Description of Coherent Beam Instabilities in the presence of Space-charge Forces using the Circulant Matrix Model (CMM)

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Goal

Improve the understanding of beam instabilities in space-charge dominated machines by investigating the limitation of space charge on the TMCI due to EM wake fields

Motivation

• No current model reflects reality [1]
• First work done by M. Blaskiewicz using an airbag square-well distribution [2]
• New approach: Using a Gaussian beam charge distribution
• Previous work with Gaussian only applicable for high space charge [3]
• Our approach can be used for any strength space charge
Collective Effects

- **Space Charge**: the direct electromagnetic interaction between beam particles—changes the beam dynamics.
- Relativistic energies mean that space charge forces are constrained to the longitudinal plane.
- **Wake fields**: generated by the interaction of the beam with its surroundings—causes head-tail instabilities.
- The wake fields are felt by all particles behind the source.
Extending BimBim

- **BimBim** calculates one turn matrix using the Circulant Matrix Model (CMM)
- CMM requires polar composition of beam in longitudinal phase space
- BimBim already considered:
  1. Transport of beam through the lattice
  2. Wake field coupling
- Need to add effect of Space charge to get:
  \[ M_{\text{OneTurn}} = M_{\text{Lattice}} \cdot M_{\text{SpaceCharge}} \cdot M_{\text{WakeFields}} \]
- Diagonalise and use eigenvalues to calculate real and imaginary tune shifts: \( Q = i \log(\lambda)/(2\pi) \)
Derivation of Space Charge Kick

\[ \begin{bmatrix} x_{1,k+1} \\ x'_{1,k+1} \\ \vdots \\ x_{n,k+1} \\ x'_{n,k+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ A_{1,1} & 1 & \ldots & A_{1,n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ A_{n,1} & 0 & \ldots & A_{n,n} & 1 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x'_{1,k} \\ \vdots \\ x_{n,k} \\ x'_{n,k} \end{bmatrix} \]

Space Charge Kick

\[ \Delta x' = A_{1,1} x_{1,k} \]

• Noting: \( \Delta x' = \Delta p_x / p_0 \) we start with

\[ \frac{dp_x}{dt} = F \]

• Linearise

\[ F_x = q E_x / \gamma^2 \]

• Transverse gaussian charge distribution:

\[ E_x = \frac{1}{2 \pi \epsilon_0 \sigma_{x,y}} \frac{x}{(\sigma_x + \sigma_y)^2} \delta(z) \]

Implementation of Space Charge in BimBim

LPAP meeting
Implementation of Space Charge in BimBim: Derivation

- Integrate over one turn of the electric field and divide by $\beta_r c$

$$p_x = \frac{q}{2\pi \epsilon_0 c \gamma_r^2 \beta_r} \delta(z) \int \frac{x}{\sigma_{x,y}(\sigma_x + \sigma_y)} ds$$

- Using the smooth approximation where the optical beta function is constant over the ring and $\sigma_{x,y} = \sqrt{\beta_{x,y}(s)\epsilon_{x,y}}$

$$p_x = \frac{qC}{4\pi \epsilon_0 m_0 c \beta_r^2 \beta \epsilon} x\delta(z)$$

- Now to get the kick between two particles

$$\Delta x' = \Delta p_x / p_0, \quad p_0 = \gamma_r \beta_r m_0 c$$

$$\Delta x' \approx \frac{q_s q_w C}{4\pi \epsilon_0 m_0 c^2 \gamma_r^3 \beta_r^2 \beta \epsilon} (x_w - x_s) \delta(z_w - z_s)$$
• Now if we consider the kick from a cell $j$ with normalized distribution $N\Psi(z_j, \delta_j)$ in phase space area $\Omega_j$ on a particle:

$$\Delta x'_j(z_w) = A_0(x_w - x_j) \int_{\Omega_j} dz_j d\delta_j \Psi(z_j, \delta_j) \delta(z - z_j)$$

$$A_0 = \frac{N q^2 C}{4\pi \epsilon_0 m_0 c^2 \gamma_r^3 \beta_r^2 \beta \epsilon}$$

• The kick of a cell $j$ on cell $i$, considering mass normalization

$$\Delta x'_{i,j} = A_0(x_i - x_j) \frac{\int_{\Omega_i} dz_i d\delta_i \int_{\Omega_j} dz_j d\delta_j \Psi(z_i, \delta_i) \Psi(z_j, \delta_j) \delta(z_i - z_j)}{\int_{\Omega_i} dz_i d\delta_i \Psi(z_i, \delta_i)}$$

• CMM requires polar decomposition of beam

$$z_i \rightarrow r_i \sigma_z \cos(\theta_i)$$

$$\delta_i \rightarrow r_i \sigma_\delta \sin(\theta_i)$$

$$z_j \rightarrow r_j \sigma_z \cos(\theta_j)$$

$$\delta_j \rightarrow r_j \sigma_\delta \sin(\theta_j)$$

Implementation of Space Charge in BimBim: Derivation
• This produces the space charge kick

\[
\Delta x'_{i,j} = A(x_i - x_j) \frac{\int_{\Omega'_i} dr_i d\theta_i \int_{\Omega'_j} dr_j d\theta_j \Psi_p(r_i, \theta_i) \Psi_p(r_j, \theta_j) \delta(K(r_i, \theta_i, r_j, \theta_j))}{\int_{\Omega'_i} dr_i d\theta_i \Psi_p(r_i, \theta_i)}
\]

• The distributions in polar coordinates:

\[
\Psi_G(r, \theta) = \frac{r}{2\pi} e^{-\frac{r^2}{2}} \quad \Psi_{AB}(r, \theta) = \frac{1}{2\pi} \delta(r - R_0)
\]

• The Dirac becomes:

\[
K(r_i, \theta_i, r_j, \theta_j) = r_i \cos(\theta_i) - r_j \cos(\theta_j)
\]

• With constant:

\[
A = \frac{A_0}{\sigma_z} = \frac{Nq^2C}{4\pi\varepsilon_0 m_0 c^2 \gamma_r^3 \beta_r^2 \beta \varepsilon \sigma_z}
\]
• For numerical computation we had to discretize the integrals and Dirac delta function

\[
\int_a^b f(y) \, dy = \lim_{\Delta y \to 0} \sum_{y=a}^{b} f(y) \Delta y , \quad \int f(y) \delta(y - y_i) \, dy = f(y_i) \approx \lim_{\Delta y \to 0} \sum f(y_i) \Pi(y - y_i) \Delta y.
\]

\[
\Delta x'_{i,j} = A(x_i - x_j) \frac{\sum \Psi_p(r_i, \theta_i) \Psi_p(r_j, \theta_j) \Pi(K(r_i, \theta_i, r_j, \theta_j)) \Delta z \Delta r_i \Delta \theta_i \Delta r_j \Delta \theta_j}{\sum \Psi_p(r_i, \theta_i) \Delta r_i \Delta \theta_i}
\]

• Change dependent parameters from step size to number of steps
  • \( \Delta z, \Delta r, \Delta \theta \rightarrow N_z, N_r \) and \( N_\theta \)

• Finite width of airbag

\[
\Psi_{AB}(r, \theta) = \begin{cases} 
\frac{1}{2\pi W}, & \text{if } (R_0 - W/2) < r < (R_0 + W/2) \\
0, & \text{otherwise.}
\end{cases}
\]
• Visually we have the distributions to be calculated with the set up:

(a) Airbag distribution

(b) Gaussian distribution

Configurations for method of convergence work

• 20 slices 1 ring
• Widths: 0.1, 0.01, 0.005

• 20 Slices 1 ring
• 20 Slices 10 rings
• 40 Slices 20 rings
Method of Convergence

- Looking at maximum value of space charge interaction matrix
- First do a scan of $N_r$ and $N_\theta$ for $nZ=10$ e.g. For Airbag, $W=0.1$

(a) $N_\theta$ vs max interaction matrix value for different values of $N_R$

(b) $N_R$ vs max interaction matrix value for different values of $N_\theta$
• Check value is converged by doubling and seeing if the max matrix value changes by <1%
• Check for increasing $N_z$
• Again, until doubled $N_z$ max matrix value changes by less than 1% e.g. for $N_r = 70$ and $N_\theta = 80$ AB, $W=0.1$
Space Charge Tune Shift Analysis

• Plot theoretical space charge tune shift against simulated tune

Maximum theoretical space charge tune shift reached at $z=0$, using constant $A$ from derivation of space charge

\[
\Delta Q^{sc}(z = 0) = \frac{A \beta}{4\pi \sqrt{2\pi}}.
\]

\[
A = \frac{Nq^2C}{4\pi \epsilon_0 m_0 c^2 \gamma_1^3 \beta_r^2 \beta \epsilon \sigma_z}
\]

• To apply to any machine:

\[
Q_{sc}^{\text{theor}} = \Delta Q^{sc}(z = 0)/Q_s
\]

BimBim Simulated tune shifts:

\[
Q_{sc}^{\text{sim}} = \frac{Re(i \log(\lambda))}{2\pi Q_s}
\]

Airbag square-well [2]:

\[
\frac{\Delta Q_{na}}{Q_s} = -\frac{Q_{sc}^{\text{theor}}}{c} \pm \sqrt{\frac{Q_{sc}^{\text{theor}}^2}{4} + (n_a)^2}
\]
Space Charge Tune Shift Analysis

Airbag vs Airbag Square-well

Red: W=0.01, Blue: W=0.005

- Convergence check: Widths produce very similar tune shift modes signifying 0.005 is sufficient
- Similar trend of mode evolution between AB and ABS
- For high space charge, positive modes seem to pair, unlike theory
**Space Charge Tune Shift Analysis**

**Gaussian**

- Similar trend to AB, no divergence of modes -10 and -9
- Modes 1 & 2 cross mode 0
- Considers radial modes
- Space charge lifts degeneracy of radial modes

20 slices 1 ring

20 slices 10 rings
Instabilities due to Space Charge and Wake Fields

• **Wake fields**
  • Broad band resonator
  • Round vacuum chamber, $\kappa=0$

• Plotted coherent tunes for increasing normalized intensity [4]

• Varying $f_r \tau_b$ – resonant frequency x bunch length

• **TMCI** – transverse mode coupling instability
  • Different TMCI regimes expected
TMCI for an Airbag distribution

- $f_r \tau_b = 1$
- Space Charge seems to suppress TMCI due to wake fields
- Region of instability appears for high space charge
- Let’s look deeper at these regions of instability
Testing effect of finite width airbag

- $l_{\text{norm}} = 25.0$, $f_r \tau_b = 1$
Testing effect of finite width airbag

- $I_{\text{norm}} = 25.0$, $f_r \tau_b = 1$

Modes 0 and -1 decouple at same space charge parameter of 3.3

Positive mode coupling at higher space charge seem dependent on width
Testing effect of $f_r \tau_b$ on TMCI threshold

- $I_{\text{norm}} = 25.0$, $W=0.005$

• Increasing $f_r \tau_b$ lifts the TMCI due to wake fields

• For same width, instability due to high space charge not affected by $f_r \tau_b$ of wake field
Comparison of wake field coupling for $f_r \tau_b = 1 \& 2$

(a) $f_r \tau_b = 1$

(b) $f_r \tau_b = 2$
Comparison of wake field coupling for $f_r \tau_b = 1$ & 2

- Different mode coupling for different $f_r \tau_b$
- As expected by [4]
TMCI for Gaussian distribution

• Starting with 20 slices 1 ring

• Agrees with space charge suppressing TMCI due to wake fields
TMCI for Gaussian distribution

• Starting with 20 slices 1 ring

• Agrees with space charge suppressing TMCI due to wake fields

• Stronger instability for high space charge compared to airbag

• Let’s check with 10 rings to consider radial modes
Gaussian: 20 slices 10 rings

- $f_r \tau_b = 1$ does not have clear suppression of TMCI due to wake fields
- $f_r \tau_b = 2$ has a mesh like pattern for increased space charge
  - Need to check regions of high space charge if they are valid TMCI
TMCI for Gaussian distribution

Gaussian: $I_{\text{norm}} = 25$, $f_r \tau_b = 1$

- Modes 0 and -1 coupled due to wake fields
- Positive mode coupling due to high space charge
- Considering 10 radial modes reveals more instabilities

- Instability stemming from 0 doesn’t seem to be completely suppressed
TMCI for Gaussian distribution

Gaussian: \( I_{\text{norm}} = 25, f_r \tau_b = 1 \)

- Increased rings and slices very similar plots
- Confirms TMCI due to wake fields completely suppressed by TMCI
- Confirms validity of positive mode coupling due to space charge
- Instability ‘bubbles’ appearing for high space charge
Gaussian 40 slices 20 rings: $I_{\text{norm}} = 25$

- Instabilities due to space charge have a lower imaginary tune
- More ‘bubbles’ of instability
Comparison between TMCI for Airbag and Gaussian distributions

• For both distributions
  • Space charge lifts the TMCI due to wake fields
  • Increasing $f_r \tau_b$ increases the TMCI threshold

• Gaussian: increasing rings and slices confirmed TMCI generated between positive modes for higher space charge
• Airbag model: positive mode coupling attributed to numerical artefacts of finite ring width

• Distributions differ for high space charge
• Gaussian distribution seems more compatible with reality [1]
Outlook

To improve:
- method of convergence was very numerically demanding
- Find converged number of rings and slices for Gaussian distribution

Future work:
- Confirm predictions with experimental data
- Mitigation of instabilities
  - Transverse feedback
  - chromaticity
References


Check convergence of Airbag width 0.005

- Mostly zeros
- Each cell only interacts with one other for a sufficiently thin ring
Gaussian 20 slices 10 rings: $I_{\text{norm}}=25$

(a) $f_r \tau_b = 2$

(b) $f_r \tau_b = 3$