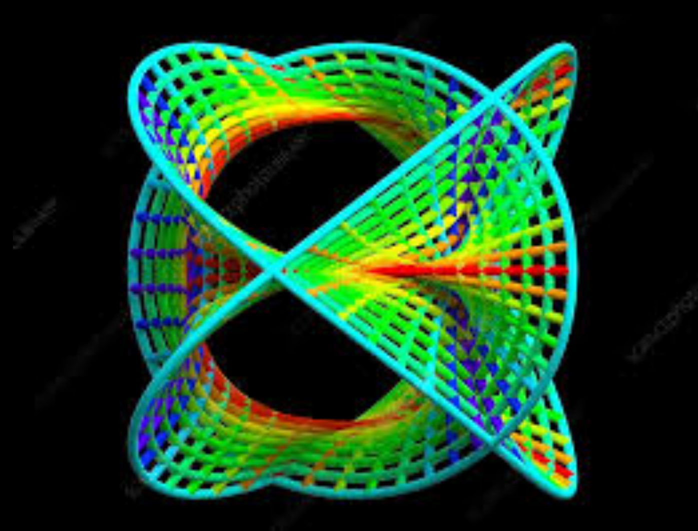
The hidden geometry of space-time



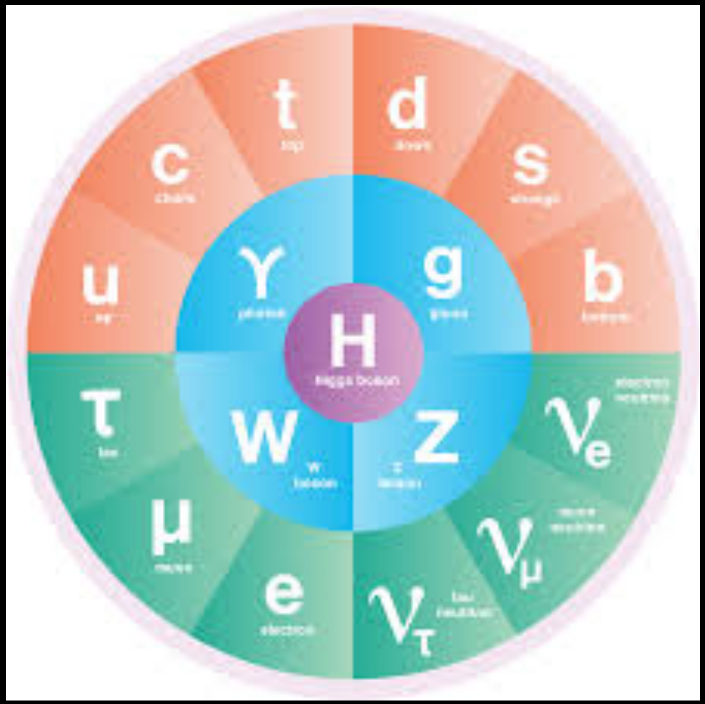
Strings 2020 public talk

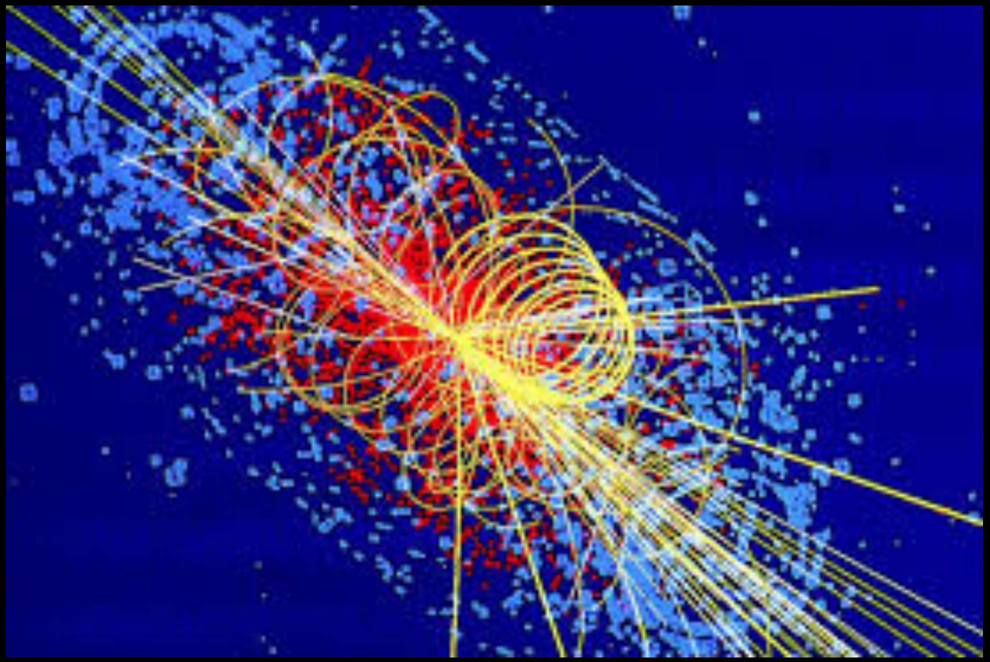
Shamit Kachru

SIMONS FOUNDATION

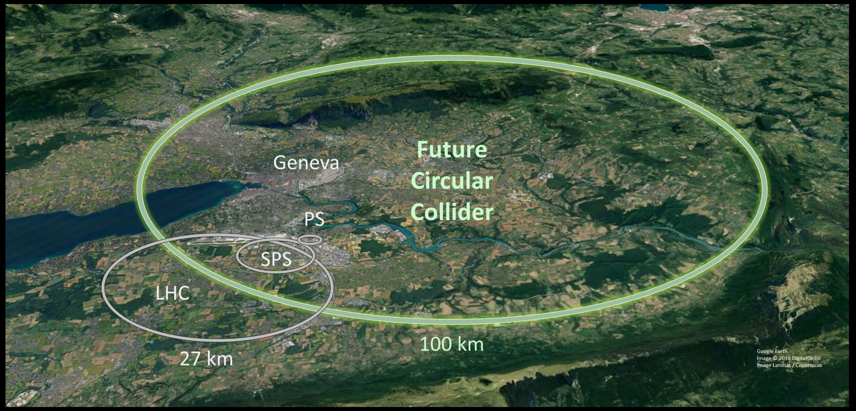
I. Introduction

Modern physics has revealed to us a bewildering array of fundamental particles and forces:





Most recently — and excitingly — the Higgs boson was discovered at CERN.



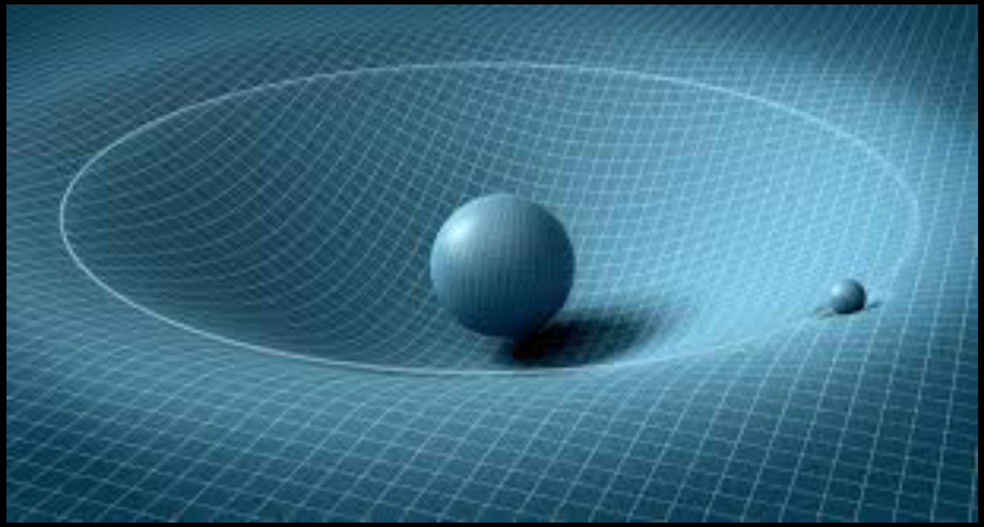
Plans are in the works for a bigger machine. What will we discover there!??

Happily, we don't need to know for this talk. I am going to discuss the force we understood (in some quantitative detail) first — gravity.

Newton's understanding (built on his discovery of calculus):

$$F = G\frac{m_1 m_2}{r^2}$$

The next big advance came with Einstein.



His general theory of relativity is, at heart, a geometric theory.

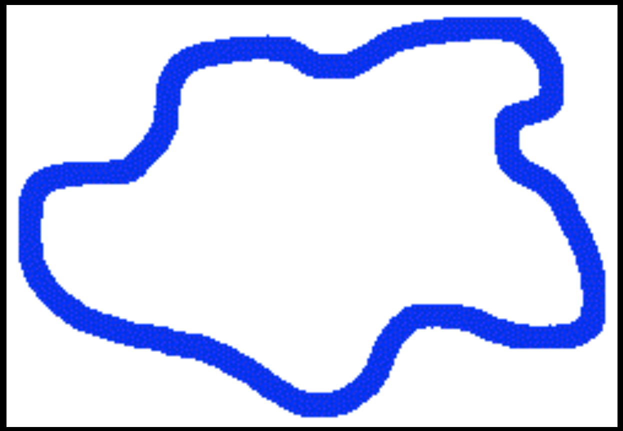
— Matter/energy tells space how to curve
— curved space tells matter how to move

Einstein’s theory evolved, in part, using the language of Riemannian geometry, a theory of curved spaces.

My goal in this talk is to update you on some of the striking new directions the study of connections between physics and geometry has gone in the past few decades.

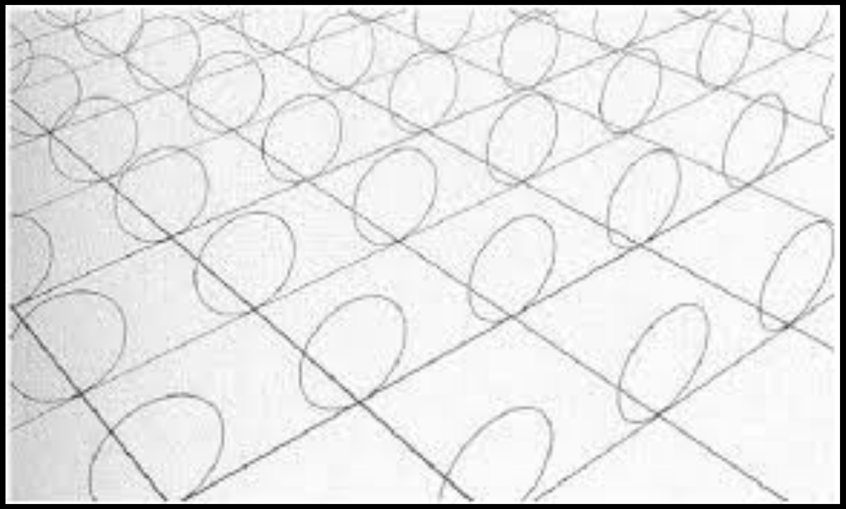
The hero of our story isn’t a person. Instead, it will be string theory, which serves as a magical generator of such connections.

II. (Brief) introduction to string theory



A theoretical structure built on tiny loops of string instead of elementary particles.

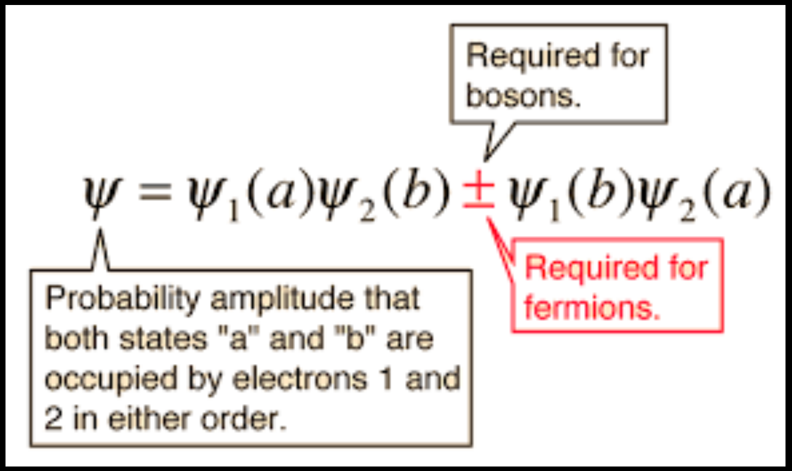
It is a fact that the theory naturally lives in higher dimensions — 9+1 in some versions.



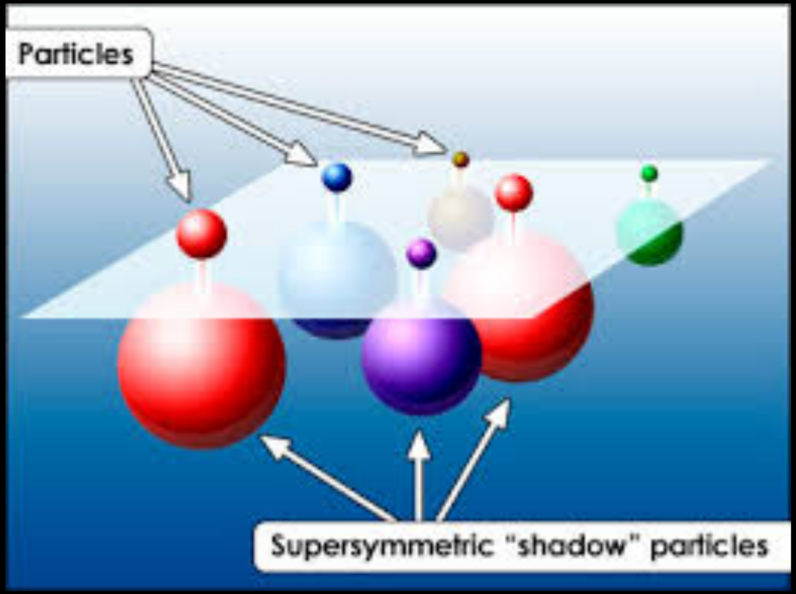
As a consequence, one considers "compactifications" of the theory — a notion first discussed by Kaluza and Klein almost a century ago.

The higher dimensional string theory has some important physical properties:

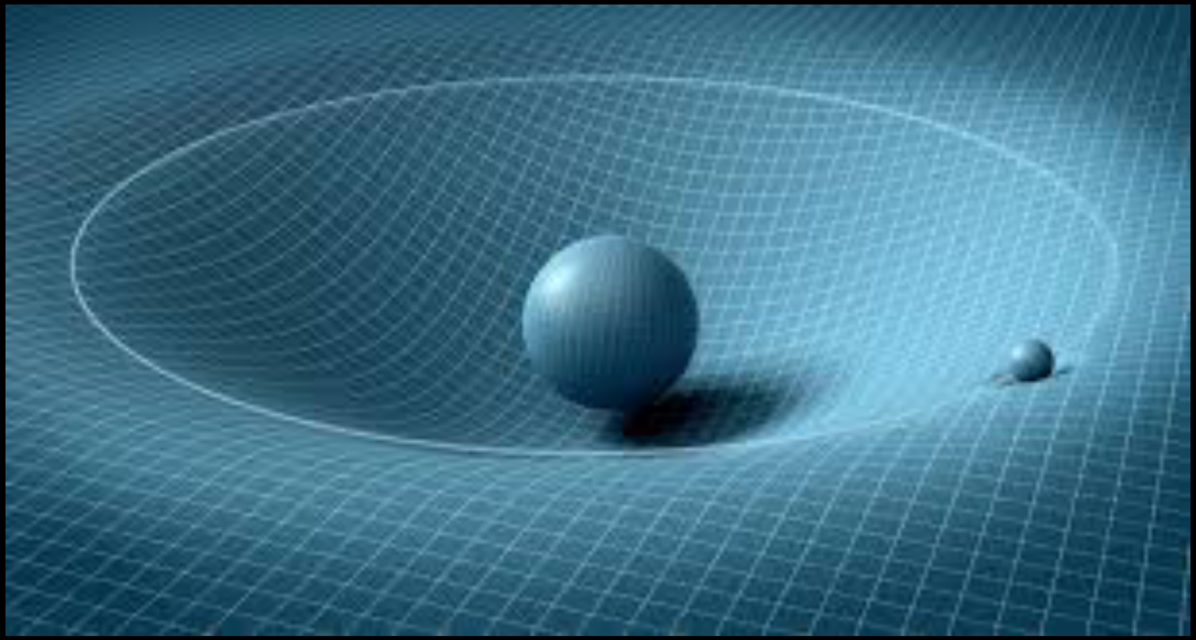
— Supersymmetry:



(e.g. electrons are fermions photons are bosons)

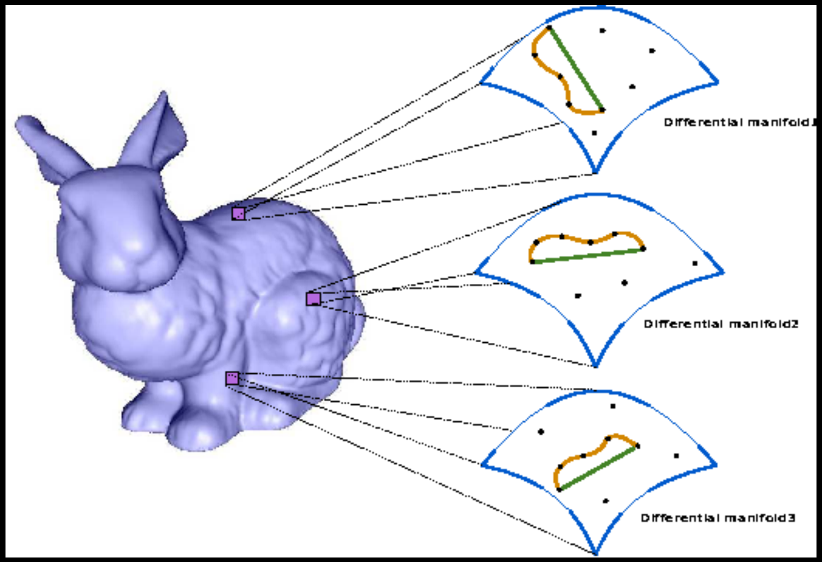


— Einstein gravity



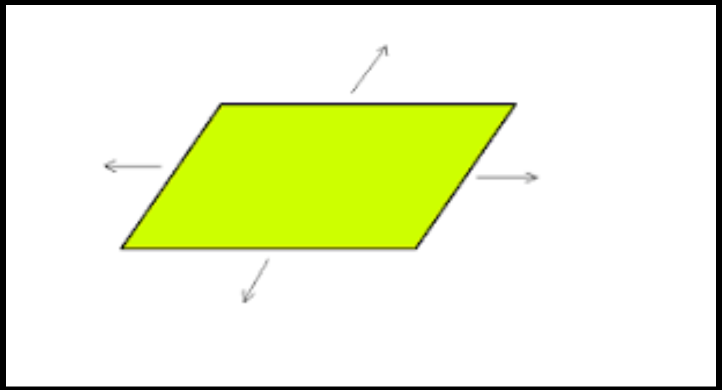
Matter tells space how to curve;
curved space tells matter how to move.

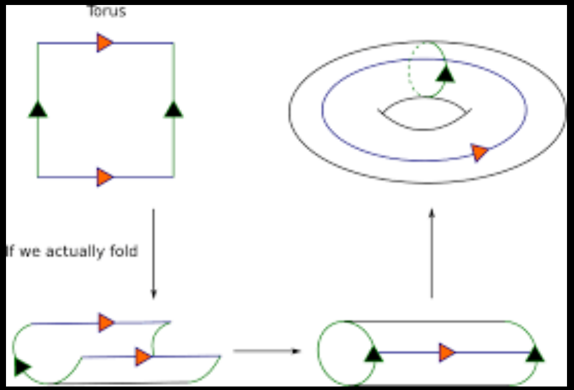
The basic mathematical object in general relativity is the "metric" on a space. It tells one how to find distances between distinct points on the space:



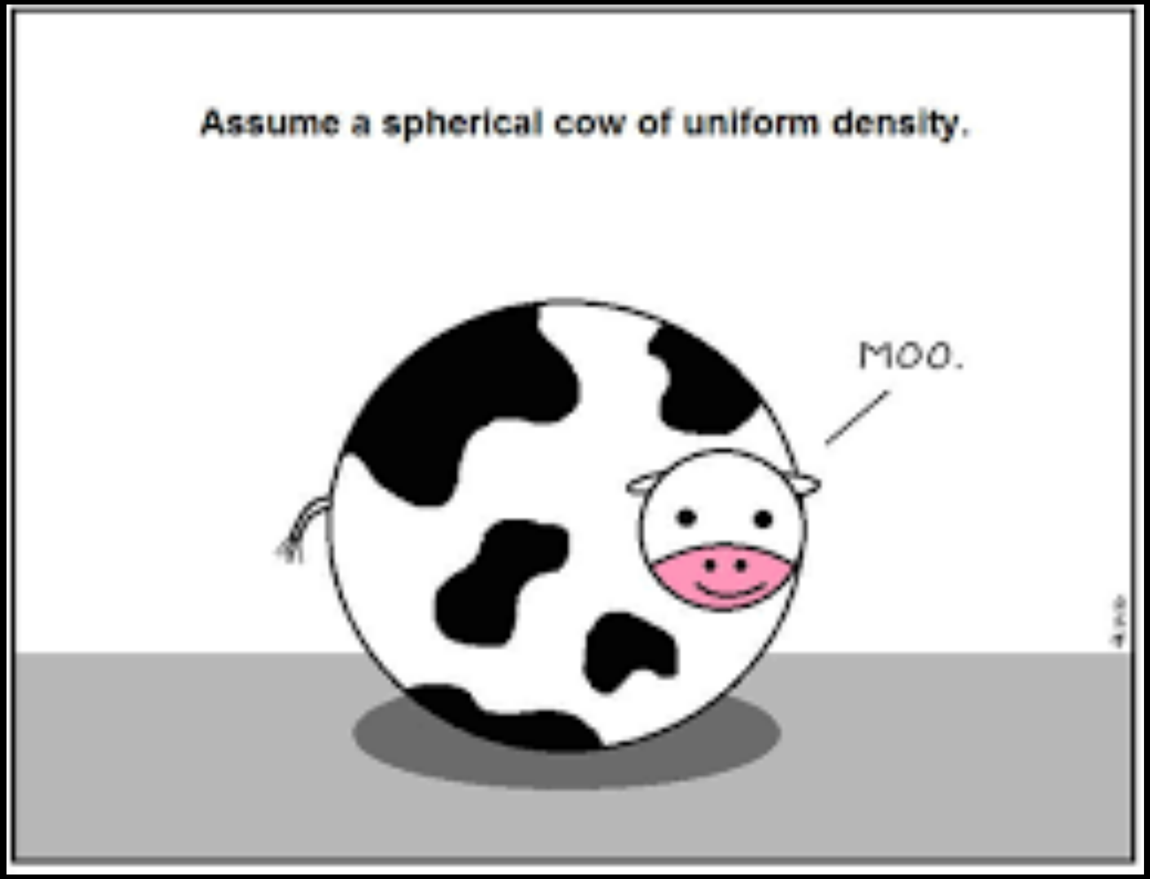
A notable feature of Einstein's equations is that its vacuum solutions are "Ricci flat".

Examples:

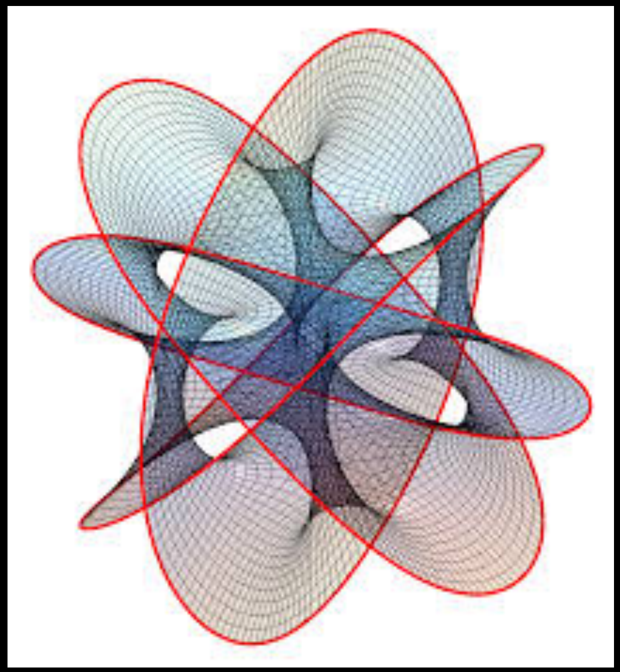




Now, physicists like to study simplified models: "spherical cows"



The spherical cow models of string compactification will involve "Ricci flat," supersymmetry preserving extra dimensions.

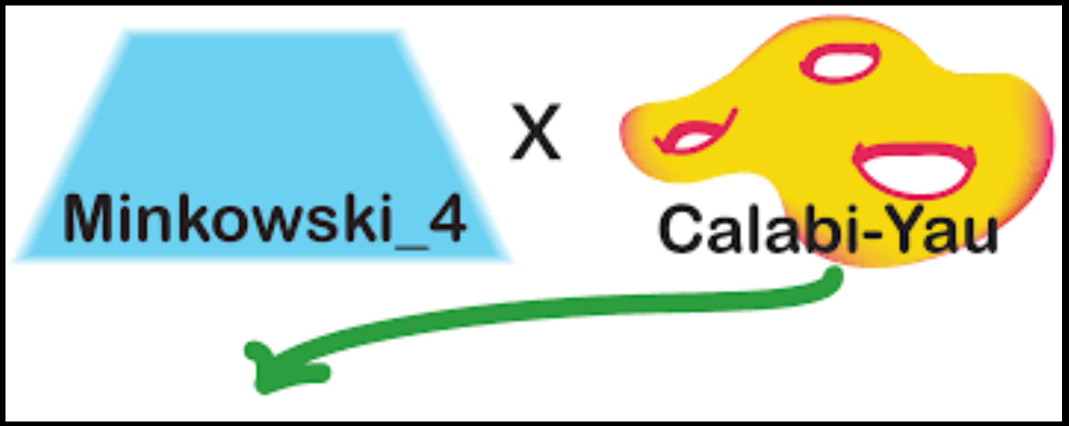


Calabi conjecture: There is a unique Ricci flat Kahler metric on a Kahler manifold of vanishing first Chern class for each choice of the Kahler form.

Yau: this is true.

(not constructive)

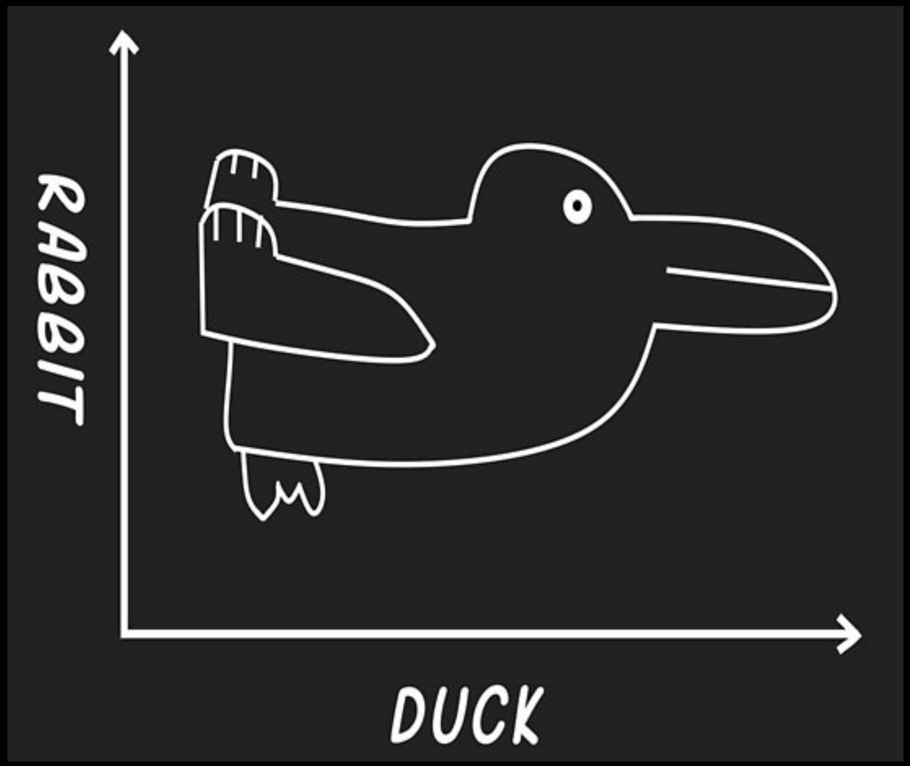
Result — the spherical cow models are:



In this talk, I will discuss geometry problems that we have encountered in studying these spherical cow models of hidden dimensions. The main interest will be the rich connections to various subjects in physics and mathematics: duality, gauge theory, Einstein's equations, algebraic geometry, and differential geometry.

III. Duality

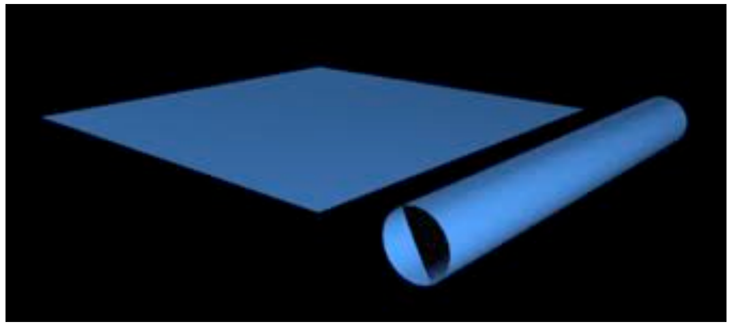
We are now going to explore some simple dualities in string theory. What is duality?

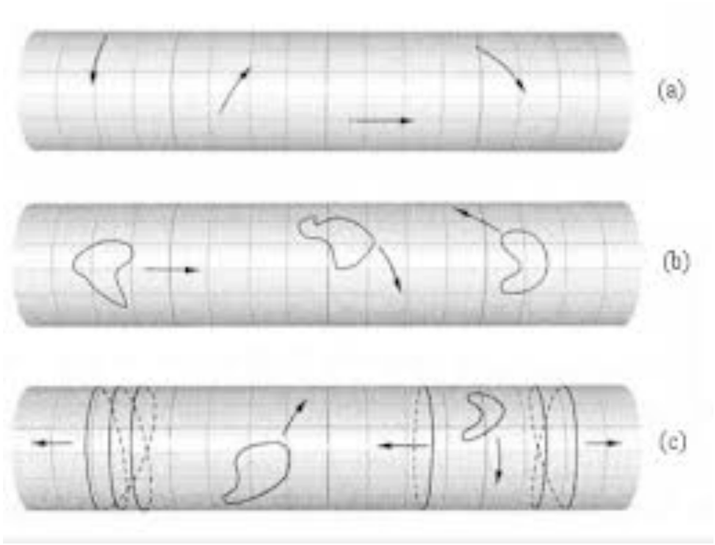


Duality is what happens when there are two equally valid ways of viewing the same system.

In string theory, duality often arises because strings "see" geometry a different way than point particles do.

Let us start by considering string compactification on a circle. Beyond oscillating, the string has two other means of excitation:





Momentum &
Winding
modes

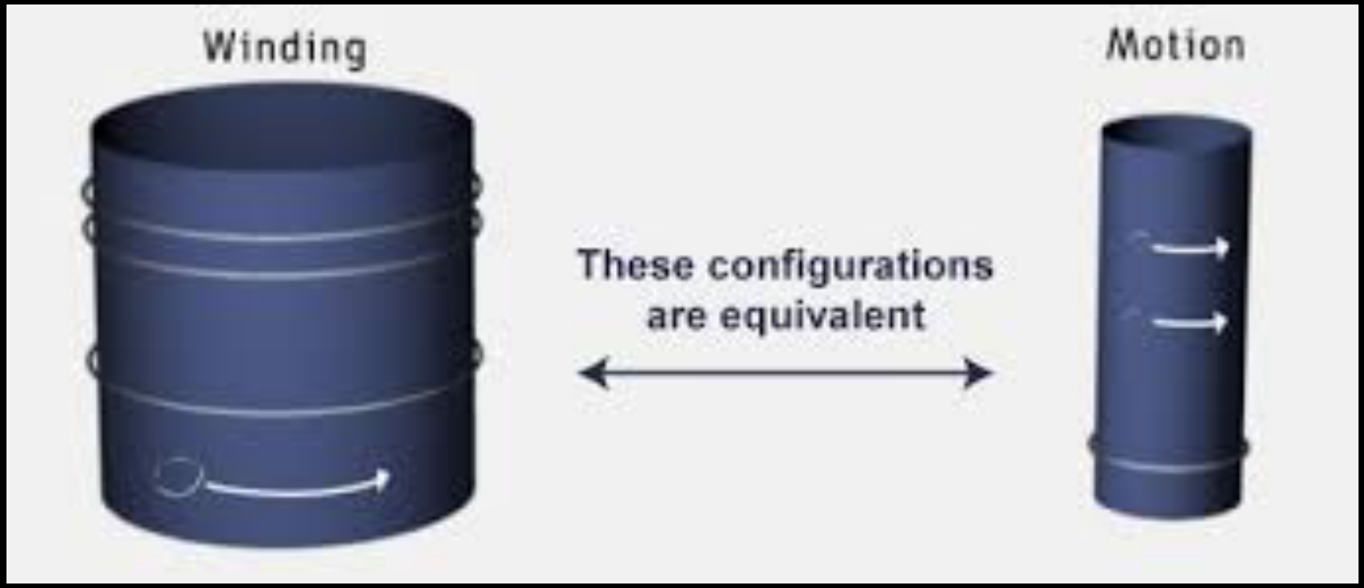
It is an old fact from quantum mechanics that single-valuedness of wavefunctions requires momenta on a circle to be quantized:

$$p = \frac{1}{R}, \frac{2}{R}, \frac{3}{R}, \dots$$

On the other hand, in units of the string scale, winding modes have energies:

$$E = R, 2R, 3R \dots$$

This set of energy levels exhibits an exchange symmetry: winding on a circle of size R is like momentum on a circle of size “1/R”:

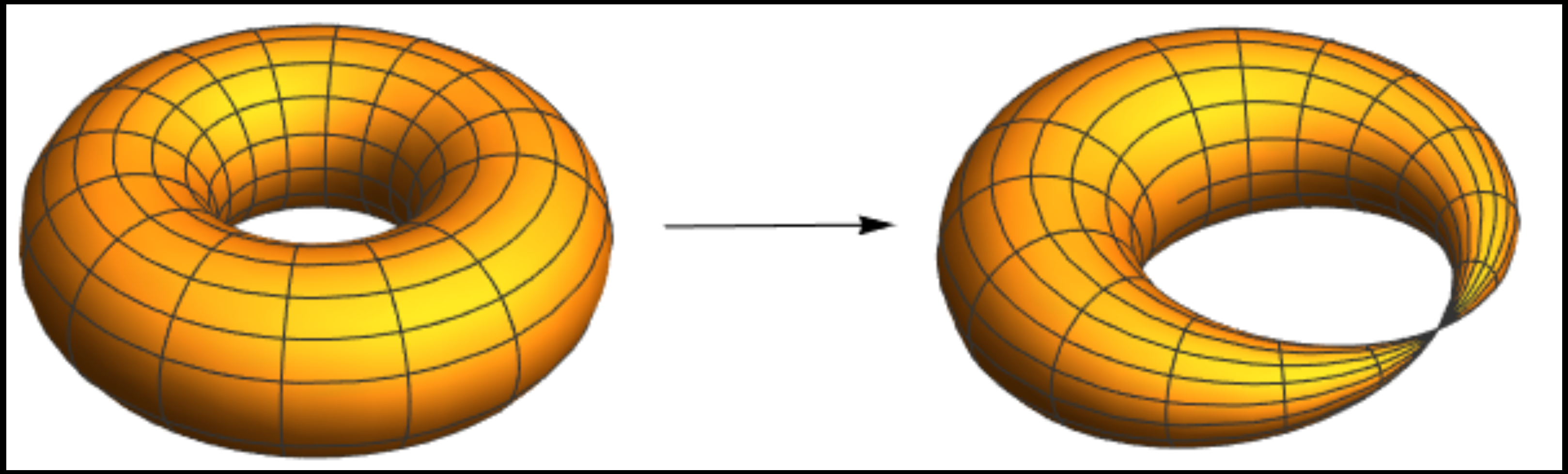


String theory on a circle of radius R is the same as string theory on a circle of radius “1/R”!

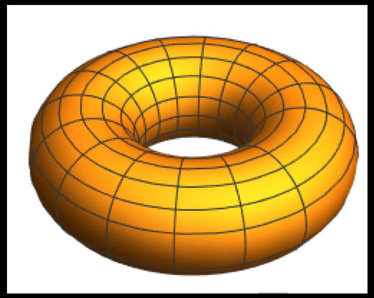
You might think that this kind of bizarre “fuzziness” of the geometry only happens for very simple spaces, like the circle. Not so.

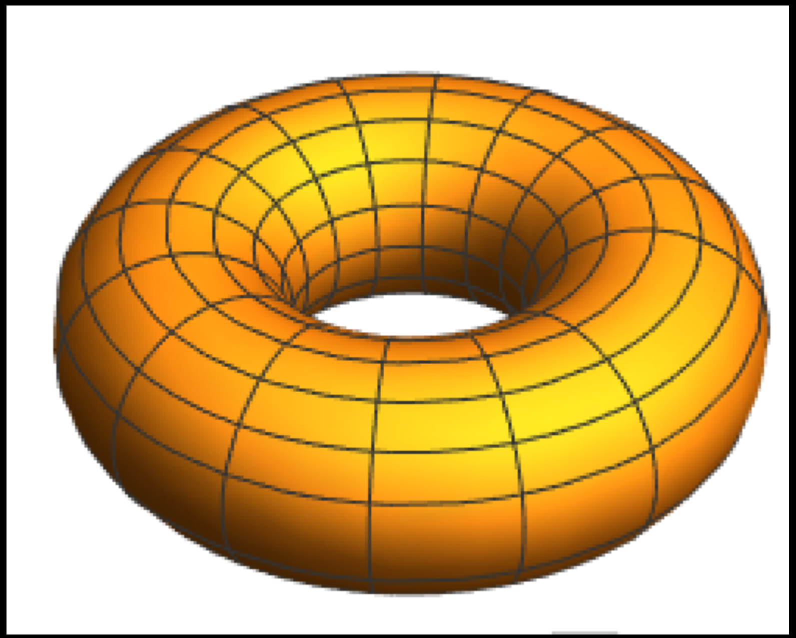
To discuss this, let us consider strings on a Calabi-Yau space. Consider one such topology.

Yau's theorem says that such a topology admits many different metrics which solve Einstein's equation.



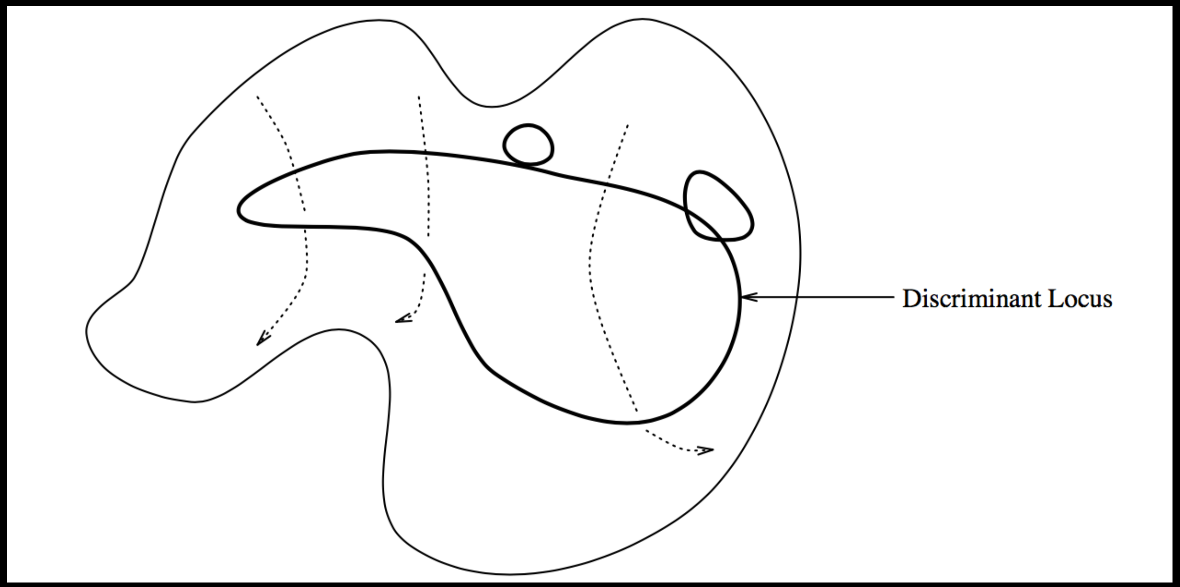
You can vary
the "shape" or
"complex structure"





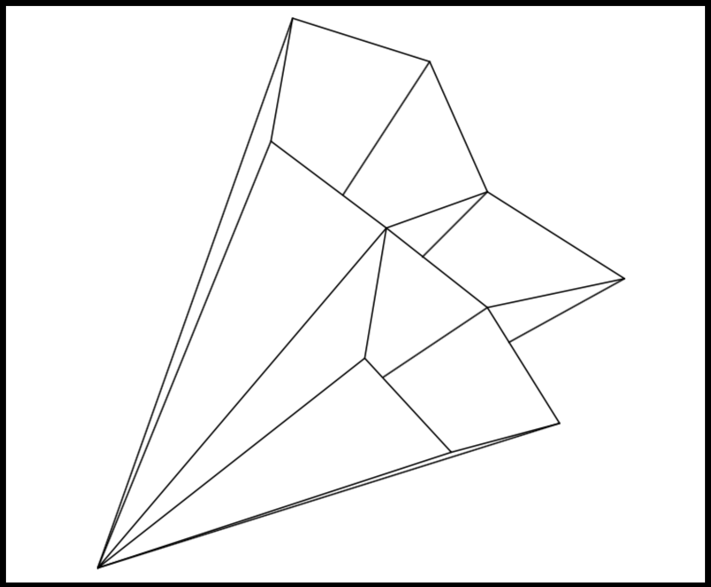
You can also
vary the "size"
or "Kahler structure,"
keeping fixed shape.

Physicists talk about the "moduli space of complex structures" and "moduli space of Kahler structures" on a Calabi-Yau. These are the parameter spaces of possible shapes and sizes.



$\mathcal{M}_{\text{complex}}(X)$

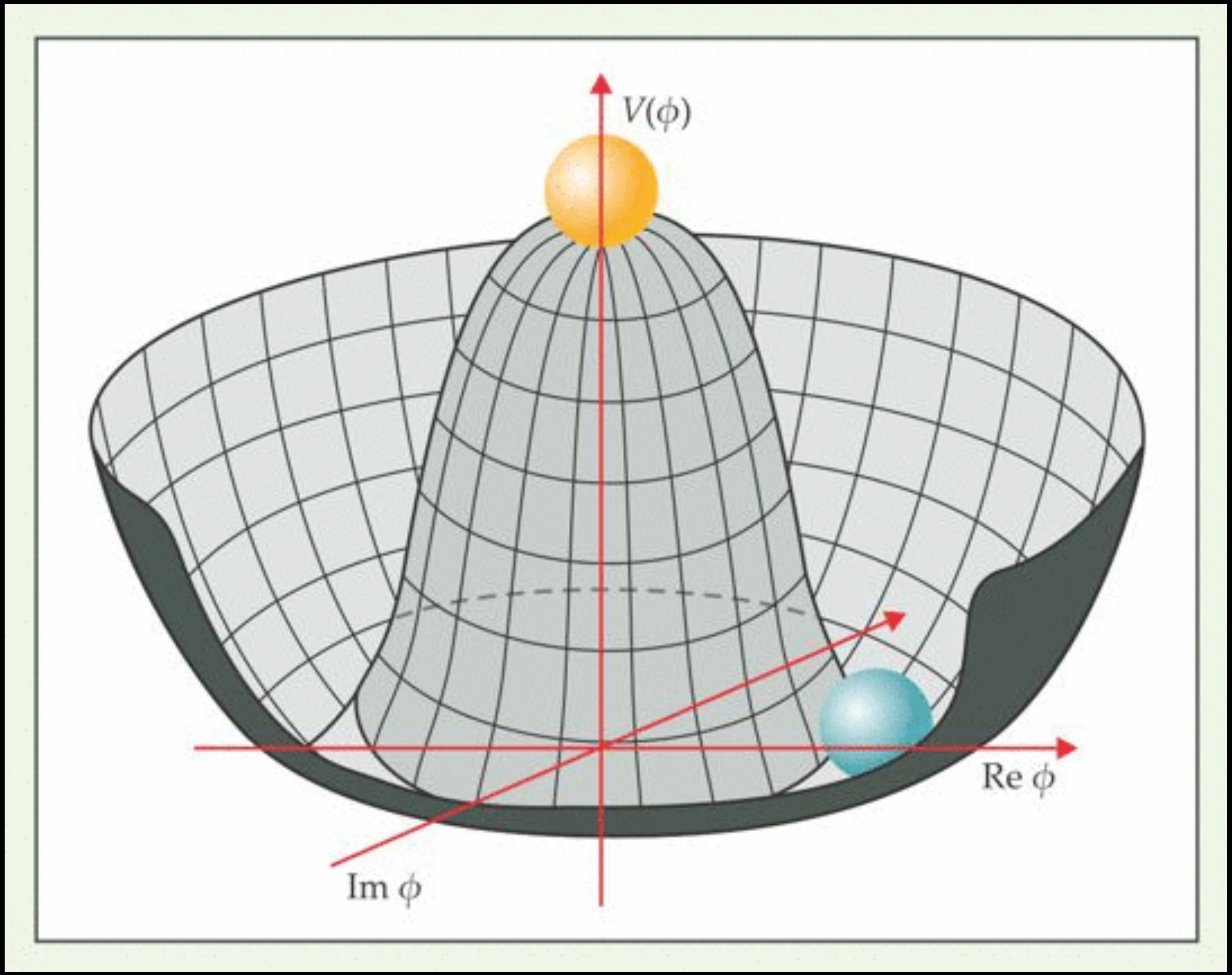
(set of possible shapes)



$\mathcal{M}_{\text{Kahler}}(X)$

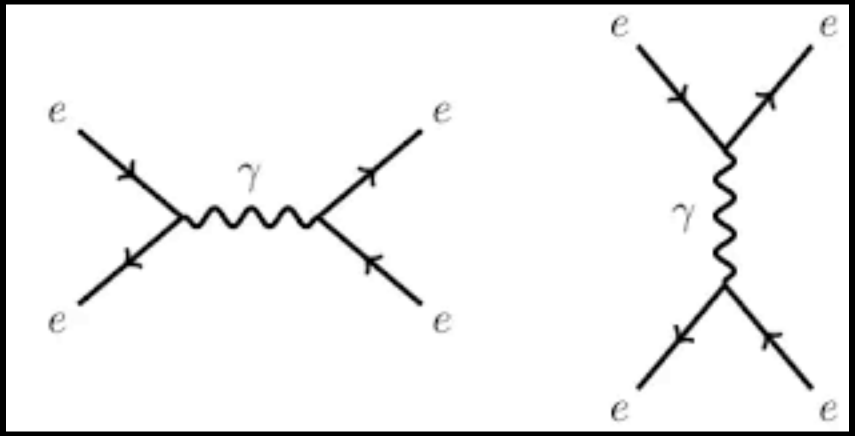
(set of possible sizes)

These parameters show up in 4d physics as the expectation values of scalar fields:



You've probably seen pictures like this of the Higgs potential. The moduli fields move along multi-dimensional analogues of the flat valley.

These fields appear in the 4d physics in an interesting way. For instance, they control the values of “gauge coupling constants” — which determine the strength of low-energy interactions.



The fine structure constant controls electron-photon interactions in QED.

But there is an important detail:

— there are different versions of 10d string theory

IIA string theory

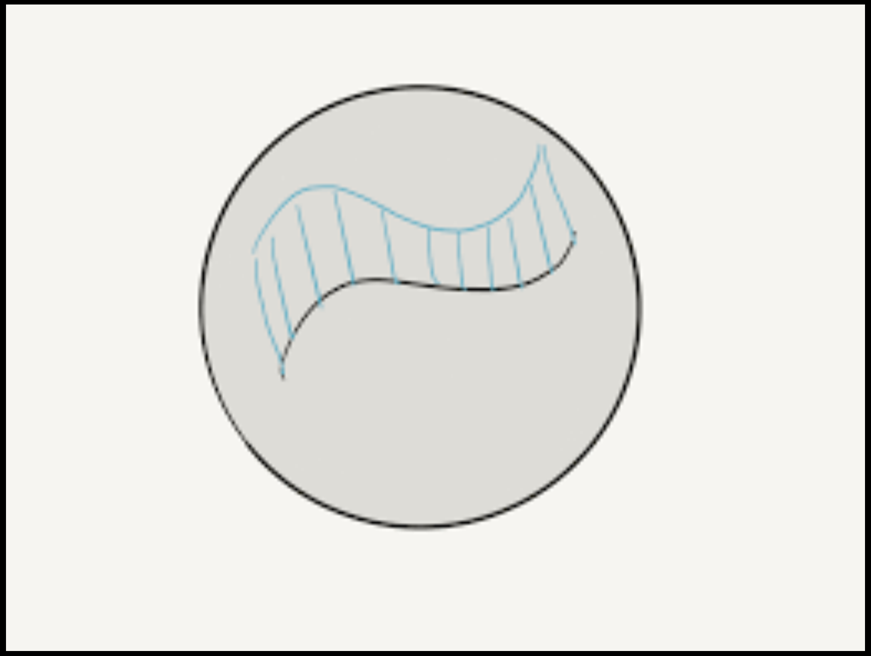
coupling controlled
by size moduli

quantum corrections
exist!

IIB string theory

coupling controlled
by shape moduli

classical result is
exact!



The quantum corrections, when they exist, are horrendous to calculate.

They correspond to counts of spheres of minimal area embedded in the Calabi-Yau manifold.

In simple cases, such spheres are labelled by an integer “degree” (roughly controlling the area).

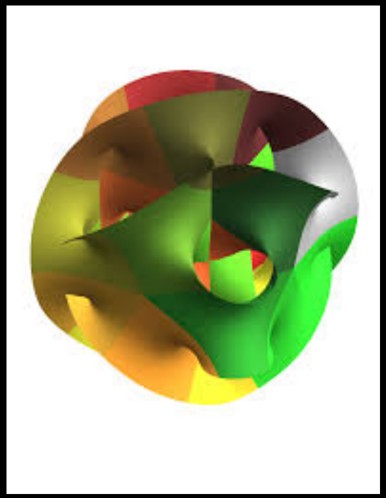
In one simple Calabi-Yau manifold, mathematicians had worked for years and had almost made it to degree three.

degree	# of curves
1	2,875
2	609,250

"The universe that God chose to exist is the best of all possible worlds."

— Leibniz

At least in this case, things work out that way. It turns out that Calabi-Yau manifolds come in pairs!



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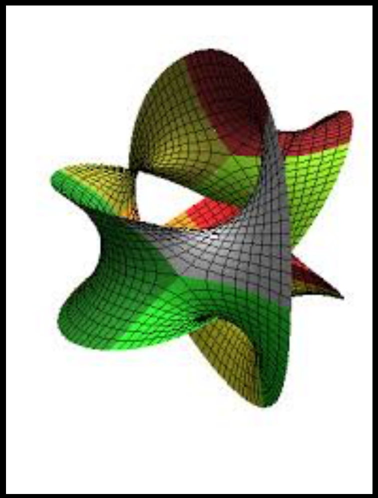


IIA on X

IIB on Y

Physics of string theory on two manifolds with drastically different topologies can be identical!

The coupling constants controlled by quantum corrections on X, can be discovered by doing classical computations on the “mirror manifold” Y.

Candelas, de la Ossa, Green, Parkes:

TABLE 4
The numbers of rational curves of degree k for $1 \leqslant k \leqslant 10$

k	n_k
1	2875
2	6 09250
3	3172 06375
4	24 24675 30000
5	22930 58888 87625
6	248 24974 21180 22000
7	2 95091 05057 08456 59250
8	3756 32160 93747 66035 50000
9	50 38405 10416 98524 36451 06250
10	70428 81649 78454 68611 34882 49750

Results in striking (and now verified) mathematical predictions of string theory!

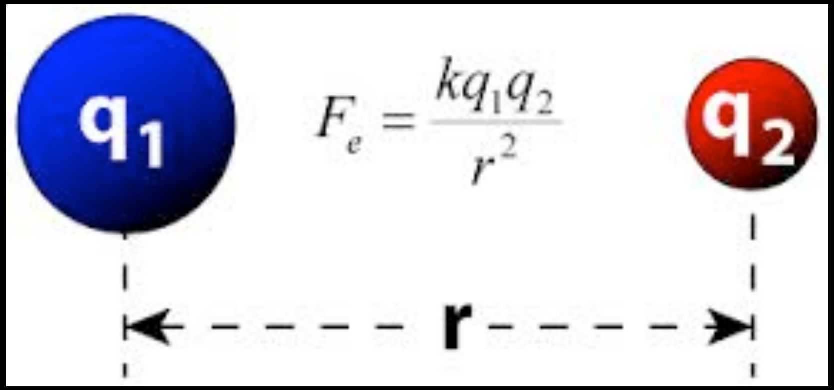
The subject of mirror symmetry — and understanding how to get Y from X — remains very active and fruitful!

Mirror symmetry first emerged in the late 1980s as an unexpected duality between seemingly unrelated quantum field theories. Important work of Yau and of Kontsevich suggested that these dualities were manifestations of deep mathematical connections between previously disparate mathematical disciplines, including algebraic geometry, symplectic topology and category theory. The Simons Collaboration on Homological Mirror Symmetry is motivated by the idea that the time is now ripe to prove fundamental theorems establishing the existence of mirror symmetry in full generality, and to explore the applications of this symmetry.

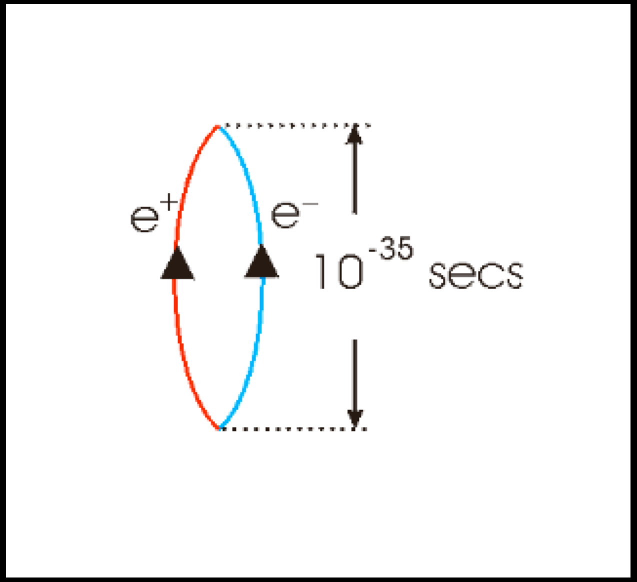
IV. Gauge theory

We now turn to a naively distinct subject; it will circle back.

We are all forced to learn Coulomb's law in high school.

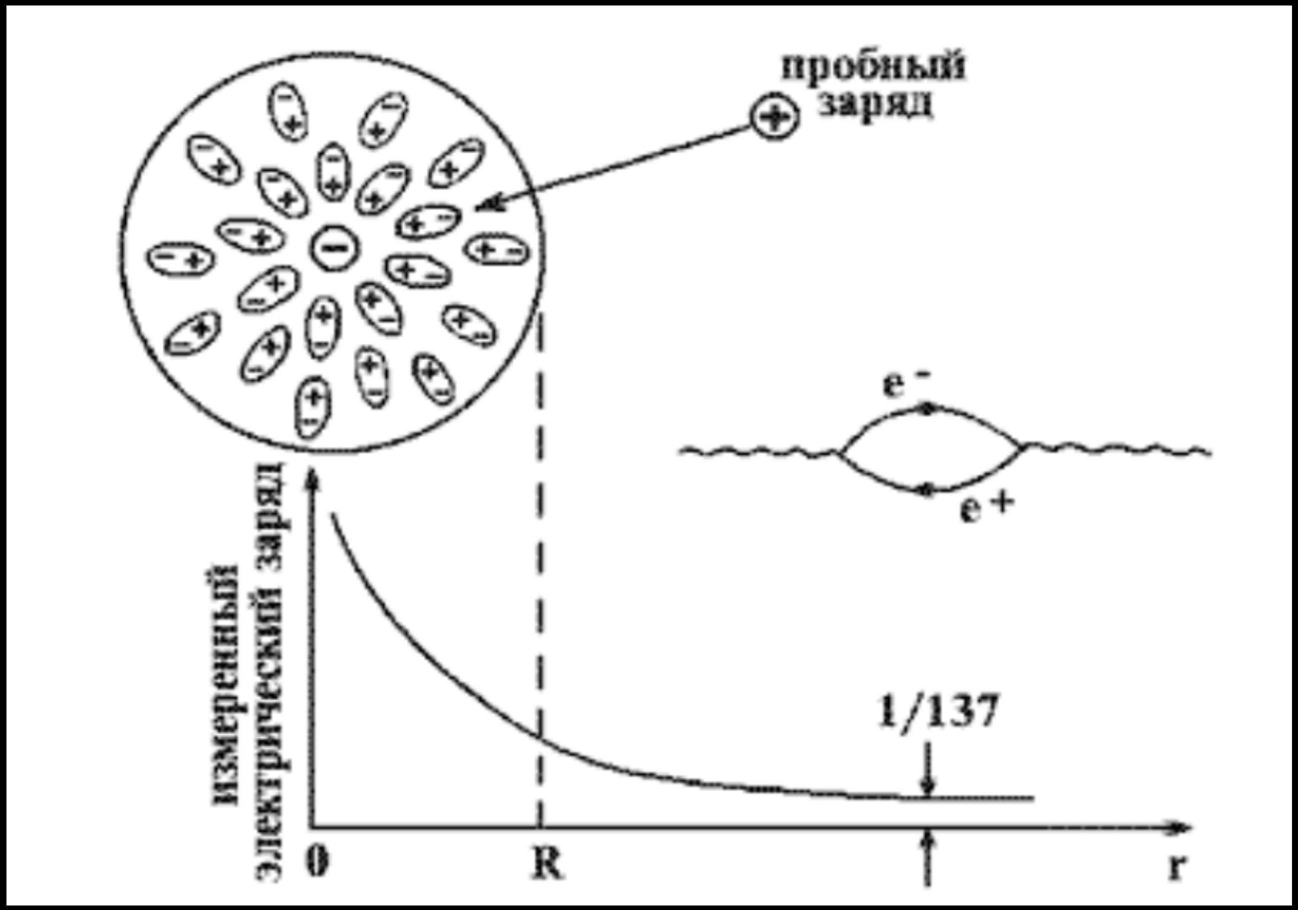


What we don't learn then is that the coupling "constant" appearing in the law isn't constant.



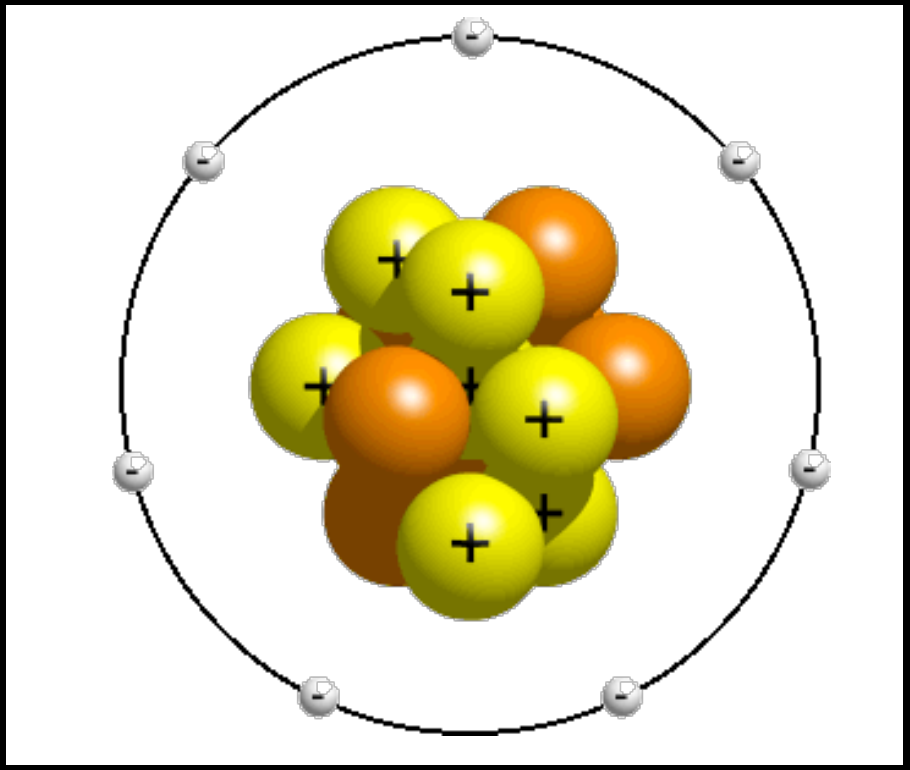
In quantum mechanics, electron/positron pairs can "virtually" appear from the vacuum and screen the Coulomb force.

Result: the QED coupling gets weaker at long distances (where we do experiments!).



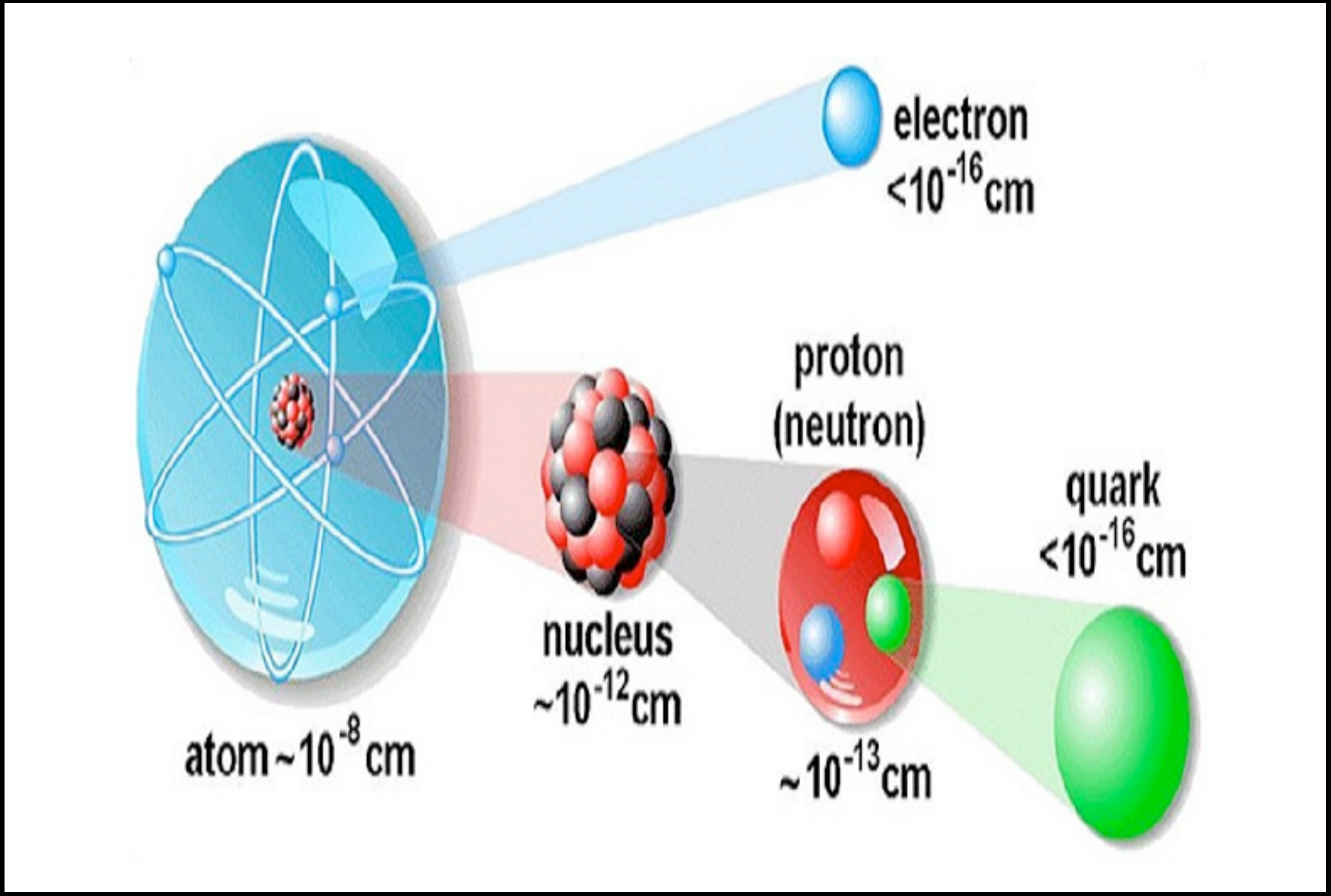
This just makes our life easier. It is easy to understand very weak interactions.

Unfortunately, Nature also contains strong interactions — e.g., nuclear forces.

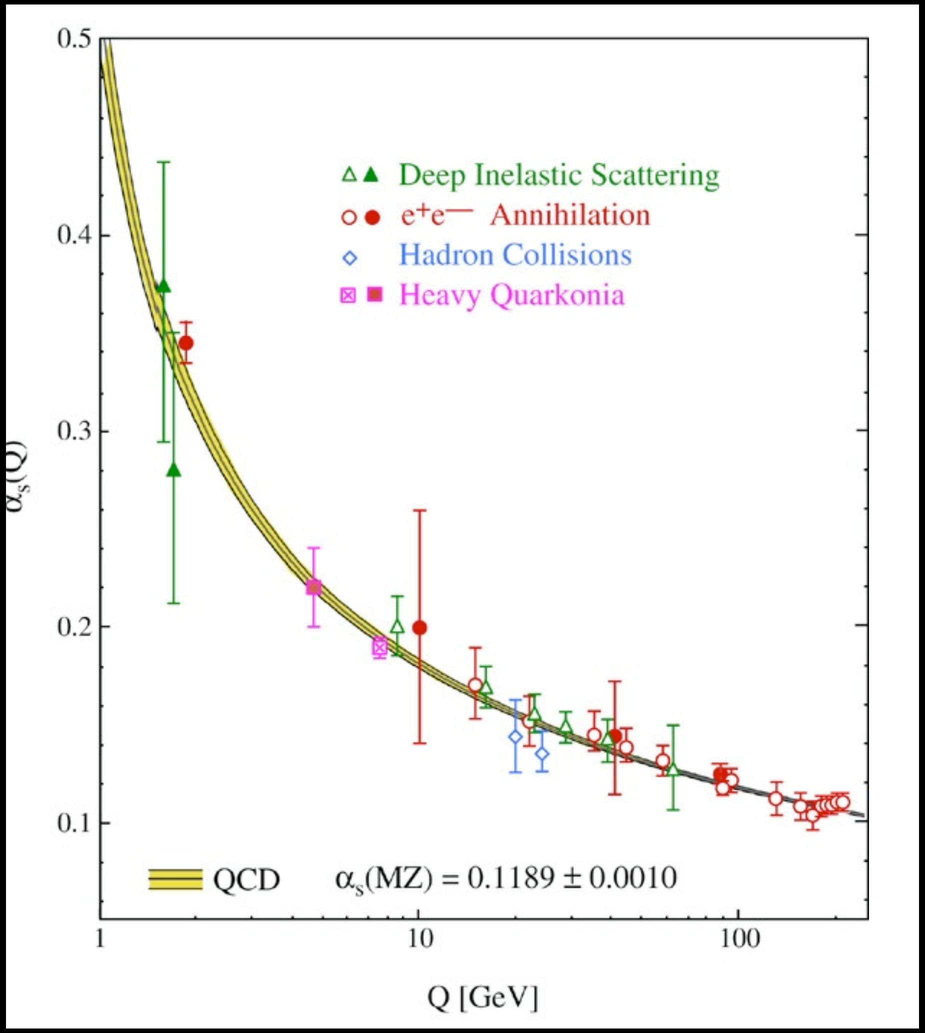


We see nucleons pictured here — the neutron and the proton.

The powerful accelerators we've built over the past decades have acted as microscopes to let us see ever further into the structure of matter:



Because of “anti-screening” of the strong force, free quarks are never seen.



Instead, quarks are confined.

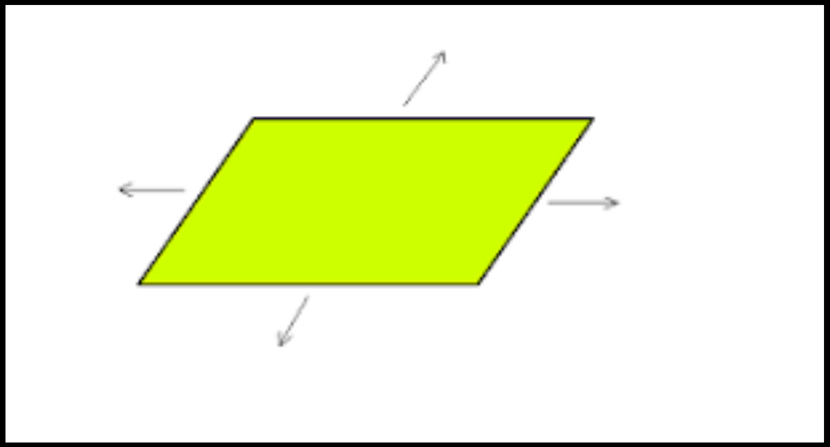
Both electromagnetism and the strong interactions are described by theories known as gauge theories.

The examples like the strong force — with interaction strength that grows strong at long distance — are too hard for us to solve.

By adding supersymmetry, we can imagine "spherical cow" examples of such gauge theories. Can we solve them?

What do we mean by "solve them"?

— These supersymmetric gauge theories again have "moduli spaces of vacua."



simplest examples: the moduli space has one complex dimension. Generically get QED-like theories.

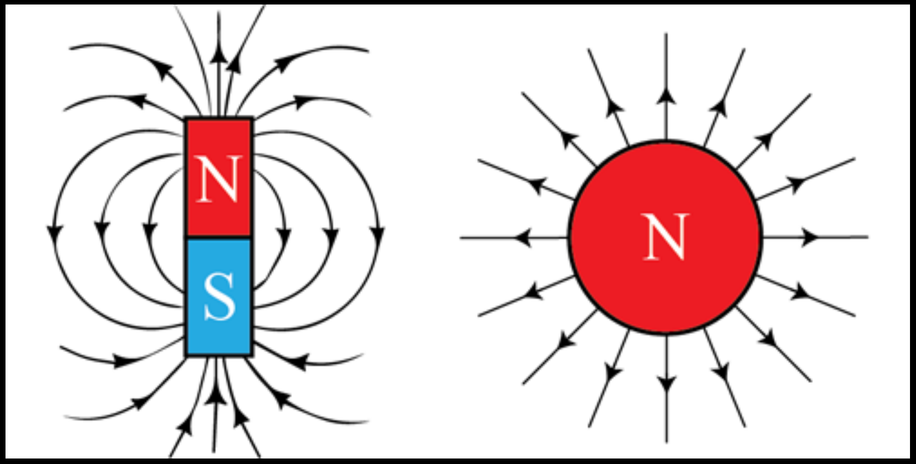
We will say we've solved the theory if we can tell you the precise masses and charges of particles at all points on the space of vacuum states.

Supersymmetry gives us a tool to do this.

— in e.g. QED, you can imagine

* particles with electric charge

* particles with magnetic charge



If we call the complex parameter parametrizing our 1d moduli space “a,” then supersymmetry says that there is a special function

$$\mathcal{F}(a)$$

“prepotential”

which controls the particle masses, the coupling “constant,” and the metric on moduli space.

In particular,

$$M(n_e, n_m) \geq |n_e a + n_m \frac{\partial \mathcal{F}}{\partial a}|$$

To “solve” the theory, we need to:

— determine this function $\mathcal{F}(a)$

— determine the degeneracies at a given value of the charges

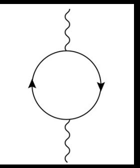
$$\#(n_e, n_m; a)$$

These are both challenging problems that have been subjects of tremendously fruitful research!

The main challenge in determining the prepotential:

$$\mathcal{F} = \text{classical} \ + \ 1 - \text{loop} \ + \ \text{instantons}$$

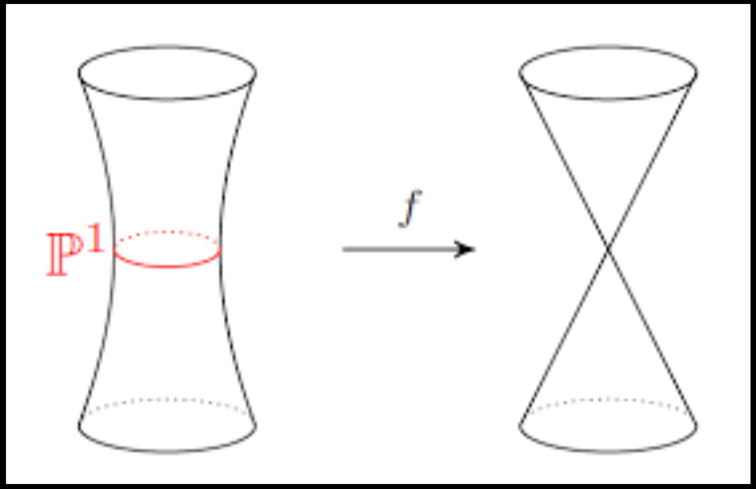
(easy)



"Instantons" are field configurations of the gluon field of the strongly interacting theory. There are 1,2,3,… instanton configurations. Determining their contributions - a counting problem of sorts - is very difficult!

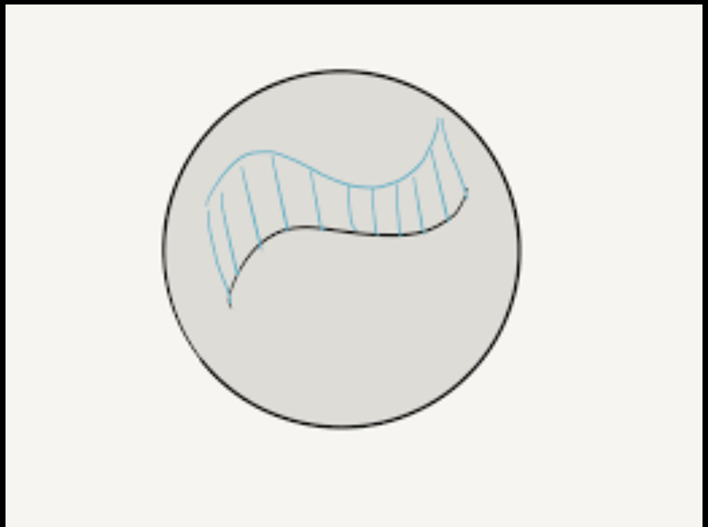
This was first done in seminal work of Seiberg and Witten in quantum field theory.

We soon found that string theory also gives a striking way of solving such theories!



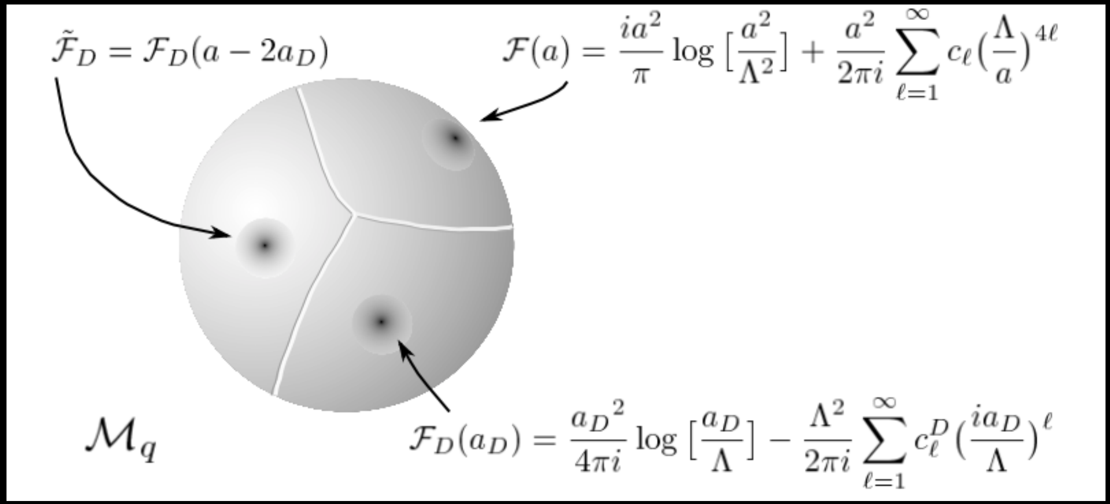
Certain string compactifications on singular Calabi-Yau manifolds give supersymmetric strongly interacting gauge theories.

The string model that makes the interpretation in terms of gauge theory obvious is a IIA model.



It geometrizes the instantons we wish to count as minimal area spheres in the Calabi-Yau space.

We can count these using mirror symmetry as in the previous part of the talk!



S.K., Vafa;
S.K., Klemm, Lerche, Mayr, Vafa; …

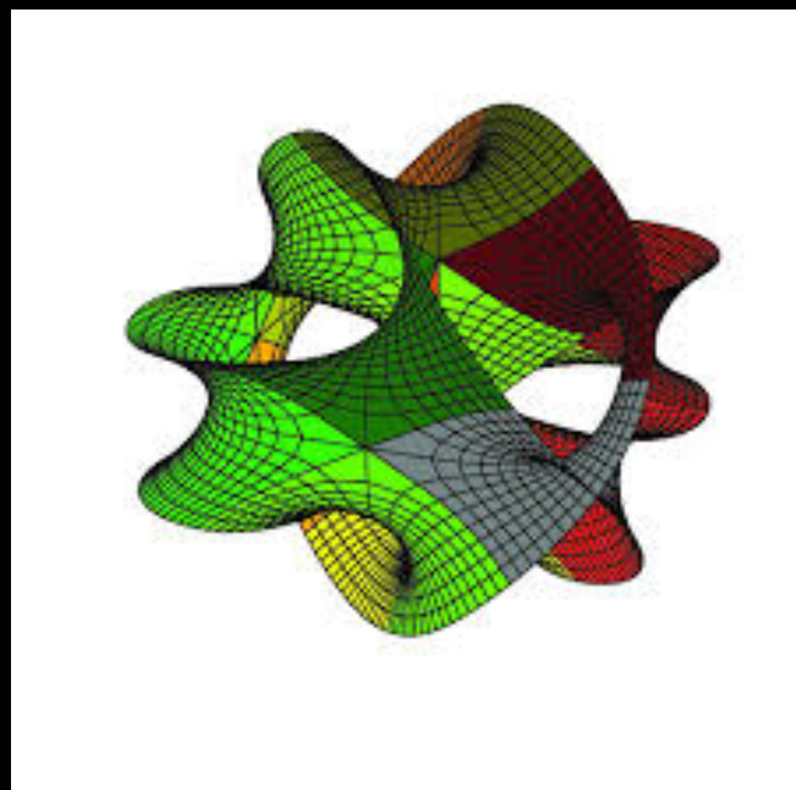
V. Conclusion: the geometry of inner space

So far, we've discussed two kinds of geometric problems and their applications to physics (and equally, the "unreasonable effectiveness of physics in mathematics"):

— enumerative geometry and the physics of Calabi-Yau compactification in string theory

— dynamics of gauge theories, counting instantons, and geometry

A central role has been played by Calabi-Yau spaces:



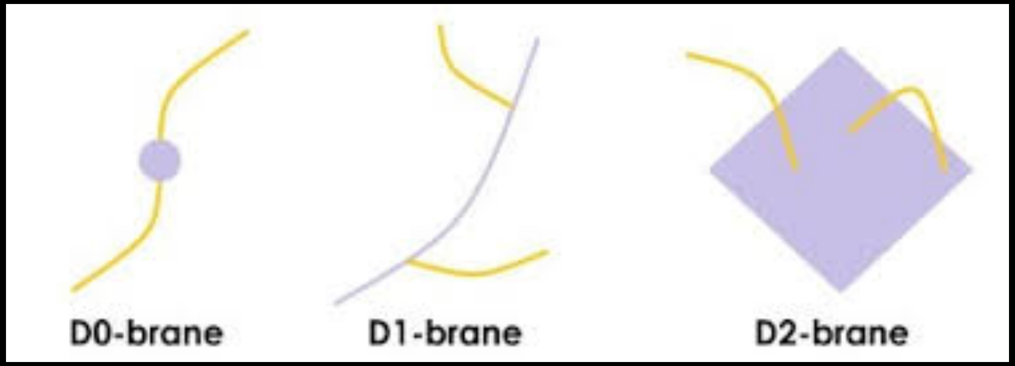
Slice of a K3 surface, the unique non-trivial Calabi-Yau space in two complex dimensions

We haven't discussed the most basic piece of data in specifying such a space in general relativity: its Ricci-flat metric.

For K3, we can combine the two stories I've told so far to obtain explicit analytical expressions for the metric which solves Einstein's equations!

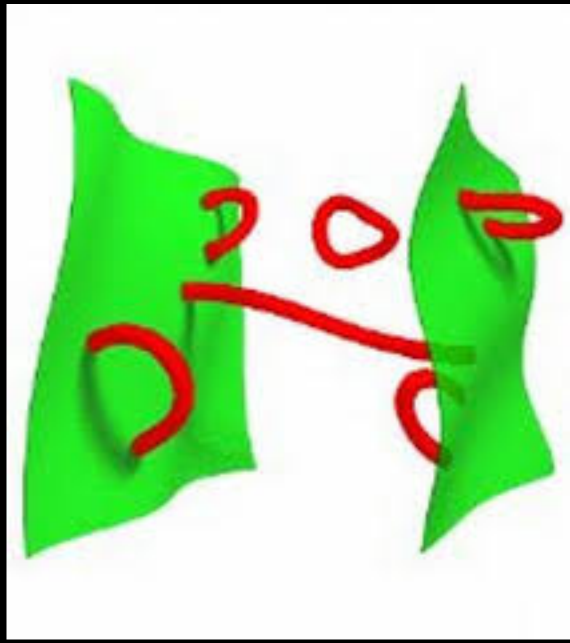
S.K., Tripathy, Zimet

The basic observation is rather simple.



String theory is a highly constrained structure, but within its list of ingredients, it includes "D-branes."

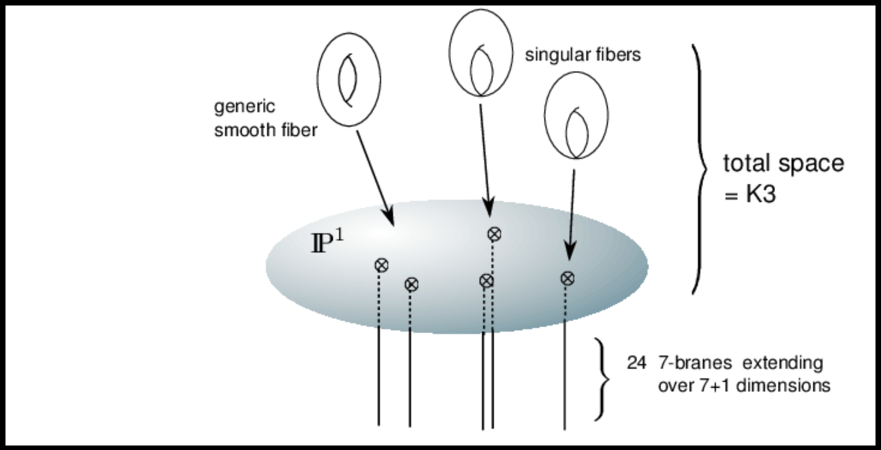
At low energies, the theories of stuff “living on” the D-branes are ordinary gauge theories of the sort we’ve discussed.

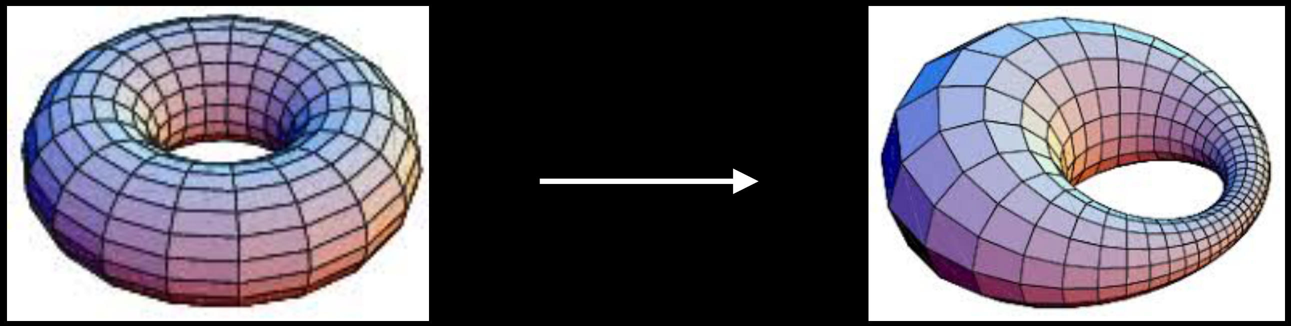


The gauge theories on D-branes involve particle species arising from open strings that stretch from one brane to another.

There is a particularly useful packaging of the data of a K3 surface (in some limit), as the geometry arising from a certain collection of D-branes.

One views the K3 surface as a collection of two-tori ("donuts") varying over a base two-sphere.

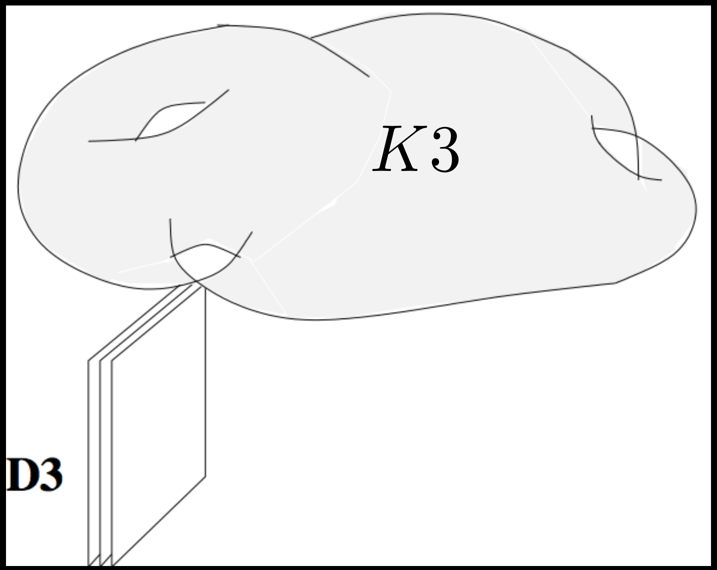




A solution of string theory can be obtained by compactifying on the two-sphere, and inserting D7-branes transverse to the two-sphere, at the points where the "donuts" degenerate as shown.

We can consider the quantum field theory on a D3-brane "probing" this K3 surface.

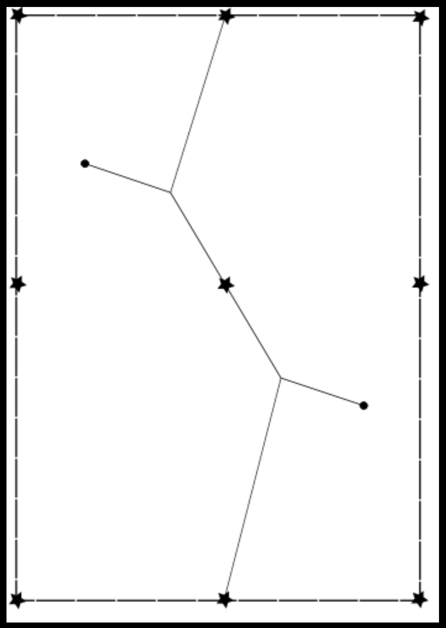
Sen; Banks, Douglas, Seiberg



Really, the D3-brane is moving around on the two-sphere base.

— It has a moduli space of vacua, much like the gauge theories we considered in section IV.

— The BPS states in this theory arise from strings that stretch from the D3-brane to the singular fibers:



Shown is a simple example of such a BPS state; the stars are singular fibers, and the dot is the D3-brane.

To “solve” the D3 brane theory, we need:

— a prepotential: $\mathcal{F}(a)$

— a BPS state count: $\#(n_e, n_m; a)$

In fact, if we can determine this data, we do more than just solve the theory:

$$\mathcal{F} \to \text{metric on moduli space}$$

$$\mathcal{F} + \# \to \text{metric on K3}$$

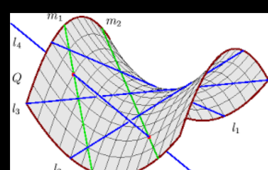
This is a very complicated problem.
Happily, the ideas of section III of the talk rescue us.

There is a dual problem which allows us to determine the integer BPS degeneracies (analogous to "counting curves" in section III), and write the metric in this formalism.

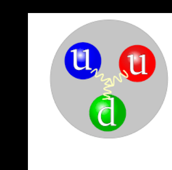
Doing this in detail would take me far beyond the confines of a popular lecture!

What I hope I did convey:

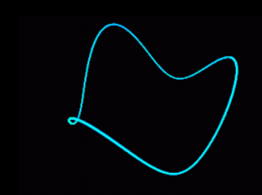
algebraic geometry



strongly interacting quantum field theory



string theory



differential geometry

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$