Replica Wormholes and the Black Hole Interior

Geoff Penington, Stanford University and UC Berkeley

Work with Steve Shenker, Douglas Stanford and Zhenbin Yang

See also: Almheiri, Hartman, Maldacena, Shaghoulian, Tajdini
The Claim

- We can successfully calculate the Page curve for an evaporating black hole just using a **gravitational path integral**.

- In this talk, I will focus on a **highly simplified model**, where we can do the most precise calculations, but the basic story seems to be very general.

- The arguments came out of ideas (QES prescription, entanglement wedge reconstruction) from AdS/CFT. **However**, don’t need a CFT, string theory, anti-de Sitter space, etc.

- All we need is an effective low energy gravitational path integral, where you sum over arbitrary topologies, including ‘**spacetime wormholes**’.

- Very similar arguments can be used to show that information **escapes** from the interior of the black hole into the Hawking radiation (see Douglas’ talk).
Part 1: A Very Simple Information Paradox
The Information Paradox in Evaporating Black Holes

- Hawking radiation is entangled with interior partner modes inside the black hole.
- Eventually, the entanglement entropy seems to become larger than the Bekenstein-Hawking entropy of the black hole (the Page time).
- If the BH entropy is truly the statistical entropy of black hole microstates (true in string theory, AdS/CFT), this is a paradox: not enough BH degrees of freedom to be able to purify the Hawking radiation.
- Possible resolutions: a) information loss or b) entanglement entropy starts decreasing at/before Page time (Page curve).
A Very Simple Model: Pure JT gravity plus EOW Branes

Analogue for Hawking radiation: add internal degrees of freedom to the EOW brane (interior modes) that are maximally entangled with a reference system.
A Very Simple Information Paradox

\[ |\psi_i\rangle_B = \text{Euclidean} = \text{Lorentzian} \]

Orthogonal basis for internal state of EOW brane

\[ |\Psi\rangle = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} |\psi_i\rangle_B |i\rangle_R. \]

(Simple) information paradox when \( k \gg e^{S_{BH}} \). Entanglement entropy seemingly becomes larger than the Bekenstein-Hawking entropy.
Part 2: Finding the Page Curve
The Island Rule/QES Prescription

Rule for computing entanglement entropies in gravity:
1. Find all (quantum) extremal surfaces
2. Entanglement entropy is the generalised entropy of the minimal QES.

Two extremal surfaces: the empty set \( S_{gen} = \log k \) and the bifurcation surface \( S_{gen} = S_{BH} \)

\[
S = \min(\log k, S_{BH})
\]
Deriving the Island Rule from the Gravitational Path Integral

This rule wasn’t just **made up** to give the ‘right’ answer! It can be **derived** from the gravitational path integral (just like the **BH entropy** can be derived using a Euclidean path integral)

**Rest of this talk:** derive the Page curve, directly from the gravitational path integral at three increasing levels of precision.

<table>
<thead>
<tr>
<th>Level of Precision</th>
<th>Formula for Page Curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>Topology</td>
<td>$S = \min (\log k, S_0)$</td>
</tr>
<tr>
<td>Geometry of leading classical saddle</td>
<td>$S = \min (\log k, S_{BH})$</td>
</tr>
<tr>
<td>Full path integral</td>
<td>$S = - \int d\lambda \lambda \log \lambda \text{Im}[R(\lambda)]/\pi$  &lt;br&gt; where $\lambda R = k + e^{S_0} \int ds \rho(s) \frac{y(s)R}{k Z_1 - y(s)R}$</td>
</tr>
</tbody>
</table>
The Replica Trick

• How do you calculate **von Neumann entropies** using a path integral?

• **Answer:** the integer **n Renyi entropies**

\[
\frac{1}{1 - n} \log \text{Tr} \rho_R^n
\]

are proportional to the logarithm of an observable on **n copies of the system**.

• We can calculate the von Neumann entropy by **analytically continuing the Renyi entropies** to **n=1**.

• **The key idea:** the gravitational path integral includes topologies that connect the different replicas via **spacetime wormholes**.
Calculating the Purity

Calculate the purity $Tr(\rho_R^2)$ using a **Euclidean path integral**, where we sum over all topologies with the correct boundary conditions:

$$
Tr(\rho_R^2) = \frac{1}{k^2} \sum_{i,j=1}^{k} \left| \langle \psi_i | \psi_j \rangle \right|^2.
$$

$$
= \frac{1}{k}
$$

Very small but doesn’t decay at large $k$

Two copies of the original black hole

Two black holes connected by a wormhole

Einstein-Hilbert term

$\sim$ Euler characteristic
Factorisation and Averaging

We didn’t have to sum over k! The saddles exist even without the sum

Wormhole topology means $|\langle \psi_i | \psi_j \rangle|^2$ is non-zero even when $i \neq j$

However, we do have $\langle \psi_i | \psi_j \rangle = 0$ when $i \neq j$.

The theory has a factorisation problem

Explanation: the gravity calculation (at least in this simple case) is not computing the inner product in a single quantum theory, it is computing an average inner product, across an ensemble of theories
Calculating the von Neumann Entropy I: Topology

In general, there are a lot of topologies that can contribute to $\text{Tr}(\rho_R^n)$. However, in the limit where $k$ is very large/small one of two families of topologies dominates.

Small $k$

\[ \frac{1}{k^{n-1}} = \text{Disconnected topology} \]

Gives $S = \log k$

Large $k$

\[ \text{Connected topology} \]

Gives $S \sim S_0$

\[ \sim e^{-(n-1)S_0} \]

Dotted lines = summed-over indices
Calculating the von Neumann Entropy II: The Leading Classical Saddle

• Connected topology has a $\mathbb{Z}_n$ replica symmetry.
• After *quotienting* by this symmetry, we get *roughly* the original black hole geometry, except that there is a *conical singularity* at the fixed point of the replica symmetry.
• In the limit $n \to 1$, the singularity *vanishes* (get original unbackreacted geometry).
• Von Neumann entropy given by the “area” of replica fixed point (in this case the *bifurcation surface*)
Calculating the von Neumann Entropy III: The Full Path Integral

Assume $k, e^{S_0} \to \infty$, but ratio arbitrary, suppressed by $e^{-2S_0}$ and $1/k^2$.

Only planar topologies survive.

**Problem:** we need to not only evaluate this sum over topologies for finite integer $n$ (doable) but also **analytically continue** the formula to non-integer $n$ (hard).
Calculating the von Neumann Entropy III: Resolvents to the Rescue

\[ R_{ij}(\lambda) = \left( \frac{1}{\lambda \mathbb{1} - \rho_R} \right)_{ij} = \frac{1}{\lambda} \delta_{ij} + \sum_{n=1}^{\infty} \frac{1}{\lambda^{n+1}} (\rho_R^n)_{ij}. \]

Graphically, 

Summing over bulk planar topologies,
Calculating the von Neumann Entropy III: Resolvents to the Rescue

Summing over bulk planar topologies,

\[ Z_n = e^{S_0} \int ds \, \rho(s) y(s)^n \]

Reorganising the sum over topologies,

\[ \rho(s) = \frac{s}{2\pi^2} \sinh 2\pi s \]

\[ y(s) = e^{-\frac{\beta s^2}{2}} 2^{1-2\mu} \left| \Gamma \left( \mu - \frac{1}{2} + is \right) \right|^2 \]

All terms with first boundary connected to no other boundaries
Calculating the von Neumann Entropy III: The Density of States

Evaluating this infinite sum (as a geometric series), gives the equation

$$\lambda R = k + e^{S_0} \int ds \frac{y(s)R}{kZ_1 - y(s)R}$$

$$\sum_i R_{ii}(\lambda)$$

The resolvent $R$ encodes not just the entanglement entropy, but the entire entanglement spectrum of the state, since the density of states

$$D(\lambda) = \frac{1}{\pi} Im(R(\lambda))$$

We don’t have an explicit solution to the resolvent equation, but it can be solved numerically, and is also well approximated by a shifted, cut-off thermal spectrum.
Calculating the von Neumann Entropy III: The Final Answer

Using this approximate solution, one can calculate the Page curve up to at most $O(G_N)$ corrections (and much more accurately far away from the phase transition)

Far away from the transition, the corrections to the naïve Page curve we found before are exponentially small (although they can still be computed)

The transition is smoothed out by quite large $O\left(\frac{1}{G_N}\right)$ corrections. These corrections are much larger in gravity than the $O(1)$ corrections in toy models such as Haar random states, with the enhancement coming from the fluctuations in the energy of the black hole.
Why bother with the more complicated calculation?

• We already got (basically) the right answer for the Page curve, just by looking at a single family of classical saddles. Why bother with all this more complicated technology when it’s just a stupid toy model?
• Because we can. And it’s fun
• The enhanced correction from the energy fluctuations at the transition is interesting (and universal?) physics [Marolf, Wang, Wang; Dong, Wang]
• The nonperturbative corrections far from the transition are crucial to make the result consistent [Hayden, GP]
• It suggests the full gravitational path integral should be taken seriously [Saad, Shenker, Stanford]
• Very close correspondence between the gravity calculations and calculations in random tensor networks [Hayden, Nezami, Qi, Thomas, Walter, Yang]
Thank you!