Electroweak Factorization at High Energies

Parton Distributions and Showers/Fragmentations

Keping Xie

Pittsburgh Particle Physics, Astrophysics, and Cosmology Center,
Department of Physics and Astronomy,
University of Pittsburgh, Pittsburgh, PA 15260, USA

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In collaboration with **Tao Han** and **Yang Ma**

Inherit from [1611.00788] and move forward

- Parton distributions [2007.14300, 2103.09844]
- Showers/Fragmentations [2203.11129](still ongoing)

DISCLAIMER

- I generalize the topic to the electroweak factorization, which includes initial-state radiations as parton distributions and the final-state radiations as showers/fragmentations.
- Some results are taken from Tao's old work [1611.00788].
- The main work is still ongoing. Some results are very preliminary.

EW physics at high energies

At high energies, all the Standard Model essentially become massless

$$\frac{v}{E} : \frac{v(250 \quad \text{GeV})}{10 \text{ TeV}} \sim \frac{\Lambda_{\text{QCD}}(0.3 \quad \text{GeV})}{10 \quad \text{GeV}}$$

$$\frac{v}{E}, \frac{m_t}{E}, \frac{M_W}{E} \to 0!$$

- The splitting phenomena dominate due to the large logarithm enhancement.
- The EW symmetry is restored:

$$SU(2)_L \times U(1)_Y \text{ unbroken.}$$

- The electroweak symmetry breaking effect: power corrections $\mathscr{O}(v/E)$
  $\to$ Higher twist effects in QCD

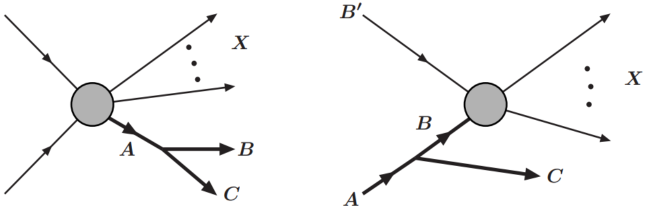
[Chen, TH, Tweedie, arXiv:1611.00788]

[Cuomo, Wulzer, arXiv:1703.08562; 1911.12366]

[Ciafaloni et al., hep-ph/0004071; 0007096; Manohar et al., 1803.06347]

[Bauer, Ferland, Webber et al., arXiv:1703.08562; 1808.08831]

Splitting phenomena



$$d\sigma_{X,BC} \simeq d\sigma_{X,A} \times d\mathscr{P}_{A\to B+C},$$

$$E_B \approx zE_A, \quad E_C \approx \bar{z}E_A, \quad k_T \approx z\bar{z}E_A\theta_{BC}$$

$$\frac{d\mathscr{P}_{A\to B+C}}{dzdk_T^2} \simeq \frac{1}{16\pi^2}\frac{z\bar{z}|\mathscr{M}^{(\text{split})}|^2}{(k_T^2+\bar{z}m_B^2+zm_C^2-z\bar{z}m_A^2)^2}, \quad \bar{z}=1-z$$

- On the dimensional ground: $|\mathscr{M}^{(\text{split})}|^2 \sim k_T^2$ or m^2
- Integrating out the k_T ends up with $\alpha_W \log(Q^2/M_V^2) P_{A\to B+C}$

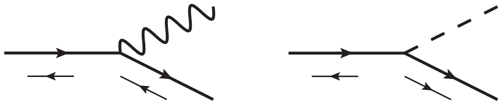
[Ciafaloni et al., hep-ph/0004071; 0007096; Bauer, Ferland, Webber et al., arXiv:1703.08562;1808.08831; Manohar et al., 1803.06347; Han, Chen & Tweedie, arXiv:1611.00788]

EW Splitting functions

- Starting from the unbroken phase: all massless

$$\mathscr{L}_{SU(2)\times U(1)} = \mathscr{L}_{\text{gauge}} + \mathscr{L}_{\phi} + \mathscr{L}_{f} + \mathscr{L}_{\text{Yukawa}}$$

- Particle contents:
  - Chiral fermions $f_{L,R}$
  - Gauge bosons: B, $W^{0,\pm}$
  - Higgs $H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = \begin{pmatrix} \phi^+ \\ \frac{h - i\phi^0}{\sqrt{2}} \end{pmatrix}$
- Splitting functions [See Ciafaloni et al. hep-ph/0505047, Han et al. 1611.00788 for complete lists.]



	$\frac{1}{8\pi^2}\frac{1}{k_T^2}\frac{1+\bar{z}^2}{z}$		$\frac{1}{8\pi^2}\frac{1}{k_T^2}\frac{z}{2}$	
	$\to V_T f_s^{(\prime)}$	$[BW]_T^0 f_s$	$H^{0(*)} f_{-s}$	$\phi^{\pm} f'_{-s}$
$f_{s=L,R}$	$g_V^2 (Q_{f_s}^V)^2$	$g_1 g_2 Y_{f_s} T_{f_s}^3$	$y^2_{f_R^{(\prime)}}$	
	Soft & collinear singularites (P_{gq})		Collinear singularity chirality-flip, Yukawa	

Electroweak symmetry breaking

Goldstone Boson Equivalence Theorem (GBET)

[Lee, Quigg, Thacker (1977); Chanowitz & Gailard (1984)]

- At high energies $E \gg M_W$, the longitudinally polarized gauge bosons V_L behave like the corresponding Goldstone bosons
  They remember their origin!
- Scalarization of V_L

$$\varepsilon_L^{\mu}(k) = \frac{E}{M_W}(\beta_W, \hat{k}) \simeq \frac{k^{\mu}}{M_W} + \mathscr{O}(M_W/E)$$

- The GBET violation can be counted as power corrections v/E
  $\rightarrow$ Higher twist effects in QCD (Λ_{QCD}/Q)

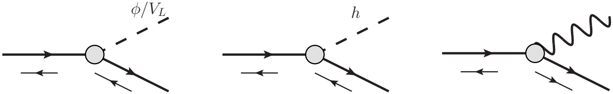
[Han et al. 1611.00788, Cuomo, Wulzer, 1703.08562; 1911.12366].

New splitting in a broken gauge theory

- Fermion splitting into longitudinal gauge boson $f \to V_L$

$$P \sim \frac{v^2}{k_T^2}\frac{\mathrm{d}k_T^2}{k_T^2} \sim 1 - \frac{v^2}{Q^2}$$

- V_L is of IR, h has no IR [Han et al. 1611.00788]



	$\frac{1}{16\pi^2}\frac{v^2}{k_T^4}\frac{1}{z}$	$\frac{1}{16\pi^2}\frac{v^2}{k_T^4}$	$\frac{1}{16\pi^2}\frac{v^2}{k_T^4}$
$f_s \to$	$V_L f_s^{(\prime)} (V \neq \gamma)$	$h f_s$	$V_T f_{-s}^{(\prime)}$
	Chirality conserving non-zero for massless f		Chirality flipping $\sim m_f$

- The PDFs for W_L/Z_L behaves as constants, which does not run at the leading log: "Bjorken scaling" restoration

$$f_{V_L/f}(x, Q^2) \sim \alpha \frac{1-x}{x}$$

Residuals of the EWSB, v^2/E^2, similar to higher-twist effects

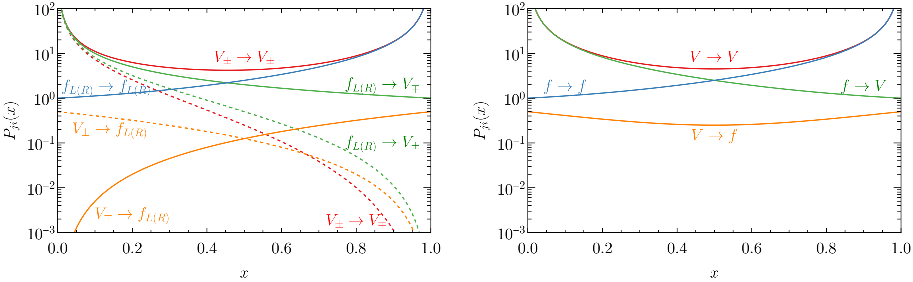
Polarizations in the EW factorization

- The EW splittings must be polarized due to the chiral nature of the EW theory

$$f_{V_+/A_+} \neq f_{V_-/A_-}, \qquad f_{V_+/A_-} \neq f_{V_-/A_+},$$
$$\hat{\sigma}(V_+B_+) \neq \hat{\sigma}(V_-B_-), \quad \hat{\sigma}(V_+B_-) \neq \hat{\sigma}(V_-B_+)$$

We are not able to factorize the cross sections in an unporlarized form.

$$\sigma \neq f_{V/A}\hat{\sigma}(VB),\ f_{V/A} = \frac{1}{2}\sum_{\lambda,s_1} f_{V_\lambda/A_{s_1}},\ \hat{\sigma}(VB) = \frac{1}{4}\sum_{\lambda,s_2}\hat{\sigma}(V_\lambda B_{s_2})$$



Definition of (QCD) PDFs

- Fast moving proton in the z direction $p^\mu = (E, 0, 0, p)$

$$n^\mu = (1,0,0,1),\ \bar{n} = (1,0,0,-1)$$
$$n^2 = \bar{n}^2 = 0, n\cdot\bar{n} = 2$$

- Light-cone coordinates

$$p^\mu = \frac{1}{2}p^- n^\mu + \frac{1}{2}p^+ \bar{n}^\mu + p_\perp^\mu,$$

where

$$p^- = \bar{n}\cdot p = E + p_z \approx 2E,\ p^+ = n\cdot p = E - p_z \approx \frac{m_p^2}{2E}.$$

- Boost along z axis,

$$p^+ \to \lambda p^+,\ p^- \to p^-/\lambda, p_\perp \to p_\perp.$$

- Quark PDFs: light-cone Fourier transforms [Collins & Soper, 1982]

$$f_q(x,\mu) = \langle p|O_q(r^-)|p\rangle,\ x = r^-/p^-$$
$$O_q(r^-) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi\, e^{-i\xi r}[\bar{q}(\bar{n}\xi)\mathscr{W}(\bar{n}\xi)]\bar{\not{n}}[\mathscr{W}^\dagger(0)q(0)],$$

Similar expressions for antiquark PDFs and gluon PDFs.

- Collinear PDFs are defined at $x^- = 0$ and $x_\perp = 0$, which are boost invariant.

EW factorization different from the QCD one

- Due to confinement, QCD observables are color invariant.

$$O_q(r^-) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi\, e^{-i\xi r}[\bar{q}(\bar{n}\xi)\mathscr{W}(\bar{n}\xi)]\bar{\not{n}}\begin{bmatrix}1\\ T^a\end{bmatrix}[\mathscr{W}^\dagger(0)q(0)],$$

$$\langle p|\bar{q}\cdots q|p\rangle = f_q(x,\mu),\ \langle p|\bar{q}\cdots T^a\cdots|p\rangle = 0.$$

  Equal probabilities to find the different colors, q_1, q_2, q_3

- EW symmetry is broken.

$$\langle p|\bar{q}\cdots t^a\cdots|p\rangle \neq 0$$

- Non-singlet PDFs $(I \neq 0)$

$$\langle p|\bar{q}_L\cdots t^3\cdots q_L|p\rangle = \frac{1}{2}\left[f_{u_L} - f_{d_L}\right] \neq 0,\ f_{u_L} \neq f_{d_L},$$

  which gives non-zero non-singlet PDFs.

  [Bauer et al., 1703.08562, 1712.07147; Manohar, Waalewijn, 1802.08687; Han, Ma, KX, 2007,14300, 2103.09844]

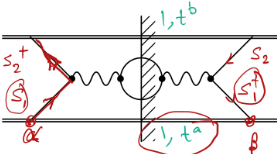
EW factorization

- A inclusive cross section can be factorized into hard, collinear (PDFs and/or FFs) and soft functions [Manohar, 1802.08687]

$$\sigma(AB \to X) = \sum_{a,b} \mathscr{C}_{a/A} \mathscr{C}_{b/B} \mathscr{S}_{ab} \mathscr{H}_{ab},$$

where the soft function

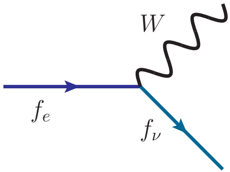
$$\mathscr{S}_{ab} = \langle 0 | S_1^\dagger t^a S_1 S_2^\dagger t^b S_2 \cdots | 0 \rangle.$$

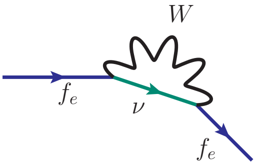


Drell-Yan

- In the QCD case, $t^a \to T^a$ vanish unless $T^A = 1$.
  $S^\dagger S = 1$ leaves a trivial soft function $\mathscr{S}_{ab} = 1$.
- In the EW theory, $\mathscr{S}$ is not trivially identity, leads a angular dependence
  $\to$Rapidity divergence $\Longrightarrow$ Collins-Soper Equation [1981] (Work in progress)
  Equivalent to the rapidity RGE in SCET [Chiu et al. 1104.0881, 1202.0814]
- EW PDFs/FFs involve both singlet and non-singlet components $(I = 0, 1, 2)$.
- DGLAP equation in $I \neq 0$ sector will gives double-log evolution.

Bloch-Nordsieck theorem violation

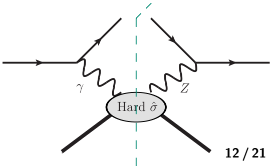




- When $SU(2)$ quantum numbers are not summed/averaged (non-singlet case), collinear factorization formalism may NOT hold solely
- A natural consequence of the infrared divergence is not canceled in $f_{I\neq 0}$, (individually in f_e or f_ν)
- We can cut the violation hardly, such as M_V/Q, [Han, Ma, KX, 2007.14300]
- A more complete framework needs a fully inclusive observables, in which the rapidity divergence is canceled with a soft function. [Manohar et al., 1803.06347]
- The interference gives the mixed PDFs or FFs/showers

$$f_{\gamma Z} \sim \langle \Omega | A^{\mu\nu} Z_{\mu\nu} | \Omega \rangle$$

similarly for f_{hZ_L} [Bauer '17, '18, Manohar '18 , Tao '16].



Parton distributions vs Showers/Fragmentations

- Initial state radiation (ISR), PDF (DGLAP) [Han, Ma, KX, 2007.14300, 2103.09844]

$$f_B(z,\mu^2) = \sum_A \int_z^1 \frac{d\xi}{\xi} f_A(\xi) \int_{m^2}^{\mu^2} d\mathscr{P}_{A\to B+C}(z/\xi, k_T^2)$$

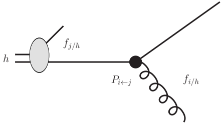
$$\frac{\partial f_B(z,\mu^2)}{\partial \mu^2} = \sum_A \int_z^1 \frac{d\xi}{\xi} \frac{d\mathscr{P}_{A\to B+C}(z/\xi,\mu^2)}{dz dk_T^2} f_A(\xi,\mu^2)$$

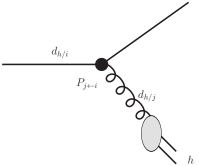
- Final state radiation (FSR): Fragmentations [Bauer et al. 1806.10157, Han, Ma, KX, 2203.11129]

$$\frac{\mathrm{d} d_i}{\mathrm{d}\log Q^2} = \sum_I \frac{\alpha_I}{2\pi} \sum_j d_j \otimes P_{ji}^I$$

Equivalent to the Sudakov in parton showers [Han et al., 2203.11129]

$$\Delta_A(t) = \exp\left[-\sum_B \int_{t_0}^t \int dz \mathscr{P}_{A\to B+C}(z) \right],$$



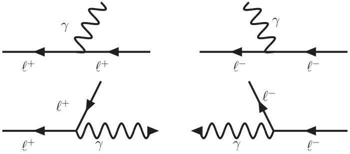


Go beyond the leading logarithm

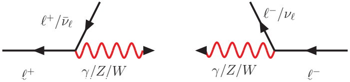
We have been doing:

- $\ell^+\ell^-$ annihilation

- EPA and ISR

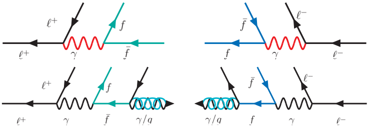


- "Effective W Approx." (EWA)
  [Kane, Repko, Rolnick, PLB 148 (1984) 367]
  [Dawson, NPB 249 (1985) 42]

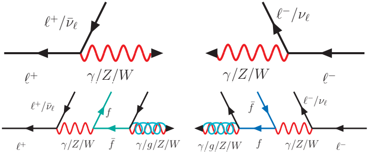


We complete:

- Above $\mu_{\rm QCD}$: QED$\otimes$QCD
  q, g become active [Han, Ma, KX, 2103.09844]



- Above $\mu_{\rm EW} = M_Z$: EW$\otimes$QCD
  EW partons emerge [Han, Ma, KX, 2007.14300]



In the end, every content is a parton, i.e. the full parton distributions or fragmentations.

The PDF evolution

- The DGLAP equations

$$\frac{\mathrm{d}f_i}{\mathrm{d}\log Q^2} = \sum_I \frac{\alpha_I}{2\pi} \sum_j P_{ij}^I \otimes f_j$$

- The initial conditions for a lepton beam

$$f_{\ell/\ell}(x, m_\ell^2) = \delta(1-x)$$

- Three regions and two matchings [Han, Ma, KX, 2007.14300, 2103.09844]
  - $m_\ell < Q < \mu_{\rm QCD}$: QED
  - $Q = \mu_{\rm QCD} \lesssim 1$ GeV: $f_q \propto P_{q\gamma} \otimes f_\gamma, f_g = 0$
  - $\mu_{\rm QCD} < Q < \mu_{\rm EW}$: QED$\otimes$QCD
  - $Q = \mu_{\rm EW} = M_Z$: $f_\nu = f_t = f_W = f_Z = f_{\gamma Z} = 0$
  - $\mu_{\rm EW} < Q$: EW$\otimes$QCD.

$$\begin{pmatrix} f_B \\ f_{W^3} \\ f_{BW^3} \end{pmatrix} = \begin{pmatrix} c_W^2 & s_W^2 & -2c_W s_W \\ s_W^2 & c_W^2 & 2c_W s_W \\ c_W s_W & -c_W s_W & c_W^2 - s_W^2 \end{pmatrix} \begin{pmatrix} f_\gamma \\ f_Z \\ f_{\gamma Z} \end{pmatrix}$$

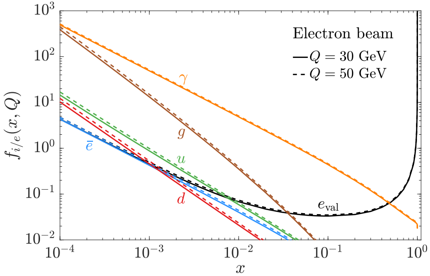
- We work in the (B, W) basis. The technical details can be referred to the backup slides.

The QED⊗QCD PDFs for lepton colliders

Electron beam:

- Scale unc. 10% for $f_{g/e}$ [2103.09844]
- $\mu_{\rm QCD}$ unc. 15%
- The averaged momentum fractions $\langle x_i \rangle = \int x f_i(x) {\rm d}x$ [%]

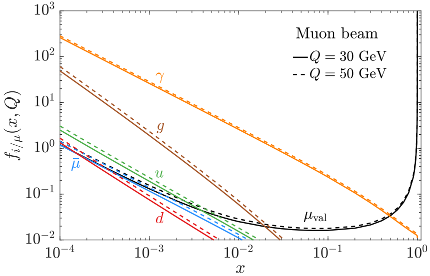
$Q(e^\pm)$	$e_{\rm val}$	γ	ℓsea	q	g
30 GeV	96.6	3.20	0.069	0.080	0.023
50 GeV	96.5	3.34	0.077	0.087	0.026
M_Z	96.3	3.51	0.085	0.097	0.028



Muon beam:

- Scale unc. 20% for $f_{g/\mu}$ [2103.09844]
- $\mu_{\rm QCD}$ unc. 5% [2106.01393]

$Q(\mu^\pm)$	$\mu_{\rm val}$	γ	ℓsea	q	g
30 GeV	98.2	1.72	0.019	0.024	0.0043
50 GeV	98.0	1.87	0.023	0.029	0.0051
M_Z	97.9	2.06	0.028	0.035	0.0062

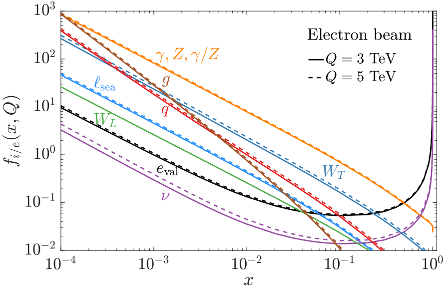


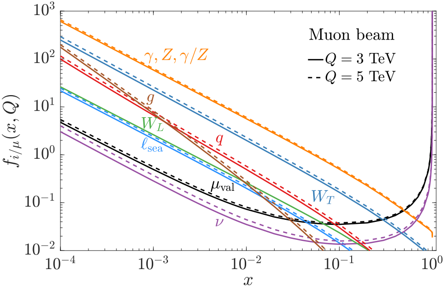
EWPDFs of a lepton

- The sea leptonic and quark PDFs

$$\nu = \sum_i (\nu_i + \bar{\nu}_i), \ \ell\text{sea} = \bar{\ell}_{\text{val}} + \sum_{i \neq \ell_{\text{val}}} (\ell_i + \bar{\ell}_i), \ q = \sum_{i=d}^{t} (q_i + \bar{q}_i)$$

Even neutrino becomes active.





- **All SM particles are partons** [Han, Ma, KX, 2007.14300]
- $W_L(Z_L)$ does not evolve: **Bjorken-scaling restoration**: $f_{W_L}(x) = \frac{\alpha_2}{4\pi}\frac{1-x}{x}$.
- The EW correction can be large: $\sim 50\%$ (100%) for $f_{d/e}$ $(f_{d/\mu})$ due to the relatively **large SU(2) gauge coupling**. [Han, Ma, KX et. al, 2106.01393]
- Scale uncertainty: $\sim 15\%$ (20%) between $Q = 3$ TeV and $Q = 5$ TeV

Fragmentation functions

- Backward evolution

$$\frac{\mathrm{d}d_i}{\mathrm{d}\log Q^2} = \sum_I \frac{\alpha_I}{2\pi} \sum_j d_j \otimes P_{ji}^I$$

- Initial conditions for fermions [Bauer et al. 1806.10157, Han, Ma, KX, 2203.11129]

$$d_{f_L}^f(x, Q_0^2) = d_{f_R}^f(x, Q_0^2) = \delta(1-x),$$
$$d_{\nu_L}^{\nu}(x, Q_0^2) = \delta(1-x),\ d_i^f = 0 \text{ for } i \neq f.$$

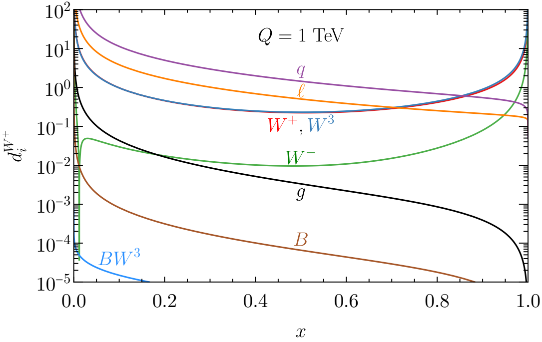
  vector bosons

$$d_{V_+}^V(x, Q_0^2) = d_{V_-}^V(x, Q_0^2) = \delta(1-x),\ d_i^V(x, Q_0^2) = 0 \text{ for } i \neq V.$$

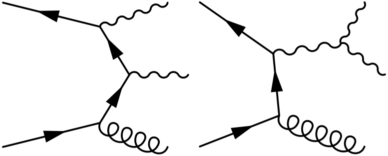
- We take the same techniques developed for PDF evolution. See backup slides for details.
- The DGLAP evolution resumms the EW logarithms, which is equivalent to the Sudakov in showering.

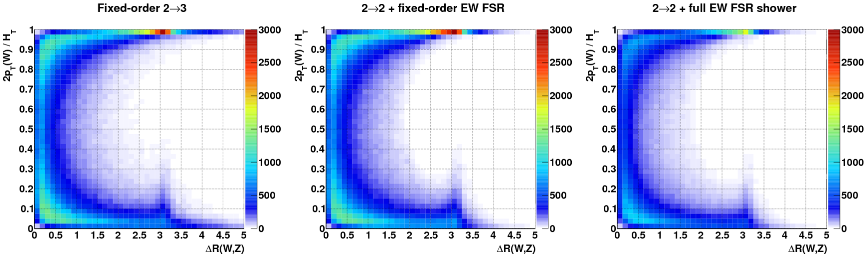
An example: a final-state W^+

- At future high-energy colliders, collinear splittings also happen to energetic final state particles ⇒ **EW jets**
- Electroweak fragmentation function (EW FF) $d_i^{W^+}$, defined as the probability of finding a W^+ in the mother particle i (i.e., $i \to W$) [Han, Ma, KX, 2203.11129]



Applications: $pp \to WZj$





[Han et al. 1611.00788]

- $(\Delta R_{WZ}, 2p_T^W/H_T) \sim (\pi, 0), (\pi, 1)$ due to the back-to-back of W, Z
- $\Delta R_{WZ} \sim 0$ due to $W \to WZ$ splitting
- The systematic comparison between fragmentation and parton shower is still ongoing.

Summary and prospects

- At high energies, all SM particle essentially become massless. The EW symmetry is asymptotically restore.
- The EW splitting phenomena dominate, due to the logarithm enhancement.
- The ISR can be factorized as the PDF, the FSR as Fragmentations (parton shower).
- the EW factorization approach allows for decomposition of polarized partonic subprocesses, including the γZ_T and hZ_L mixing.
- Bloch-Nordsieck theorem violation: Factorization breaks down for the insufficiently inclusive processes.
  - Cutoff (M_W) to regulate the divergence (easy to implement),
  - Fully inclusive to cancel all the divergence (consistent treatment).
- High-energy behavior of longitudinal gauge boson $\varepsilon_L^\mu = \frac{E}{m}(\beta, \hat{k})$.
  - Goldstone equivalence gauge: $\varepsilon_n^\mu(k) \equiv \frac{-\sqrt{|k^2|}}{n(k)\cdot k} n^\mu(k) \stackrel{\text{on-shell}}{\rightarrow} \frac{m_W}{E+|\vec{k}|}(-1, \hat{k})$, [Han et al. 1611.00788]
  - Match to the scalar mode in the unbroken EW phase [Han, Ma, KX, ongoing]

Backup

The QED⊗QCD DGLAP evolution

- The singlets and gauge bosons

$$\frac{\mathrm{d}}{\mathrm{d}\log Q^2}\begin{pmatrix} f_L \\ f_U \\ f_D \\ f_\gamma \\ f_g \end{pmatrix} = \begin{pmatrix} P_{\ell\ell} & 0 & 0 & 2N_\ell P_{\ell\gamma} & 0 \\ 0 & P_{uu} & 0 & 2N_u P_{u\gamma} & 2N_u P_{ug} \\ 0 & 0 & P_{dd} & 2N_d P_{d\gamma} & 2N_d P_{dg} \\ P_{\gamma\ell} & P_{\gamma u} & P_{\gamma d} & P_{\gamma\gamma} & 0 \\ 0 & P_{gu} & P_{gd} & 0 & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} f_L \\ f_U \\ f_D \\ f_\gamma \\ f_g \end{pmatrix}$$

- The non-singlets

$$\frac{\mathrm{d}}{\mathrm{d}\log Q^2} f_{NS} = P_{ff} \otimes f_{NS}.$$

- The averaged momentum fractions of the PDFs: $f_{\ell_{\rm val}}, f_\gamma, f_{\ell_{\rm sea}}, f_q, f_g$

$$\langle x_i \rangle = \int x f_i(x)\mathrm{d}x, \ \sum_i \langle x_i \rangle = 1$$

$$\frac{\langle x_q \rangle}{\langle x_{\ell\rm sea} \rangle} \lesssim \frac{N_c\left[\sum_i (e_{u_i}^2 + e_{\bar{u}_i}^2) + \sum_i (e_{d_i}^2 + e_{\bar{d}_i}^2)\right]}{e_{\bar{\ell}_{\rm val}}^2 + \sum_{i\neq\ell{\rm val}} (e_{\ell_i}^2 + e_{\bar{\ell}_i}^2)} = \frac{22/3}{5}$$

The EW isospin (T) and charge-parity (CP) basis

- The leptonic doublet and singlet in the (T,CP) basis

$$f_\ell^{0\pm} = \frac{1}{4}\left[(f_{\nu_L} + f_{\ell_L}) \pm (f_{\bar{\nu}_L} + f_{\bar{e}_L})\right], \quad f_\ell^{1\pm} = \frac{1}{4}\left[(f_{\nu_L} - f_{\ell_L}) \pm (f_{\bar{\nu}_L} - f_{\bar{e}_L})\right].$$

$$f_e^{0\pm} = \frac{1}{2}[f_{e_R} \pm f_{\bar{e}_R}]$$

- Similar for the quark doublet and singlets.
- The bosonic

$$f_B^{0\pm} = f_{B_+} \pm f_{B_-},\; f_{BW}^{1\pm} = f_{BW_+} \pm f_{BW_-},$$

$$f_W^{0\pm} = \frac{1}{3}\left[\left(f_{W_+^+} + f_{W_+^-} + f_{W_+^3}\right) \pm \left(f_{W_-^+} + f_{W_-^-} + f_{W_-^3}\right)\right],$$

$$f_W^{1\pm} = \frac{1}{2}\left[\left(f_{W_+^+} - f_{W_+^-}\right) \mp \left(f_{W_-^+} - f_{W_-^-}\right)\right],$$

$$f_W^{2\pm} = \frac{1}{6}\left[\left(f_{W_+^+} + f_{W_+^-} - 2f_{W_+^3}\right) \pm \left(f_{W_-^+} + f_{W_-^-} - 2f_{W_-^3}\right)\right].$$

The EW PDFs in the singlet/non-singlet basis

Construct the singlets and non-singlets

- Singlets

$$f_L^{0,1\pm} = \sum_i^{N_g} f_\ell^{0,1\pm}, \quad f_E^{0\pm} = \sum_i^{N_g} f_e^{0\pm},$$

- Non-singlets

$$f_{L,NS}^{0,1\pm} = f_{\ell_1}^{0,1\pm} - f_{\ell_2}^{0,1\pm}, \quad f_{E,NS}^{0\pm} = f_{e_1}^{0\pm} - f_{e_2}^{0\pm}$$

- The trivial non-singlets

$$f_{L,23}^{0,1\pm} = f_{E,23}^{0\pm} = 0$$

Reconstruct the PDFs for each flavors

- The leptonic PDFs

$$f_{\ell_1}^{0,1\pm} = \frac{f_L^{0,1\pm} + (N_g - 1) f_{L,NS}^{0,1\pm}}{N_g}, \quad f_{\ell_2}^{0,1\pm} = f_{\ell_3}^{0,1\pm} = \frac{f_L^{0,1\pm} - f_{L,NS}^{0,1\pm}}{N_g},$$

$$f_{e_1}^{0\pm} = \frac{f_E^{0\pm} + (N_g - 1) f_{E,NS}^{0\pm}}{N_g}, \quad f_{e_2}^{0\pm} = f_{e_3}^{0\pm} = \frac{f_E^{0\pm} - f_{E,NS}^{0\pm}}{N_g}.$$

- The quark components can be constructed as singlets/non-singlets, and reconstructed correspondingly as well.

The DGLAP in the singlet and non-singlet basis

$$\frac{\mathrm{d}}{\mathrm{d}L}\begin{pmatrix} f_L^{0\pm} \\ f_Q^{0\pm} \\ f_E^{0\pm} \\ f_U^{0\pm} \\ f_D^{0\pm} \\ f_B^{0\pm} \\ f_W^{0\pm} \\ f_g^{0\pm} \end{pmatrix} = \begin{pmatrix} P_{LL}^{0\pm} & 0 & 0 & 0 & 0 & P_{LB}^{0\pm} & P_{LW}^{0\pm} & 0 \\ 0 & P_{QQ}^{0\pm} & 0 & 0 & 0 & P_{QB}^{0\pm} & P_{QW}^{0\pm} & P_{Qg}^{0\pm} \\ 0 & 0 & P_{EE}^{0\pm} & 0 & 0 & P_{EB}^{0\pm} & 0 & 0 \\ 0 & 0 & 0 & P_{UU}^{0\pm} & 0 & P_{UB}^{0\pm} & 0 & P_{Ug}^{0\pm} \\ 0 & 0 & 0 & 0 & P_{DD}^{0\pm} & P_{DB}^{0\pm} & 0 & P_{Dg}^{0\pm} \\ P_{BL}^{0\pm} & P_{BQ}^{0\pm} & P_{BE}^{0\pm} & P_{BU}^{0\pm} & P_{BD}^{0\pm} & P_{BB}^{0\pm} & 0 & 0 \\ P_{WL}^{0\pm} & P_{WQ}^{0\pm} & 0 & 0 & 0 & 0 & P_{WW}^{0\pm} & 0 \\ 0 & P_{gQ}^{0\pm} & 0 & P_{gU}^{0\pm} & P_{gD}^{0\pm} & 0 & 0 & P_{gg}^{0\pm} \end{pmatrix} \otimes \begin{pmatrix} f_L^{0\pm} \\ f_Q^{0\pm} \\ f_E^{0\pm} \\ f_U^{0\pm} \\ f_D^{0\pm} \\ f_B^{0\pm} \\ f_W^{0\pm} \\ f_g^{0\pm} \end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}L}\begin{pmatrix} f_L^{1\pm} \\ f_Q^{1\pm} \\ f_W^{1\pm} \\ f_{BW}^{1\pm} \end{pmatrix} = \begin{pmatrix} P_{LL}^{1\pm} & 0 & P_{LW}^{1\pm} & P_{LM}^{1\pm} \\ 0 & P_{QQ}^{1\pm} & P_{QW}^{1\pm} & P_{QM}^{1\pm} \\ P_{WL}^{1\pm} & P_{WQ}^{1\pm} & P_{WW}^{1\pm} & 0 \\ P_{ML}^{1\pm} & P_{MQ}^{1\pm} & 0 & P_{MM}^{1\pm} \end{pmatrix} \otimes \begin{pmatrix} f_L^{1\pm} \\ f_Q^{1\pm} \\ f_W^{1\pm} \\ f_{BW}^{1\pm} \end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}L} f_W^{2\pm} = P_{WW}^{2\pm} \otimes f_{WW}^{2\pm}$$

The splitting functions can be constructed in terms of Refs. [Han et al. 1611.00788, Bauer et al. 1703.08562,1808.08831]