Zeta and L-functions, old and new

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Introduction

Crazy question

Is there a *formula* for period integrals

$$I = \int_\sigma \omega \qquad ?$$

Can we generate, by pure thought, all the periods which will show up in a given setting (e.g., your favourite QFT)?

What could we mean by this?

Example 1

Let

$$I = \int_0^1 \int_0^1 \frac{dxdy}{1 - xy}$$

It is given by the formula

$$I = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right)^{-1}$$

(Hint: $I = \zeta(2)$, by Leibniz).

Example 2

Consider

$$\int_0^\infty \int_0^\infty \int_0^\infty \left(\frac{1}{(1+x+y+z)(1+x^{-1}+y^{-1}+z^{-1})-1} \right) \frac{dx}{x}\,\frac{dy}{y}\,\frac{dz}{z}$$

This integral equals

$$4\,\pi\sqrt{15}\,L(f,2)$$

where

$$L(f,s) = \prod_{p \text{ prime}} \left(1 - a_p p^{-s} + \left(\frac{p}{15}\right) p^{2-2s}\right)^{-1} \qquad \mathrm{Re}(s) > 2$$

The a_p are integers, related to the number of solutions of

$$(1+x+y+z)(1+x^{-1}+y^{-1}+z^{-1}) - 1 = 0$$

over finite fields $\mathbb{F}_p$. (Uses modularity of a certain K3 surface [Peters, Top, van der Vlugt]).

L-functions

A very deep conjectural programme in mathematics [Euler, ..., Beilinson, Deligne, Zagier, ..., Bloch-Kato,...] is, roughly, to interpret values of L-functions as periods:

$$\{\text{Special values of L-functions}\} \longrightarrow \{\text{Some period integrals}\}$$

Only *very special* periods have any chance whatsoever to be L-values.[1] In general, we do not expect factorisation over primes.

[1] Only expected for the single-valued periods of certain special combinations of motivic periods which are primitive for the motivic Galois coaction - i.e., a single letter in f-alphabet decomposition.

Towards mixed L-functions?

In this talk I want to raise the question: do there exist 'mixed' L-functions whose special values capture *all* periods?

$$\{\text{Special values of } \textit{mixed} \text{ L-functions}\} \longleftrightarrow \{\text{Period integrals}\}$$

Problem: there is no definition of a mixed L-function, and very few classes of period integrals are well-understood!

I shall give a first *tentative* definition of a mixed L-function and give many examples where their 'critical' values are indeed periods. This is not the final word on this question! But they generate interesting new objects which are connected to the modular graph functions and multiple elliptic polylogs discussed in this conference.

Plan

- Quick recap on L-functions
- Iterated Mellin transforms
- Examples and multiple Jacobi values
- L-functions associated to real analytic modular forms

Classical theory of L-functions

Recap on L-functions

We shall consider Dirichlet series which 'come from geometry':

$$L_M(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \qquad\qquad \mathrm{Re}(s) \gg 0$$

This means that they have an Euler product over all primes p

$$L_M(s) = \prod_p L_p^{-1}(s)$$

where $L_p(s)$ is a polynomial in p^{-s} related to counting points on algebraic varieties over finite fields.

Serre gave a recipe to define the *completed* L-function

$$\Lambda_M(s) = N^{-s/2} \pi^{as+b} \left(\prod_{i=1}^{r} \Gamma\left(\frac{s+n_i}{2}\right)\right) L_M(s) \ .$$

where N is called the conductor, a, b and n_i are integers.

Properties

Some deep and difficult conjectures:

1. $\Lambda_M(s)$ has a meromorphic continuation to $\mathbb{C}$.
2. There is a functional equation (self-dual case only)

$$\Lambda_M(s) = \varepsilon_M \Lambda_M(w-s) \qquad \varepsilon_M = \pm 1$$

They are classical theorems in the simple cases we will look at.

The values of $L_M(s)$, and hence $\Lambda_M(s)$, at integers $s = n$ are very interesting and encode arithmetic information.

Deligne made a conjecture about the values for *critical* values: when n is not a pole of either the gamma-factors of $\Lambda_M(s)$ or $\Lambda_M(w-s)$. Beilinson's conjecture is about *non-critical* values. Roughly, $\Lambda_M(n)$ is expected to be a period.

Trivial example: the L-function of a point

Example 1

Riemann zeta function

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1} = \sum_{n \geq 1} \frac{1}{n^s}$$

Completed version $\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ satisfies

$$\xi(s) = \xi(1-s)$$

Its values for $n > 1$ are periods:

$$\zeta(n) = \int_{[0,1]^n} \frac{dx_1 \ldots dx_n}{1 - x_1 \ldots x_n}$$

These values are critical when n is even. Deligne's conjecture predicts that $\zeta(2n)$ is proportional to $(2\pi i)^{2n}$ (Euler).

Modular forms (Hecke)

Consider a modular form of weight $2w$ and level 1:

$$f = \sum_{n \geq 0} a_n q^n \qquad q = e^{2\pi i \tau}$$

which is an eigenfunction for Hecke operators.

Example 2

$$L(f,s) = \prod_p \left(1 - a_p p^{-s} + p^{2k-1-2s}\right)^{-1} = \sum_{n \geq 1} \frac{a_n}{n^s}$$

Its completed version is

$$\Lambda(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s)$$

and satisfies

$$\Lambda(f,s) = (-1)^w \Lambda(f, 2w - s)$$

Towards mixed L-functions

We wish to speculate, based on no evidence whatsoever, that there exists such a thing as a 'mixed' L-function. What on earth could it be? It needs to be related to periods.

Non-example I

The simplest *mixed* family of periods are MZVs:

$$\zeta(k_1, \dots, k_r) = \sum_{1 \le n_1 < \dots < n_r} \frac{1}{n_1^{k_1} \dots n_r^{k_r}}$$

for $k_r \ge 2$. Periods of mixed Tate motives over $\mathbb{Z}$ are of this form (with $2\pi i$). We certainly need to capture these with our theory.

Naive answer: multiple zeta functions.

$$\zeta(s_1, \dots, s_r) = \sum_{1 \le n_1 < \dots < n_r} \frac{1}{n_1^{s_1} \dots n_r^{s_r}}$$

They do have a meromorphic continuation but have poles along infinitely many hyperplanes (bad!). No known functional equation.

Multiple zeta functions are *not* examples of our mixed L-functions.

Non-example II

Similarly, one might take two series

$$L_1(s) = \sum_n \frac{a_n}{n^s} \qquad \text{and} \qquad L_2(s) = \sum_n \frac{b_n}{n^s}$$

and simply try to consider multiple Dirichlet series

$$L(s_1, s_2) = \sum_{1 \leq m \leq n} \frac{a_m b_n}{m^{s_1}(m+n)^{s_2}}$$

or variants on this theme. This is *not* the definition we will take.

(In some circumstances, our forthcoming definition can actually give back linear combinations of such series - e.g. if L_1 and L_2 correspond to Hecke eigenforms and $s_1 \in \mathbb{Z}$. In this case we can actually prove meromorphic continuation and a functional relations. This is an accident).

Mellin transforms (Zagier, Dokchister)

Assume all properties of $\Lambda_M(s)$ and write as a Mellin transform

$$\Lambda_M(s) = \int_0^\infty \theta_M^0(t)\, t^s \frac{dt}{t}$$

where

$$\theta_M^0(t) = \sum_{n \geq 1} a_n \, \phi\left(\frac{nt}{N}\right)$$

and $\phi(t)$ is the inverse Mellin transform of a product of Gamma functions (this can be computed once and for all).

There exists a unique polynomial

$$\theta_M^\infty(t) \in \mathbb{C}[t]$$

such that $\theta_M(t) = \theta_M^\infty(t) + \theta_M^0(t)$ satisfies the functional equation

$$\theta_M\left(\frac{1}{t}\right) = \varepsilon_M \, t^w \, \theta_M(t)$$

Example 3

For the Riemann zeta function, we get

$$\theta_{\mathbb{Q}}(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t^2} = \underbrace{1}_{\theta_{\mathbb{Q}}^{\infty}(t)} + \underbrace{2 \sum_{n \geq 1} e^{-\pi n^2 t^2}}_{\theta_{\mathbb{Q}}^{0}(t)}$$

We have $\theta_{\mathbb{Q}}(1/t) = t\theta_{\mathbb{Q}}(t)$

Example 4

For a cuspidal eigenform $f = \sum a_n q^n$ of weight $2k$ we get

$$\theta_f(t) = \theta_f^0(t) = \sum_{n \geq 1} a_n e^{-\pi n t} \ .$$

In this case $\theta_f^{\infty} = 0$. We have

$$\theta_f(1/t) = (-1)^k t^{2k} \theta_f(t)$$

Mixed Mellin transforms

Let $\theta_1, \ldots, \theta_r$ be theta functions satisfying functional equations

$$\theta_i(1/t) = \varepsilon_i t^{w_i} \theta_i(t)$$

and

$$\theta_i = \theta_i^\infty + \theta_i^0$$

where θ_i^∞ is a polynomial, and θ_i^0 has rapid decay at infinity.

Definition 5

Consider the regularised iterated integral for $\mathrm{Re}(s_i)$ large

$$\Lambda(\theta_1, \ldots, \theta_r; s_1, \ldots, s_r) = \int_0^{\vec{1}_\infty} \theta_1(t_1) t_1^{s_1} \frac{dt_1}{t_1} \ldots \theta_r(t_r) t_r^{s_r} \frac{dt_r}{t_r}$$

where $\vec{1}_\infty$ denotes a tangential base point at infinity.

Example: cases $r = 1, 2$

Working out the definition of $\overrightarrow{1}_\infty$, one gets

$$\Lambda(\theta; s) = \int_0^\infty \theta^0(t) t^s \frac{dt}{t}$$

For modular forms, this gives back Hecke's (strange) definition!

The case $r = 2$ is given by

$$\Lambda(\theta_1, \theta_2; s_1, s_2) = \int_{0 \le t_1 \le t_2 \le \infty} \theta_1(t_1) t_1^{s_1} \frac{dt_1}{t_1} \theta_2^0(t_2) t_2^{s_2} \frac{dt_2}{t_2}$$

$$- \int_{0 \le t_1 \le t_2 \le \infty} \theta_2^\infty(t_1) t_1^{s_2} \frac{dt_1}{t_1} \theta_1^0(t_2) t_2^{s_1} \frac{dt_2}{t_2}$$

The general case is completely explicit. We can write Λ as combinations of rapidly converging integrals.

Properties

Theorem 6

$\Lambda(\theta_1, \ldots, \theta_r; s_1, \ldots, s_r)$ has a meromorphic continuation to $\mathbb{C}^r$ with poles along finitely many hyperplanes in the s_i.

It has a functional equation

$$\Lambda(\theta_1, \ldots, \theta_r; s_1, \ldots, s_r) = \varepsilon_1 \ldots \varepsilon_r \Lambda(\theta_r, \ldots, \theta_1; w_r - s_r, \ldots, w_1 - s_1)$$

and shuffle product formula

$$\Lambda(\theta_1, \ldots, \theta_p; s_1, \ldots, s_p) \Lambda(\theta_1, \ldots, \theta_q; s_1, \ldots, s_q) = \sum_{\sigma \in S_{p,q}} \Lambda(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(p+q)}; s_{\sigma(1)}, \ldots, s_{\sigma(p+q)})$$

Examples, and totally critical values

Critical values

Definition 7

Call $n_1, \ldots, n_r$ totally critical if each n_i is critical for $\Lambda(\theta_i; s)$.

Totally critical values give rise to lots of numbers:

$$\Lambda(\theta_1, \ldots, \theta_r; n_1, \ldots, n_r)$$

Are they related to periods? In many cases, yes.

Example 1: Modular forms

When $f_1, \ldots, f_r$ are Hecke eigenforms of weights $w_1, \ldots w_r$ then

$$\Lambda(f_1, \ldots, f_r; s_1, \ldots, s_r) = \Lambda(f_r, \ldots, f_1; w_r - s_r, \ldots, w_1 - s_1)$$

The totally critical values are the integers inside the box

$$1 \leq s_i \leq w_i - 1$$

Theorem 8

They are periods of the relative completion of the fundamental group of a modular curve, aka multiple modular values.

Manin considered such L-functions in the case when all f_i are cusp forms (no tangential base point regularisation required).

Example 1 cont.

Theorem (Alex Saad 2020)

All multiple zeta values can be written as a $\mathbb{Q}[2\pi i]$ linear combination of regularised iterated integrals of Eisenstein series.

So we get all multiple zeta values from totally critical values

$$\Lambda(\mathbb{G}_{2n_1}, \ldots, \mathbb{G}_{2n_r}; k_1, \ldots, k_r)$$

where $\mathbb{G}_{2n_1}$ is the holomorphic Eisenstein series of weight $2n_1$.

$$\zeta(3) = -(2\pi)^3 \Lambda(\mathbb{G}_4, 3)$$

All periods of mixed Tate motives over $\mathbb{Z}$ (MZV's) are totally critical values of mixed L-functions.

Example 2: Multiple Riemann ξ

Recall the 'trivial motive' gives rise to the theta function

$$\theta_{\mathbb{Q}}(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t^2}$$

whose (regularised) Mellin transform is $\xi(s)$.

Define the multiple Riemann ξ function by

$$\xi(s_1, \ldots, s_r) = \Lambda(\theta_{\mathbb{Q}}, \ldots, \theta_{\mathbb{Q}}; s_1, \ldots, s_r) \ .$$

Functional equation: $\xi(s_1, \ldots, s_r) = \xi(1 - s_r, \ldots, 1 - s_1)$.

E.g., for $r = 2$ we have

$$\xi(s_1, s_2) = \xi(1 - s_2, 1 - s_1)$$

$$\xi(s_1)\xi(s_2) = \xi(s_1, s_2) + \xi(s_2, s_1)$$

Values

The totally critical values are when s_i are all positive even. They are linear combinations of multiple quadratic sums, e.g.

$$Q(n_1, n_2) = \sum_{k_1, k_2 \geq 1} \frac{1}{(k_1^2 + k_2^2)^{n_1} (k_2^2)^{n_2}}$$

We can prove that:

$$\xi(2s_1, 2s_2) = \pi^{-s} \Gamma(s) \int_i^{\vec{1}_\infty} E(z, s_1 + s_2)\, y^{s_2 - s_1} \frac{dy}{y}$$

where $E(z, s)$ is the real-analytic Eisenstein series. we can deduce that $\xi(2n_1, 2n_2)$ where $n_1, n_2 > 0$ are integers, are periods related to iterated integrals on the CM elliptic curve $\mathbb{C}/(\mathbb{Z}i + \mathbb{Z})$ whose period lattice is the Gaussian integers.

A common generalisation: Multiple Jacobi values

Let Θ_J be the graded algebra generated by Jacobi's theta nulls:

$$\theta_2 = \sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})^2} \qquad \theta_3 = \sum_{n\in\mathbb{Z}} q^{n^2} \qquad \theta_4 = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} \ .$$

(NB: they have functional *relations* with half integer weights so $\theta_i^\infty(t) \in \mathbb{C}[\sqrt{t}]$. We can replace $\sqrt{t}$ with t).

Definition 9

A *multiple Jacobi value* is a totally critical value of

$$\Lambda(\theta_1, \ldots, \theta_r; s_1, \ldots, s_r) \qquad \theta_i \in \Theta_J \ .$$

Satisfy amazingly rich tapestry of identites. Since Θ_J contains modular forms of level 1, this subsumes the two previous examples.

Example 3: MZV's as multiple Jacobi values

The two elements θ_3^4, $2\theta_4^4 - \theta_3^4$ define a pair of weight 2 functions

$$\theta_+(1/t) = t^2\theta_+(t) \qquad \text{and} \qquad \theta_-(1/t) = -t^2\theta_-(t)\ .$$

Then $\Lambda(\theta_\pm; s) = \pm\Lambda(\theta_\pm; 2-s)$ have no pole at $s = 1$.

Theorem 10

All multiple zeta values (and $\log 2$) are $\mathbb{Q}[\pi^\pm]$-linear combinations of the central values $\Lambda(\theta_1, \ldots, \theta_r; 1, \ldots, 1)$ where $\theta_i \in \{\theta_+, \theta_-\}$

So multiple Jacobi values contain MZV's in two completely different ways, this time related to the modularity of $\mathbb{P}^1\backslash\{0, 1, \infty\}$.

Multiple Jacobi values contain periods coming from both

$$\mathcal{M}_{0,4} = \mathbb{P}^1\backslash\{0, 1, \infty\} \qquad \text{and also} \qquad \mathcal{M}_{1,1}\ .$$

Non-standard L-functions associated to real analytic modular forms

We can associate yet another type of L-function to modular graph functions occuring in string perturbation theory, for example.

Real analytic modular forms

Hecke defined an L-function of a classical holomorphic modular form. The same is possible for *real analytic modular forms*

Let $f : \mathcal{H} \longrightarrow \mathbb{C}$ be a real analytic function such that

- It is modular of two weights (w, w'), i.e.

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{w}(c\overline{\tau} + d)^{w'} f(\tau)$$

  for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.
- It admits an expansion with $q = e^{2\pi i\tau}$

$$f(\tau) = \sum_{|k| \leq N} (-2\pi \, \mathrm{Im}(z))^k \sum_{m,n \geq 1} a_{m,n}^{(k)} \, q^m \overline{q}^n$$

  where the coefficients $a_{m,n}^{(k)} \in \mathbb{C}$ do not grow too rapidly.

L-functions of real analytic modular forms

For each (finitely many) k, define a Dirichlet series

$$\mathbf{L}^{(k)}(f,s) = \sum_{m,n\geq 1} \frac{a_{m,n}^{(k)}}{(m+n)^s}$$

and define the completed L-function to be

$$\mathbf{\Lambda}(f,s) = \sum_k (-1)^k (2\pi)^{-s} \Gamma(s+k) \mathbf{L}^{(k)}(f,s)$$

It does not depend on any of the zeroth Fourier modes $a_{0,0}^{(k)}$.

It is a non-classical L-function, since it is a sum of several Dirichlet series. In general, it does not have an Euler product.

Properties

Proposition

Then $\mathbf{\Lambda}(f,s)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies

$$\mathbf{\Lambda}(f,s) = i^{w-w'}\mathbf{\Lambda}(f,w+w'-s)\ .$$

It has simple poles with principle part:

$$\sum_k (-2\pi)^k\, a_{0,0}^{(k)} \left(\frac{i^{w-w'}}{s-(w+w'+k)} - \frac{1}{s+k} \right)$$

The function $\mathbf{\Lambda}(f,s)$ gives a recipe to reconstruct all zeroth Fourier modes $a_{0,0}^{(k)}$ from the non-zero ones: $a_{m,n}^{(k)}$, for $m,n \geq 1$.

Example

Consider a real analytic Eisenstein series

$$E = \sum_{m,n\in\mathbb{Z}^2}' \frac{\mathrm{Im}(\tau)}{(m\tau+n)^{w+1}(m\bar{\tau}+n)^{w'+1}}$$

of modular weights (w, w'). Its **L**-function is

$$\mathbf{\Lambda}(E, s) = *\,\xi(s+1)\xi(s-w-w')$$

a product of two completed Riemann zeta functions.

The proposition explains why

1. Its coefficients $a_{m,n}^{(k)}$ for $m, n \geq 1$ are rational
2. The zero modes $a_{0,0}^{(k)}$ are zeta values.

More generally, to any modular graph function I_G, we can associate a completed L-function $\mathbf{\Lambda}_G(s)$.

Closing the circle

These different kinds of L-functions are related.

Single-valued iterated integrals on the universal punctured elliptic curve $\mathcal{E}_\tau^\times$ define a class of real-analytic modular forms $\mathcal{MI}^E$.

Theorem 11

The function $\mathbf{\Lambda}(f;s)$ associated to any $f \in \mathcal{MI}^E$ is a linear combination of the mixed L-functions

$$\Lambda(\mathbb{G}_{2n_1}, \ldots, \mathbb{G}_{2n_k}; p_1, \ldots, p_{k-1}, s)$$

where $\mathbb{G}_{2n}$ are holomorphic Eisenstein series.

Conclusion

- Starting with the classical theory of L-functions and their conjectured relation to periods, we defined mixed L-functions

$$\Lambda(\theta_1, \ldots, \theta_r; s_1, \ldots, s_r)$$

  via iterated Mellin transforms.

- They satisfy many analytic properties including a functional equation, multiplicative structure.

- Their totally critical values are related to many interesting periods. They are deeply connected to: string amplitudes, elliptic iterated integrals, modular forms, lattice sums, and other topics which have arisen in this conference.