Fractons for Field Theorists and Field Theory of Fractons

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IAS

Thank Pranay Gorantla, Michael Hermele, Ho Tat Lam, Tom Rudelius, and Shu-Heng Shao for many discussions

Why should we discuss it here?

- Quantum Field Theory appears everywhere. It can be viewed as the language of physics.
- It is particularly prominent in various aspects of String Theory.
- Recently discovered lattice systems – including fracton models – exhibit bizarre properties. They do not seem to fit the framework of continuum QFT.
- Should the framework of continuum QFT be extended to accommodate these models?

Brief history of fracton models

- Early lattice models motivated by various considerations
  - Chamon model, "Quantum glassiness in strongly correlated clean systems: An example of topological overprotection" [Chamon, 2004]
  - Haah code, "Local stabilizer codes in three dimensions without string logical operators" [Haah, 2011]
- Realization of the significance of these models for phases of matter and more examples
  - "A new kind of topological quantum order: A dimensional hierarchy of quasiparticles built from stationary excitations" [Vijay, Haah, and Fu, 2015]
  - "Fracton topological order, generalized lattice gauge theory, and duality" [Vijay, Haah, and Fu, 2016]

Brief history of fracton models

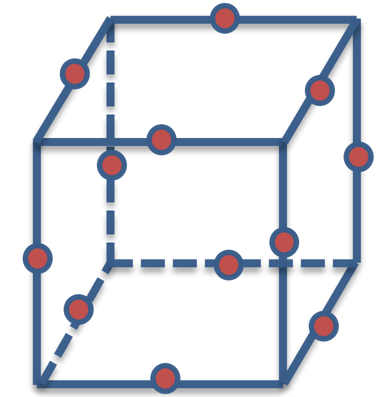
- Many more examples, including gapless models [Pretko, 2016], associated with strange gauge theories [Xu, 2006]
- Different constructions, in particular, layers of coupled $2+1d$ theories [Ma, Lake, Chen, and Hermele; Vijay, 2017]
- Connections to other fields, including elasticity, quantum information, topological quantum field theory
- It is a large field and I cannot do justice to it here.

The X-cube model [Vijay, Haah, and Fu]

- $\mathbb{Z}_2$ spin on every spatial link (qubit on every link)
- The Hamiltonian has two kinds of terms

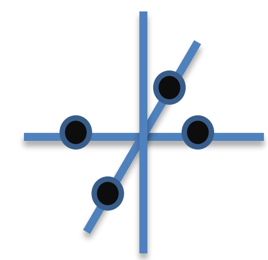
$$H = -\sum_c B_c - \sum_s \left(A_s^x + A_s^y + A_s^z\right)$$

  - Cube interaction



$$B_c = \prod_{\text{links around } c} \sigma^1$$

  - Site interactions



$$A_s^z = \prod_{\substack{\text{links at } s \\ \text{along } x \text{ and } y}} \sigma^3$$

- Innocent looking local Hamiltonian

The X-cube model [Vijay, Haah, and Fu]

- All the terms in the Hamiltonian commute with each other.
- Lowest energy when all the terms are 1.
- On a torus of with $L^x \times L^y \times L^z$ sites with periodic boundary conditions, huge ground state degeneracy $2^{2(L^x+L^y+L^z)-3}$
  - Depends on the number of lattice sites
  - Not proportional to the volume (sub-extensive)
  - Infinite in the continuum limit: lattice spacing $a \to 0$, $L^i \to \infty$ with fixed size $\ell^i = aL^i$
- The ground states reflect short distance physics – high momentum modes (of lattice scale) have zero energy – UV/IR mixing.

The X-cube model [Vijay, Haah, and Fu]

- Gap in the spectrum.
- Conserved charges (logical operators) act in the space of ground states. They are lines or strips wrapping the torus. Sub-system symmetries.
- No local operator acts in the space of ground states. Hence, it is robust under small changes of the Hamiltonian. Small deformations of the Hamiltonian do not change the low-energy physics.

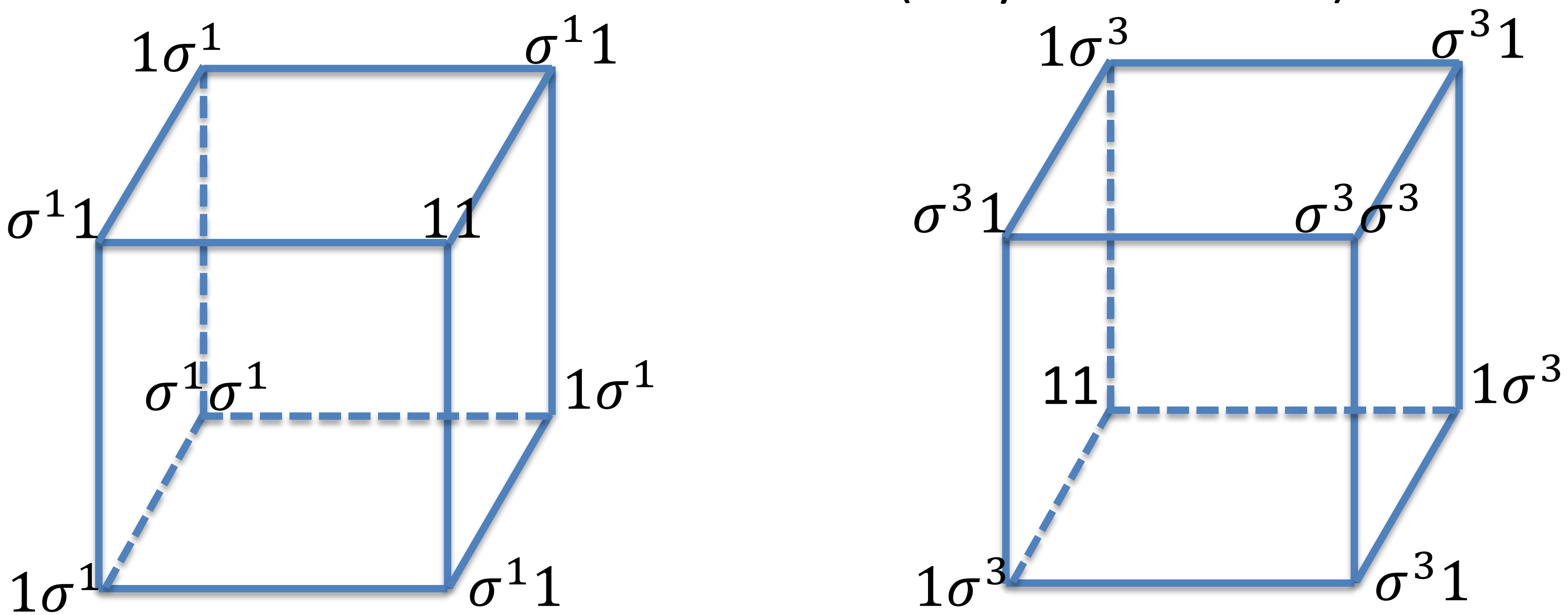
The X-cube model [Vijay, Haah, and Fu]

- Localized excitations with restricted mobility:
  - Fractons are fixed at a point
    - Not created by local operators
    - Appear at the 4 corners of “membrane operators”
    - Cannot move by a local operator (except when creating other fractons).
  - Lineons fixed to move on a line (x, y, or z)
  - Planon fixed to move on a plane (xy, yz, or xz)
    - These are dipoles of fractons or lineons

Haah code [Haah]

Two qubits at every site.

Two kinds of terms in the Hamiltonian (they all commute)



$1\sigma^1$ means the action of $1 \otimes \sigma^1$ on the two qubits.

Innocent looking local Hamiltonian.

Haah code [Haah]

Many bizarre properties.

In particular, the ground state degeneracy is $2^{k(L^x,L^y,L^z)}$ with $k(L^x,L^y,L^z)$

- a complicated number-theoretic function of L^x, L^y, L^z
- complicated even in the special case $L = L^x = L^y = L^z$
- not monotonic in L
- bounded by $\sim L$
- For some sequence of $L \to \infty$, $k(L) \to$ finite

The limit $L^i \to \infty$ is ambiguous. Is there a continuum limit?

Gapless models [Pretko]

Motivated by earlier models of symmetric tensor gauge theories in the continuum and the lattice [Xu; …]

$$A_0 \to A_0 + \partial_0 \alpha$$
$$A_{ij} \to A_{ij} + \partial_i \partial_j \alpha$$

Gauge invariant electric and magnetic fields

$$E_{ij} = \partial_0 A_{ij} - \partial_i \partial_j A_0$$
$$B_{[ij]k} = \partial_i A_{jk} - \partial_j A_{ik}$$

Lagrangian $\mathcal{L} = E^2 - B^2$

Gapless “photon”

Gapless models [Pretko]

$$A_0 \to A_0 + \partial_0 \alpha$$
$$A_{ij} \to A_{ij} + \partial_i \partial_j \alpha$$

Gauss law with matter $\sum_{ij} \partial_i \partial_j E_{ij} = \rho$

Conserved charge $\int d^3x\, \rho$

Conserved dipole charge $\int d^3x\, x^i \rho$

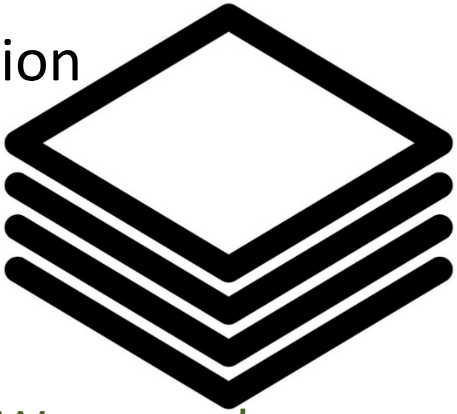
Restricted mobility because of the conservations – fractons.

Equivalently, the pure gauge theory (without matter) has defects representing probe particles

$$\exp\left(i \int dt\, A_0\right)$$

They cannot move in space because there is no A_i.

Coupled-layer constructions
[Ma, Lake, Chen, and Hermele; Vijay]



- Stack many $2+1d$ lattice models along one direction
- Repeat in the two other directions
- Couple the theories on the three stacks

Follow-up

- Place in arbitrary foliated manifolds [Shirley, Slagle, Wang, and Chen, 2017]
- Repeat with $2+1d$ continuum TQFT [Slagle, Aasen, and Williamson, 2018]

Coupled-layer constructions
[Ma, Lake, Chen, and Hermele; Vijay]

Some properties are manifest and intuitive

- Excitations with restricted mobility
- Sub-system global symmetry acts separately on each layer
- Sub-extensive ground state degeneracy
- The observables are discontinuous – changing from layer to layer.

The “continuum limit” as the distance between the layers goes to zero is challenging.

Questions

- Are there more examples, perhaps with more exotic phenomena?
- What is the underlying reason for the bizarre behavior?
- What are the possible theories/phases?
  - Organization/classification
  - How are these phases characterized in a universal way?
  - Is there a sensible continuum quantum field theory for the long-distance behavior?
- Formulate the theories on nontrivial manifolds
- Many more

Look for a low-energy continuum QFT

- The properties of fracton models, in particular, the UV/IR mixing, seem to contradict the renormalization group picture.
- Try to extend slightly the framework of continuum QFT to accommodate fractons and other models.
- Motivations:
  - Universal description of these systems.
  - Hopefully, this will also teach us something new about QFT.

Exotic continuum QFT [NS, Shao]

Elements in our continuum theories (essentially unavoidable)

- Spacetime symmetries (in addition to translations)
  - No Lorentz invariance
  - No rotation symmetry. Preserve only the subgroup generated by 90 degree rotations.
- Impose exotic global symmetries and then gauge them
- Discontinuous fields
  - Not as discontinuous as on the lattice – the allowed discontinuities are restricted
  - In the gauge theories, the gauge parameters and the transition functions can have such discontinuities
  - Universal – independent of details at the lattice scale

Tensor gauge theory

A variant of the tensor gauge theory with A_{ij} only for $i \neq j$ was studied by [Xu, Wu; …] (named "hollow tensor gauge theory" by [Ma, Hermele, Chen]). It has only the cubic symmetry S_4

$$A_0 \to A_0 + \partial_0 \alpha \qquad \mathbf{1}$$
$$A_{ij} \to A_{ij} + \partial_i \partial_j \alpha \qquad \mathbf{3'}$$

Gauge invariant electric and magnetic fields

$$E_{ij} = \partial_0 A_{ij} - \partial_i \partial_j A_0 \qquad \mathbf{3'}$$
$$B_{[ij]k} = \partial_i A_{jk} - \partial_j A_{ik} \qquad \mathbf{2}$$

The lattice theory is destabilized by monopole operators, as in $2+1d$ ordinary $U(1)$ gauge theory – Polyakov mechanism

But it does make sense as a continuum QFT…

Tensor gauge theory – spectrum

The dispersion relation (suppressing coefficients)

$$\omega^4 - \omega^2\left(k_x^2 + k_y^2 + k_z^2\right) + \left(k_x^2 k_y^2 + k_x^2 k_z^2 + k_y^2 k_z^2\right) = 0$$

2 gapless modes: 3 from A_{ij} $(i \neq j)$ minus one because of Gauss law.

For $k_x = k_y = 0$ with arbitrarily large k_z, modes with $\omega = 0$. (Similarly for the other directions.)

UV/IR mixing.

This is the key to the peculiarities of these models.

Infinite number of zero energy states. They remain at zero energy even when the quantum effects are analyzed more carefully. [NS, Shao]

Tensor gauge theory – duality [NS, Shao]

This theory is dual to a free non-gauge theory of a field $\hat{\phi}$ in $\mathbf{2}$ of S_4 (indices and coefficients suppressed)

$$E = \partial \hat{\phi} \qquad \mathbf{3}'$$
$$B = \partial_0 \hat{\phi} \qquad \mathbf{2}$$

The two gapless modes can be thought of as modes of $\hat{\phi}$.

Large ground state degeneracy from winding modes of $\hat{\phi}$.

The dangerous monopole operator is $e^{i\hat{\phi}}$.

- It is natural to leave it out of the continuum Lagrangian (as in $2+1d$ ordinary $U(1)$ gauge theory).
- Actually, in this case, $e^{i\hat{\phi}}$ is irrelevant.

Tensor gauge theory – gauge transformations [NS, Shao]

The gauge group is $U(1)$, i.e., $\alpha \sim \alpha + 2\pi$.

Usually, α is continuous.

Here, we allow certain discontinuous gauge transformations. (Justify using a careful analysis of the continuum limit.)

A typical large gauge transformation:

$$\alpha = 2\pi\left(\frac{x}{\ell^x}\,\Theta(y-y_0) + \frac{y}{\ell^y}\,\Theta(x-x_0) - \frac{xy}{\ell^x\ell^y}\right)$$
$$\alpha(x+\ell^x, y, z, t) = \alpha(x, y, z, t) + 2\pi\Theta(y-y_0)$$
$$\alpha(x, y+\ell^y, z, t) = \alpha(x, y, z, t) + 2\pi\Theta(x-x_0)$$

With these gauge transformations as transition functions, we find quantized fluxes.

Tensor gauge theory [NS, Shao]

We have to use discontinuous gauge parameters and even singular field strength.

Yet, the answers are

- unambiguous
- reproduce the lattice answers
- universal

Can be phrased as a continuum, coupled-layer construction with ordinary $2+1d$ $U(1)$ gauge theory on each layer [Gorantla, Lam, NS, and Shao]

$\mathbb{Z}_N$ tensor gauge theory in the continuum [NS, Shao]

Higgs the tensor $U(1)$ gauge theory to $\mathbb{Z}_N$

$$\mathcal{L} = b\,(\partial_0\phi - NA_0) + \sum_{i\neq j} e_{ij}\left(\partial_i\partial_j\phi - NA_{ij}\right)$$

$$A_0 \to A_0 + \partial_0\alpha$$

$$A_{ij} \to A_{ij} + \partial_i\partial_j\alpha$$

$$\phi \to \phi + N\,\alpha$$

b and e_{ij} act as Lagrange multipliers, forcing the Higgsing.

We can dualize ϕ to another exotic gauge field $\hat{A}$ and find a BF-like description (suppress indices) [Slagle, Kim]

$$\mathcal{L} = \frac{N}{2\pi}\left(\hat{A}_0^{(2)}B^{(2)} + \hat{A}^{(3')}E^{(3')}\right)$$

N is quantized for invariance under large gauge transformations.

$\mathbb{Z}_N$ tensor gauge theory – defects

The simplest defect

$$\exp\left(i\int_{-\infty}^{\infty} dt\, A_0\right)$$

represents the world-line of a probe particle at fixed position. Gauge invariance prevents it from moving. It is a fracton.

A dipole of probe particles at x_1, x_2 is represented by the gauge invariant defect (similar dipoles in the other directions)

$$\exp\left(i\int_{x_1}^{x_2} dx\ \int_{\mathcal{C}}\left(dt\, \partial_x A_0 + dy\, A_{xy} + dz\, A_{xz}\right)\right)$$

$\mathcal{C}$ is a curve in (y, z, t). The dipole is restricted to the (y, z) plane. It is a planon.

Their restricted mobility follows from the gauge symmetry.

$\mathbb{Z}_N$ tensor gauge theory – defects

More defects from winding operators of ϕ, or equivalently, from Wilson lines of $\hat{A}$

The N'th power of every defect is trivial.

$\mathbb{Z}_N$ tensor gauge theory – spectrum

$$\mathcal{L} = b\,(\partial_0\phi - NA_0) + \sum_{i\neq j} e_{ij}\left(\partial_i\partial_j\phi - NA_{ij}\right)$$

Solve $A_0 = \frac{1}{N}\partial_0\phi\,,\quad A_{ij} = \frac{1}{N}\partial_i\partial_j\phi.$

The ground states are $\{\phi\}/\phi \sim \phi + N\alpha$. They are generated by

$$\phi = 2\pi\left(\frac{x}{\ell^x}\,\Theta(y-y_0) + \frac{y}{\ell^y}\,\Theta(x-x_0) - \frac{xy}{\ell^x\ell^y}\right)$$

For every value of x_0 and for every value of y_0 we have an integer modulo N. Adding the other directions, accounting for the common zero mode, and placing on a lattice, we have

$$N^{2(L^x+L^y+L^z)-3}$$

states.

$\mathbb{Z}_N$ tensor gauge theory

- Gapped. Only zero energy states (infinitely many)
- The spectrum is in the simplest representation of the algebra of operators (logical operators)
- No local operators acting in the space of ground states. Hence, the system is robust!
- Can be phrased as a continuum, coupled-layer construction with ordinary $2+1d$ $\mathbb{Z}_N$ gauge theory on each layer [Slagle, Aasen, and Williamson; Gorantla, Lam, NS, and Shao]

This is a continuum field theory description of the $\mathbb{Z}_N$ version of the X-cube model.

Comparing $\mathbb{Z}_N$ gauge theories

	Ordinary $2+1d$	Tensor $3+1d$
Lagrangian	$U(1) \to \mathbb{Z}_N$ or $\frac{N}{2\pi}\left(\hat{A}_0 B + \hat{A} E\right)$	$U(1) \to \mathbb{Z}_N$ or $\frac{N}{2\pi}\left(\hat{A}_0^{(2)} B^{(2)} + \hat{A}^{(3')} E^{(3')}\right)$
Phase	Gapped, robust	Gapped, robust
Defects and operators	Wilson lines of A and $\hat{A}$	Wilson lines and strips of A and $\hat{A}$
Probe particles	Anyons	Fractons, lineons, planons
States on a torus	N^2	$N^{2(L^x+L^y+L^z)-3}$

Summary

- Fractons exhibit interesting properties
  - UV/IR mixing:
    - Large ground state degeneracy. Sometimes there is no well defined limit as $L \to \infty$
    - Discontinuous observables
  - Excitations with restricted mobility
- These seem incompatible with the framework of continuum QFT.

Summary

- We presented nonstandard continuum QFTs:
  - Invariant only under discrete rotations
  - Exotic global and gauge symmetries
  - The fields and the gauge transformation parameters can be discontinuous and even singular.
    - More continuous than on the lattice, but less continuous than in standard continuum theories.
- These continuum theories capture the universal properties of the lattice models. They reproduce their long-distance physics and the properties of probe particles.

Outlook

- Make the treatment of the discontinuous fields more precise
- Understand better the underlying geometry of these gauge fields
- Place these theories on more complicated manifolds
- Repeat for other known lattice models
- Find new such models

Thank you

Stay healthy