$ we get at the MUM point in the GKZ coordinates z_k linear inhomogenous differential equations for the chain integral of the form [1]:

$$\mathcal{D}_m \varpi_k = \sum_{i=1}^{h^{l-2,1}} a_i^{(k)} \log(z_i)$$

for $a_i^{(k)} =$ const. and $k = 1, \ldots |\{\widehat{\mathcal{D}}\}|$.
In general, the inhomogeniety has the form $\sum_i p_i(\underline{z}) q_i(\underline{\log(z)})$ with p_i, q_i polynomials [2].

- The homogeneous solutions to this system are in one-to-one correspondence to the Π_k and can be linearly combined to get the maximal cut integral.

- Adding the single inhomogenous solution $\varpi_{h_{l-1}+1}$ we find the Feynman amplitude as

$$\mathcal{F}_{\sigma_l}(t,\xi_i) = \sum_k \lambda_k^{(l),\,loc} \varpi_k^{loc}(t,\xi_i) \qquad \text{with} \;\; \lambda_k^{(l),\,loc} \in \mathbb{C}$$

everywhere in its physical moduli in terms of fast convergent local Frobenius bases $\varpi_k^{loc}(t,\xi_i)$ as well as connection matrices $\lambda_{ik}^{(l),\,loc}$ that characterise the Feynman amplitude.

For $l = 2$ similar solutions where found using Griffiths reduction methods in [S. Bloch, M. Kerr and P. Vanhove (2016)].

However the are shortcomings in this geometric description:

- The generic deformation space of the geometrical model $\dim\left(H^1(M_{l-1}, T_{M_{l-1}})\right) = h^{l-2,1} = l^2$ growths faster than the number of physical parameters $\sim l + 1$.
- The relevant periods are over an even tinier piece of the middle (co)homology of M_{l-1} .
- Therefore the derivations of the physical relevant periods

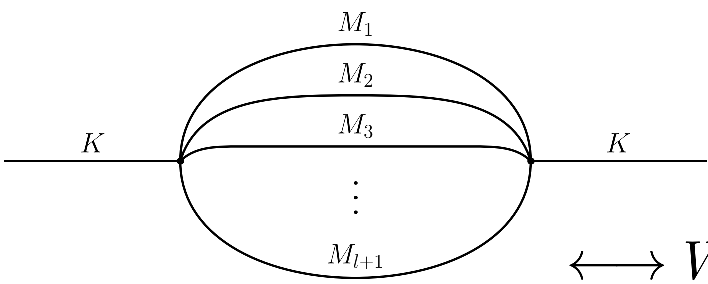
becomes more and more cumbersome.

- Moreover the cofficients $\lambda_k^{(l),\,loc}$ can only determined approximately by numerical analytic continuation.

To overcome these problems we propose a better Calabi-Yau motive for the Banana graphs and a new $\widehat{\Gamma}$-class technique that is inspired from homological mirror symmetry [2].

A better Calabi-Yau motive:

Is the primitive vertical quantum cohomology of $W_{l-1} \subset F_l$, the compact Banana CY-manifolds:



$$\longleftrightarrow W^l_{l-1} = \left(\begin{array}{c||cc} \mathbb{P}^1_1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ \mathbb{P}^1_{l+1} & 1 & 1 \end{array} \right) \subset \left(\begin{array}{c||c} \mathbb{P}^1_1 & 1 \\ \vdots & \vdots \\ \mathbb{P}^1_{l+1} & 1 \end{array} \right) = F_l \,,$$

Here W_{l-1} is the complete intersection Calabi-Yau space of two degree $(1, \ldots, 1)$ constraints $P_i = 0$, $i = 1, 2$ in $\mathbb{P}^1_1 \times \ldots \times \mathbb{P}^1_{l+1} = A_{l+1}$, while the relevant Fano ambient space F_l of W_{l-1} is the hypersurface of degree $(1, \ldots, 1)$ in A_{l+1}.

Note, that in the high energy regime of the Feynman amplitude we get an one-to-one asymptotic identification of the complexified (large volume) Kähler parameters of the $l+1$ rational curves $\mathbb{P}^1_k$ with the ratios of the mass squares by K^2

$$\mathfrak{t}^k \simeq \frac{1}{2\pi i}\int_{\mathbb{P}^1_k}(i\omega - b) + \mathcal{O}(e^{-t^k}) = \frac{\log}{2\pi i}\left(\frac{M_k^2}{K^2}\right) = \frac{\log(z_k)}{2\pi i}$$

for $k = 1, \ldots, l+1$. Away from the limit the mirror construction of [Hosono, Klemm, Theisen, and Yau (1993)] for complete intersections and the associated GKZ system provides the exact answer, including the exponentially suppressed $\mathcal{O}(e^{-t^k})$ instanton corrections, for the periods over the horizontal homology of M_{l-1} in terms of the non redundant complex parameters z_k. The latter provide a modular invariant parametrization.

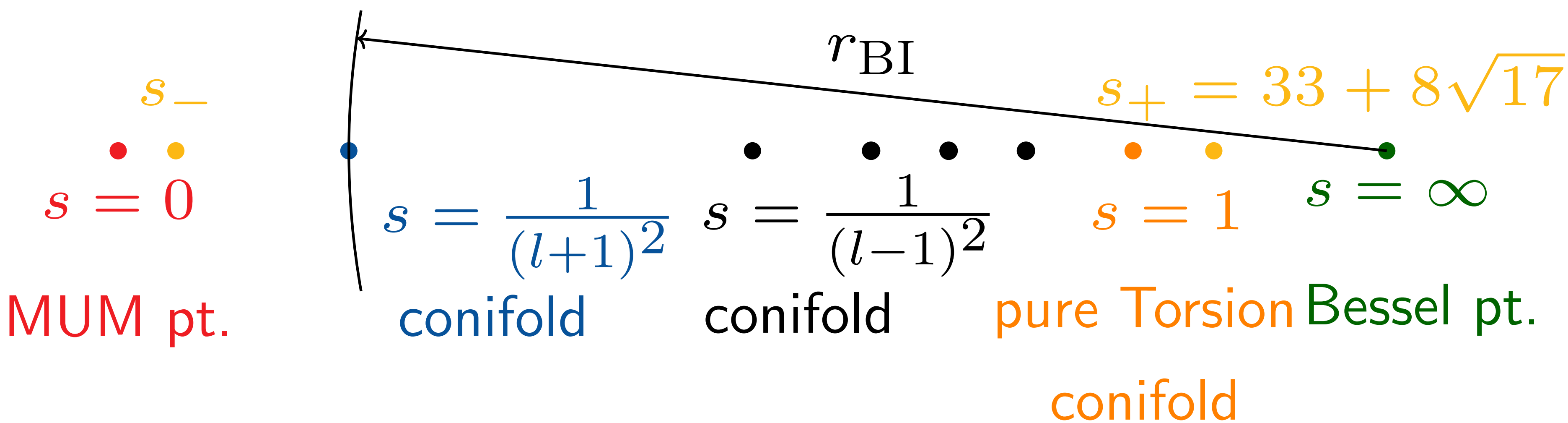
Fiberation Structure:

$E = \left(\begin{array}{c} \mathbb{P}^1_1 \\ \mathbb{P}_2 \\ \mathbb{P}^1_3 \end{array} \middle\| \begin{array}{cc} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{array} \right)$ is the elliptic curve associated to the two-loop graph. The K3 associated to the three-loop graph $K_3 = \left(\begin{array}{c} \mathbb{P}^1_1 \\ \mathbb{P}^1_2 \\ \mathbb{P}^1_3 \\ \mathbb{P}^1_4 \end{array} \middle\| \begin{array}{cc} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{array} \right)$ is fibered in four ways by E over each of its $\mathbb{P}^1_k$. The K3 fibres the Calabi-Yau threefold W_3 in five ways and so on. This fiberation structure reflects the fact that the masses M_k can be sent in arbitrary order to infinity to obtain the lower loop graphs. In these limits the contributions of the decompactified base $\mathbb{P}^1$ are exponentially suppressed leaving previously encountered special functions, like elliptic dilogarithm, in the fiber. The intersection calculations done for the $\widehat{\Gamma}$ latter reveals also that the Oguiso criteria for fiberations are fulfilled.

Analytic structure:

Roadmap to the physical moduli space:

$$s = 1/t \in \mathcal{M}_{cs}(M_{l-1}) = \mathbb{P}^1 \setminus \left(\bigcup_{j=0}^{\lfloor \frac{l+1}{2} \rfloor} \left\{ \frac{1}{(l+1-2j)^2} \right\} \cup \{0\} \right)$$



To simplify the presentation we restrict, up to some remarks, to the equal mass case, i.e. $\xi_k = 1$ for all $k = 1, \ldots, l+1$, but later we generalize the analytic properties to arbitrary masses.

The diagonal differential $\mathcal{D}_l$-module has only regular singular points at the $l+1$ locations indicated above. For example, for $l = 4$ $\mathcal{D}_4$ is the operator AESZ34 which reads with $\theta = s\frac{d}{ds}$

$$
\begin{aligned}
\mathcal{D}_4 \varpi \; & = [\textstyle\prod_{i=0}^{2}((2i+1)^2 s - 1)\theta^4 + (4 - 70s + 450s^3)\theta^3 - \\
& (6 - 63s + 26s^2 - 225s^3)\theta^2 + (4 - 28s) + 5s - 1]\varpi \\
& = (l+1)!s = 120s \; .
\end{aligned}
$$

The Feynman graph is only singular at three following points:

- The Bessel point ($s = \infty$): Over the years the Feynman amplitude integrals was expressed in this region in terms of a Bessel function integral [Laporta, Remiddi (2004), Borwein, Salvy (2007), Weinzierl (2011), & Adams, Bogner (2013), Vanhove (2014)]

$$\mathcal{F}_{\sigma_l} = 2^l \int_0^\infty z I_0(\sqrt{t}z) \prod_{k=1}^{l+1} K_0(\xi_k z)\ \mathrm{d}z\ .$$

The radius of convergence r_{BI} is limited by the conifold point $s = \frac{1}{(l+1)^2}$, i.e. $t < (l+1)^2$ (in general by a conifold divisor $t < (\sum_{k=1}^{l+1} \xi_k)^2$). Note $\mathcal{D}_l \sim \mathcal{B}^2\mathrm{Sym}_{l+1}(\mathcal{D}_{\mathrm{Bessel}})$.

- The conifold ($s = \frac{1}{(l+1)^2}$): Here a single cycle $\nu \simeq S^{l-1} \in H_{l-1}(M_{l-1}, \mathbb{Z})$ vanishes giving according to Lefshetz rise to a reflection monodromy for l odd or a symplectic reflection (shift) monodromy if l is even. By homological mirror symmetry the homology of the vanishing cycle ν is exactly determined by the $\widehat{\Gamma}$-class for the structure sheaf $\mathcal{O}_{W_{l-1}}$ on W_{l-1}. This fixes Frobenius κ-constants exactly, which in turn specify the conifold monodromy in the Frobenius basis. In the path of analytic continuation indicated above it gives rise to the imaginary part of the amplitude $\mathcal{F}_{\sigma_l}$.

- The point of maximal unipotent monodromy ($s = 0$): (in general a normal crossing point of the divisors $D_j = M_j^2/K^2 = 0 \subset \mathcal{M}_{cs}(M_{l-1})$, $j = 1, \ldots, l+1$).
  Here the monodromy matrices $\mathbb{M}_j$ fulfill $(\mathbb{M}_j - \mathbb{1})^{l+1} = 0$ and there is no lower exponent for which that is true. According to Wilfried Schmid this (maximal) unipotent behaviour is caused by a hierarchical structure of degenerating cycles. At the mirror dual point in $\mathcal{M}_{\mathsf{KS}}(W_{l-1})$, all rational curves $\mathbb{P}^1_j$ (spanning the Mori-cone) are big. In the resulting classical geometrical limit we formuate a new $\widehat{\Gamma}$-class conjecture for the Fano ambient space, which captures exactly the real part of

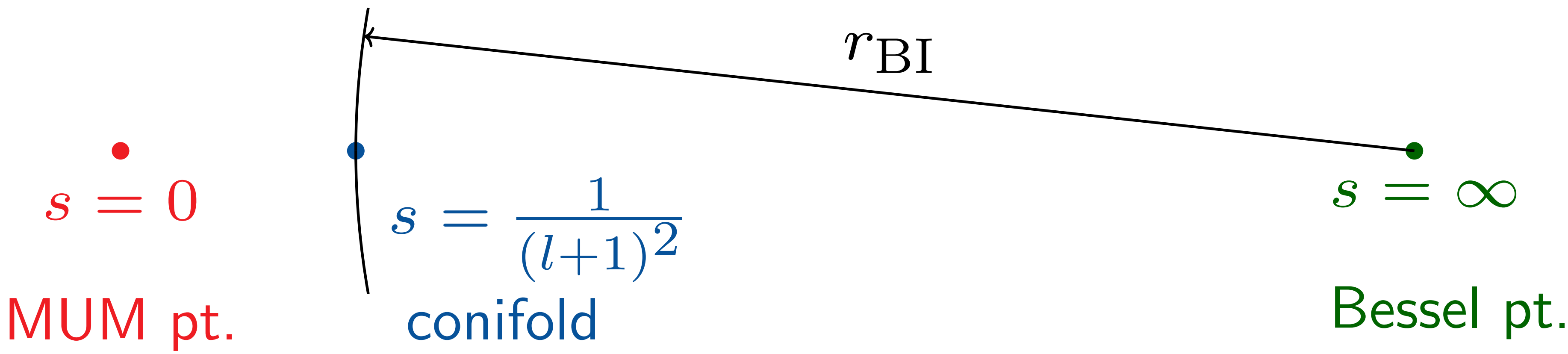
$\mathcal{F}_{\sigma_l}$. This gives a complete analytic description of the amplitude in terms of the Frobenius basis in this region.

- Fibres with special evaluations:
  - Pure torsion point ($s = 1$): At this on shell point the Feynman integral $\mathcal{F}_{\sigma_l}(1)$ is non-singular but pure torsion in the cohomology of $H_{l-1}(M_{l-1}, \mathbb{Z})$ and its evaluation leads up to known $\mathbb{Q}$ constants to special period values of Hecke eigenforms of weight l (sumarized with refs. in Table 3 in [2]).
  - Rank two attractor point for $l = 4$ ($s_\pm = 33 \pm 8\sqrt{17}$): Here (and at another point $s = -1/7$ outside the physical region) the numerator of the Hasse-Weil Zeta function

$$P_3(M_3/\mathbb{F}_p, T) = (1 - a_p(pT) + p(pT)^2)(1 - b_pT + p^3T^2)$$

splits. Here a_p and $b_p \in \mathbb{Z}$ according to the Weil conjectures $|a_p| \leq 2p$, $|b_p| \leq 2p^{\frac{3}{2}}$, are Fourier coefficients of a weight two and four Hecke eigenform in $S_*^{new}(\Gamma_0(34), \chi_{43}(33, \cdot)$ [Candelas, de la Ossa, Elmi, van Straten (2019)] whose periods $\omega^\pm, \omega^\pm$ and quasi periods $\eta^\pm, \eta^\pm$ determine all values of the period matrix of the maximal cut integrals up to known $\mathbb{Q}[\sqrt{17}]$ coefficients completely (numerical no proofs) [Masterthesis: K. Bönisch, May 2020, http://www.th.physik.uni-bonn.de/Groups/Klemm/data.php], [Talk: Broadhurst].

Exact Analytic solutions everywhere:



We follow [Hosono, Klemm, Theisen, and Yau (1993)] to construct the Frobenius solutions at the MUM point. The Mori cone of W_{l-1} is spanned by the ℓ-vectors i.e. $\ell^{(k)} = (\ell_{01}^{(k)}, \ldots, \ell_{0h}^{(k)}; \ell_1^{(k)}, \ldots, \ell_c^{(k)})$ for $k = 1, \ldots, h^{1,1}(W_{l-1}) = h^{l-2,1}(M_{l-1})$ vectors.

In the case at hand $h^{1,1}_{\text{prim vert}}(W_{l-1}) = h^{l-2,1}_{\text{prim hor}}(M_{l-1})$:

$$\begin{aligned} \ell^{(1)} &= \ (-1,-1;1,1,0,0,\cdots,0,0,0,0) \\ &\vdots \\ \ell^{(l+1)} &= \ (-1,-1;0,0,0,0,\cdots,0,0,1,1) \ . \end{aligned}$$

From these one obtains a generalized Gelfand- Kapranov- Zelevinskĭ differential system whose solutions are derived using the Frobenius deformation method (Givental's I-function) from:

$$\omega_0(\underline{z};\underline{\epsilon}) = \sum_{n_1,\ldots,n_{l+1}\geq 0} c(\underline{n}+\underline{\epsilon})\,\underline{z}^{\underline{n}+\epsilon}\,.$$

Here the underlined quantities are $(l+1)$-tuples and the series coefficients $c(\underline{n})$ are determined by the $l+1$ ℓ-vectors via

$$c(\underline{n}) = \frac{\prod_{j=1}^{2}\left(-\sum_{k=1}^{l+1} l_{0j}^{(k)}\, n_k\right)!}{\prod_{i=1}^{2l+2}\left(\sum_{k=1}^{l+1} l_{i}^{(k)}\, n_k\right)!}\,.$$

The $c(\underline{n}+\underline{\epsilon})$ are as usual defined by promoting all factorials $*!$ to $\Gamma(*+1)$ and deforming each integer n_k to $n_k+\epsilon_k$. In particular, the unique holomorphic solution at the point of maximal unipotent monodromy is given by

$$\varpi_0(\underline{z}) = \omega_0(\underline{z};\underline{\epsilon})\big|_{\underline{\epsilon}=\underline{0}} = \sum_{n_1,\ldots,n_{l+1}\geq 0} \binom{|n|}{n_1,\ldots,n_{l+1}}^2 \prod_{k=1}^{l+1} z_k^{n_k} \,,$$

where $z_k = s\xi_k^2$ for $k = 1,\ldots,l+1$. All other elements of the Frobenius basis are obtained by taking derivatives

$$\sum_{\{i_1,\ldots,i_k\}} a_{i_1,\ldots,i_k} \frac{\partial^k}{\partial\epsilon_{i_1}\cdots\partial\epsilon_{i_k}} \omega_0(\underline{z};\underline{\epsilon})\Bigg|_{\text{all }\ \epsilon_i=0} \quad \text{for } k = 1,\ldots,l \,.$$

The precise combinations of derivatives, which actually yield k-order logarithmic solutions $\varpi_l(\underline{z})$ are given by the classical intersection numbers of the Chow_k ring elements generating the primitive homology.of W_{l-1} or F_l respectively.

The dimension of the primitive homology, which maps due to an argument in [AK, Lian, Ruan, Yau ('97), Cabo-Bizet, AK, Vieira-Lopes ('14)] to the primitive horizontal middle homology of M_{l-1} and determines the number of k-order logarithmic solutions, does not grow that fast

$$h^{\mathrm{prim}}_{k,k}(F_l)=\left\{\begin{array}{ll}\binom{l+1}{k} & \text{if } k<\lceil\frac{l}{2}\rceil \\ \binom{l+1}{l-k} & \text{otherwise}\end{array}\right., \quad h^{\mathrm{prim}}_{k,k}(W_{l-1})=\left\{\begin{array}{ll}\binom{l+1}{k} & \text{if } k<\lceil\frac{l}{2}\rceil-1 \\ \binom{l+1}{l-1-k} & \text{otherwise}\end{array}\right. .$$

Let now ϖ_k for $k = 0, \ldots, l$ a canonical normalized (see [2]) Frobenius basis for the solutions at the MUM point and

$$\mathcal{F}_{\sigma_l}(s) = \sum_{k=0}^{l} \lambda_k^{(l)} \varpi_k(s)$$

the Banana amplitude to arbitrary loop order then:

Statement (Theorem I) equal mass :

(i) The $\lambda_0^{(l)}$ are given by the generating series

$$\sum_{l=0}^{\infty} \lambda_0^{(l)} \frac{x^l}{(l+1)!} = -\frac{\Gamma(1-x)}{\Gamma(1+x)} \mathrm{e}^{-2\gamma x + i\pi x} \quad .$$

(ii) The leading logarithmic terms of the Banana amplitude are determined by

$$\lambda_k^{(l)} = (-1)^k \binom{l+1}{k} \lambda_0^{(l-k)} \ .$$

(iii) The vanishing cycle ν at the conifold is given by the imaginary part of the Banana amplitude (for real physical parameters)

$$\nu(t) = \int_\nu \Omega(t) = \sum_{k=0}^{l} \mathrm{Im}(\lambda_k^{(l)}) \varpi_k \ .$$

Let us give an example for the Feynman amplitudes $\mathcal{F}_{\sigma_l}$ with $l \leq 5$:

l	ϖ_0	ϖ_1	ϖ_2	ϖ_3	ϖ_4
1	$-2\pi i$				
2	$18\zeta(2)$	$6\pi i$			
3	$-16\zeta(3) + 24i\pi\zeta(2)$	$-72\zeta(2)$	$-12\pi i$		
4	$-450\zeta(4) - 80i\pi\zeta(3)$	$80\zeta(3) - 120\pi i\zeta(2)$	$180\zeta(2)$	$20\pi i$	
5	$-288\zeta(5) + 1440\zeta(2)\zeta(3) - 540i\pi\zeta(4)$	$2700\zeta(4) + 480i\pi\zeta(3)$	$-240\zeta(3) + 360\pi i\zeta(2)$	$-360\zeta(2)$	$-30\pi i$

The idea to prove the Theorem I uses the $\widehat{\Gamma}$-class and provides an immediate generalization for the l-loop Banana amplitude with arbitrary masses:
Let ϖ_k^s for $k = 0, \ldots, l$ and $s = 1, \ldots h_{kk}^{prim}(W_{l-1})$ a canonical normalised (see [2]) Frobenius basis for the solutions at the MUM point and

$$\mathcal{F}_{\sigma_l}(s, \xi_i) = \sum_{k=0}^{l} \sum_{s=1}^{h_{k,k}^{\text{prim}}} \lambda_{k,s}^{(l)} \varpi_k^s$$

the Banana amplitude with arbitrary masses for arbitrary loop l then we can calculate the $\lambda_{k,s}^{(l)}$ from the $\widehat{\Gamma}$-class.

Instead of formulating theTheorem II for non equal math cases we just give some examples for $l \leq 4$:

l	$\lambda_{0,s}{}^{(l)}$	$\lambda_{1,s}{}^{(l)}$	$\lambda_{2,s}{}^{(l)}$	$\lambda_{3,s}{}^{(l)}$	$\lambda_{4,s}{}^{(l)}$
2	$18\zeta(2)$	$2\pi i$	1		
3	$-16\zeta(3)+24\pi\zeta(2)i$	$-18\zeta(2)$ $-18\zeta(2)$ $-18\zeta(2)$ $-18\zeta(2)$	$-2\pi i$	1	
4	$-450\zeta(4)$-$80\pi\zeta(3)i$	$16\zeta(3)$-$24\pi\zeta(2)i$ $16\zeta(3)$-$24\pi\zeta(2)i$ $16\zeta(3)$-$24\pi\zeta(2)i$ $16\zeta(3)$-$24\pi\zeta(2)i$ $16\zeta(3)$-$24\pi\zeta(2)i$	$6\zeta(2)$ $6\zeta(2)$ $6\zeta(2)$ $6\zeta(2)$ $6\zeta(2)$	$2\pi i$	1

Idea of the proofs using $\widehat{\Gamma}$-classes:

Essentially, it is a calculation in the vertical homologies of W_{l-1} and F_{l-1} and, up to some subtlety that we first guessed correctly, and Iritani recently confirmed, a variant of the $\widehat{\Gamma}$-class formalism of [Katzarkov, Kontsevich and Pantev], [Golyshev, Iritani and Galkin] for W_{l-1} and F_l.

Let the mirror map be

$$\mathfrak{t}^k = \frac{1}{2\pi i}\frac{\varpi_1^k}{\varpi_0} = \frac{1}{2\pi i}\log(z_k) + \tilde{\Sigma}_k(z)$$

and ω the symplectic form on W_{l-1} or F_l respectively.

Let further for a sheaf $\mathcal{S}$ of rank n the regularized $\widehat{\Gamma}(\mathcal{S})$-class be

$$\widehat{\Gamma}(\mathcal{S}) = \exp\left(\sum_{k\geq 2}(-1)^k(k-1)!\,\zeta(k)\,\mathrm{ch}_k(\mathcal{S})\right) \ ,$$

then [Seidel],[Katzarkov, Kontsevich and Pantev]

$$\nu(\mathfrak{t}) = \int_{W_{l-1}} e^{\omega\cdot\mathfrak{t}}\,\widehat{\Gamma}(TW_{l-1}) + \mathcal{O}(e^{\mathfrak{t}})$$

and [Golyshev, Iritani and Galkin],[Iritani 2020]

$$\mathcal{F}_{\sigma_l}(\mathfrak{t}) = \int_{F_l} e^{\omega\cdot\mathfrak{t}}\,\widehat{\Gamma}(1-c_1)^2\,\frac{\sin(2\pi c_1)}{2\pi c_1} + \mathcal{O}(e^{\mathfrak{t}})\ .$$

Then the Theorems follow using standard intersection calculus: From the adjunction formula it follows that the Chern classes c_k of W_{l-1} are given by the degree k part of

$$c_k(W_{l-1}) = \left[\frac{\prod_{i=1}^{l+1}(1+H_i)^2}{(1+\sum_{i=1}^{l+1} H_i)^2}\right]_{\deg(H)=k}\ .$$

Since the hyperplane classes in each $\mathbb{P}^1$ fulfill $H_i^2 = 0$ we can express the Chern classes c_k in terms of elementary symmetric polynomials $s_k(\underline{H}) = \sum_{i_1<\ldots<i_k} H_{i_1} \cdots H_{i_k}$ as

$$
\begin{aligned}
c_k(W_{l-1}) &= (-1)^k k! \sum_{j=0}^{k} \frac{(-2)^j (k+1-j)}{j!} s_k(\underline{H}) \\
&=: \mathcal{N}_k^{W_{l-1}} s_k(\underline{H}) \ .
\end{aligned}
$$

Similarly, considering the power one of the normal bundle in the denominator of the adjunction formula one gets

$$c_k(F_l) = (-1)^k k! \sum_{j=0}^{k} \frac{(-2)^j}{j!} s_k(\underline{H}) =: \mathcal{N}_k^{F_l} s_k(\underline{H}) \ .$$

The evaluation of the integral of a top degree product of Chern classes c_{k_n} over $X = W_{l-1}$ or $X = F_l$ is given by

$$\int_X \prod_n c_{k_n} = (l+1)! \prod_n \frac{\mathcal{N}_{k_n}^X}{k_n!} \ .$$

With coefficients of the intersection rings

$$\mathcal{R}^{W_{l-1}} = 2 s_{l-1}(\underline{H}), \quad \mathcal{R}^{F_l} = s_l(\underline{H})$$

one gets the Theorems by evaluation of $\nu(\mathfrak{t})$ and $\mathcal{F}_{\sigma_l}(\mathfrak{t})$ in terms of (finite) power series of $\mathfrak{t}^k$ □

Conclusions:

The solution of these Goldilocks amplitudes to all loop orders has revealed general features that should pave the way for further progress in higher loop calculations in more complicated situations.

Some of these features are:

- The (maximal) unipotent structure of period degenerations as predicted by mirror symmetry for Calabi-Yau motives fits perfectly with the high energy degeneration structure of Feynman amplitudes.

- The $\widehat{\Gamma}$-class formalism links the topological information (e.g. sufficient according to C.T.C Wall for the $l = 4$ CY3-fold case) to this physical degeneration. This helps to find the best Calabi-Yau motive and explains some of the transcendentality structure.

- The Banana amplitudes case are also a Goldilocks case for studying of master integrals at higher loops.

- The rich data bases of Calabi-Yau – and Fano motives provided by people like Kreuzer and Skarke

  http://hep.itp.tuwien.ac.at/~kreuzer/CY/,

  Almqvist, Enckefort, van Straten and Zudilin

  https://cydb.mathematik.uni-mainz.de/

  or Coates, Corti, Galkin, Golyshev and Kasprzyk

  http://coates.ma.ic.ac.uk/fanosearch/

  can become like the integrable tables of the future at least the ones extending elliptic integrals to Calabi-Yau integrals, which seem crucial. Combined with the $\mathcal{D}$-modules derived from the Gelfand-Kapranov-Zelevinskĭ systems one can provide super fast and precise numerical evaluations of Feynman amplitudes.

One a more speculative basis: It is curious that in the dimensional expansion of Feynman Integrals $I = \frac{I_{-1}}{\epsilon} + I_0 + \epsilon I_1 + \ldots$ difference recursions appear between the terms just like in the quantum period formalism in refined topological string theory in particular in the Nekrasov-Shatashvili limit. This could provide a deeper link of this expansion to integrable models.