Hamiltonian Dynamics Problem Sheet

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Problem 1. Hamilton's canonical equations are given by

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Apply the equations to the following two Hamiltonians:

$$H(q_1, q_2, p_1, p_2; t) = \frac{1}{2}\left(p_1^2\left(p_2^2 + q_2^2\right) + q_1^2\right)$$

$$H(r, \phi, p_r, p_\phi; t) = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2 \sin^2\alpha} + mgr\cos\alpha$$

Problem 2. The action integrated along the path between times times t_1 and t_2 is given

$$\mathcal{S} = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt$$

where L is the Lagrangian, a function of generalised coordinates $q, \dot{q}$. The *principle of least action* states that the action is stationary, i.e.

$$\delta S = 0$$

Use integration by parts to derive the Euler-Lagrange equation following this principle.

Problem 3. A mass m sliding on a frictionless table is connected by a string through a hole in the table to another mass M hanging underneath. Assume that the mass M moves in a vertical line only and that the string, of length l, always remains taut. As shown in the figure, the coordinates of the mass on the table are given by a radius r and azimuthal angle θ. Note, it is convenient to calculate the potential energy with respect to the table surface.

- Write down the Lagrangian.
- Use the Euler-Lagrange equation the find the equation of motion in r and θ.

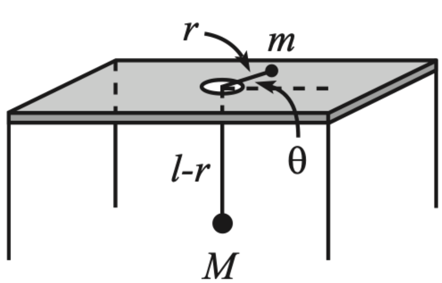

Figure 1: Mass on a table connected to one below.

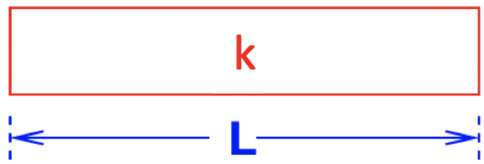

Figure 2: Schematic of a thick quadrupole of length L and strength k.

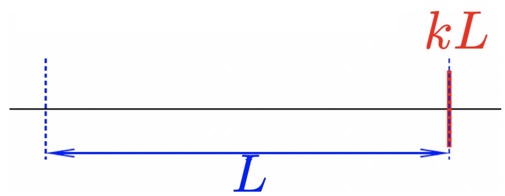

Figure 3: Schematic of the drift-kick approximation of a quadrupole

Problem 4. Find a canonical transformation defined by the following generating functions

$$\begin{aligned} F &= \ln(qt)e^P \\ F &= q\ln P \end{aligned}$$

Calculate the Poisson brackets to demonstrate that the transformations are canonical (recall for a canonical transformation the Poisson bracket must satisfy $[Q,P]_{q,p}=1$).

Problem 5. The transfer matrix of a *thick* quadrupole of normalised strength k and length L (Fig. 2) may be written

$$M_{thick} = \begin{pmatrix} \cos\sqrt{k}L & \frac{1}{\sqrt{k}}\sin\sqrt{k}L \\ -\sqrt{k}\sin\sqrt{k}L & \cos\sqrt{k}L \end{pmatrix}$$

and that of a drift

$$M_{drift} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

First, find the form of the transfer matrix of a thin-lens quadrupole in the limit $L \to 0$.

One may approximate M_{thick} through various combinations of thin-lens quadrupoles and drifts. In Fig. 3 the quadrupole is approximated by a drift of the same length folllowed by a thin-lens quadrupole of strength kL (the so called *drift-kick* approximation). In Fig. 4 the the thin-lens quadrupole is located at the centre of the drift (known as *drift-kick-drift*).

By multiplying the appropriate matrices together, investigate whether the *drift-kick* and *drift-kick-drift* arrangements are symplectic (i.e. $M^T\Omega M = \Omega$). By comparing the combined transfer matrices with the Taylor expansion of the elements of M_{thick} can you say which is the more accurate scheme?

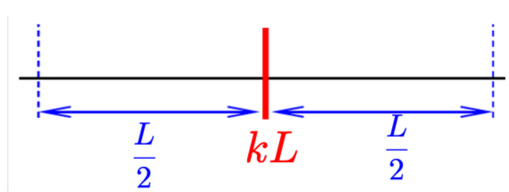

Figure 4: Schematic of the drift-kick-drift approximation of a quadrupole

Problem 6. Starting with the vector potential for a multipole magnet

$$A_x = 0, \quad A_z = 0, \quad A_l = -\mathcal{R} \sum_{n=1}^{\infty} (b_n + i a_n) \frac{(x + iz)^n}{n r_0^{n-1}} \tag{1}$$

find the Hamiltonian for an octupole (n=4) assuming small dynamic variables. We assume a normal octupole (i.e. the skew term $a_4 = 0$). The strength parameter b_n is given by

$$b_n = \frac{1}{(n-1)!} \frac{\delta^{n-1} B_y}{\delta x^{n-1}} \tag{2}$$

Note, one may follow the procedure used to calculate the quadrupole Hamiltonian in the lecture slides.