Lecture 2

Undulators and Wigglers

Contents

Dipole Radiation

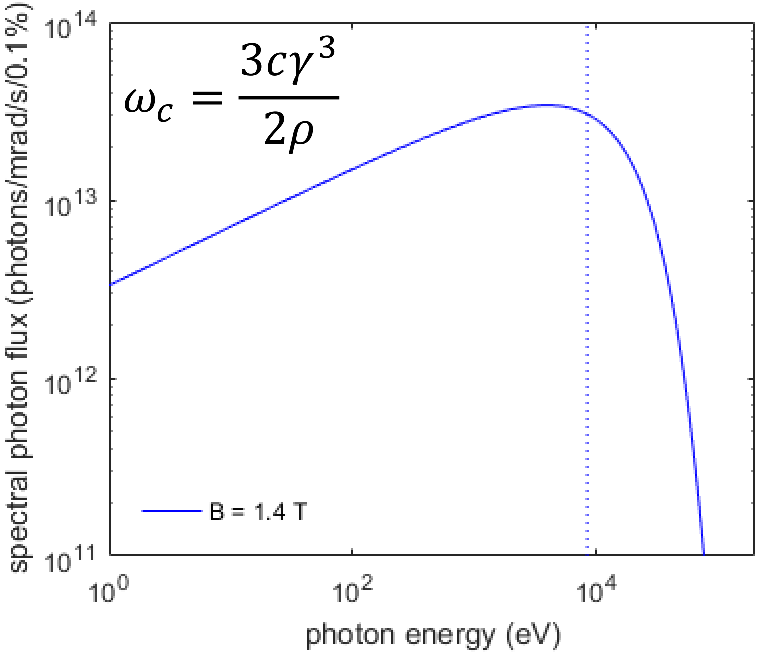
From previous lecture, total energy radiated per unit frequency is given by:

$$\frac{dW}{d\omega} = \frac{\sqrt{3}e^2}{4\pi\epsilon_0 c}\gamma\frac{\omega}{\omega_c}\int_{\omega/\omega_c}^{\infty} K_{5/3}(x)dx$$

Flux:

- Scales linearly with beam current
- Spectrum scales with critical frequency
- Fixed by design of storage ring

What if we want to change it?



Wavelength Shifter (3-pole wiggler)

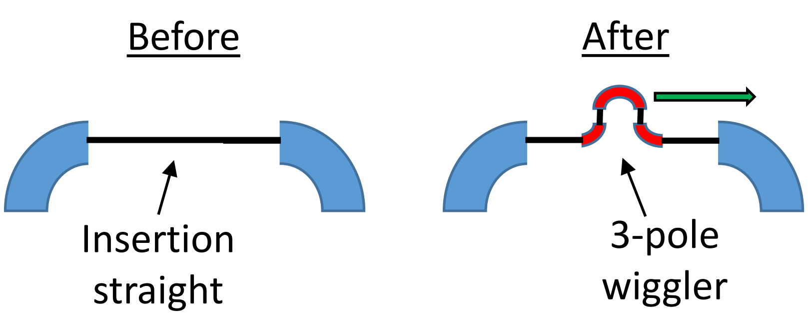
In order to change the spectrum at fixed electron energy, we need to change the critical frequency via the bend radius (i.e. the magnetic field)

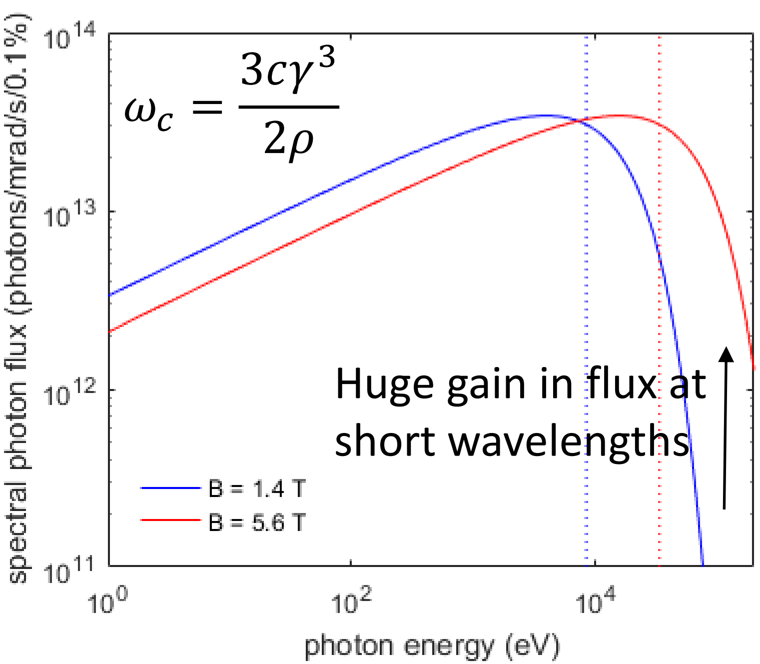
$$\omega_c = \frac{3c\gamma^3}{2\rho} \propto B_{dipole}$$

It is placed in the ring in an existing straight-section as a chicane => ‘Insertion device’

The electron beam enters and exits on-axis

Radiation at centre of dipole is parallel to central electron beam trajectory





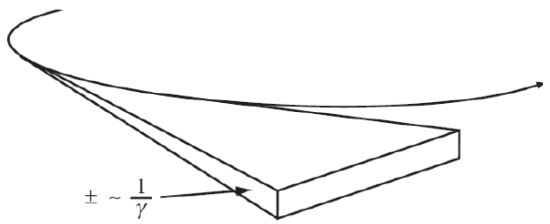
Insertion Devices

Wavelength shifters allow the critical frequency to be controlled, but the observer still only receives light from one part of the arc.

Could put several wavelength shifters in a single straight; the flux would increase linearly with the number of devices.

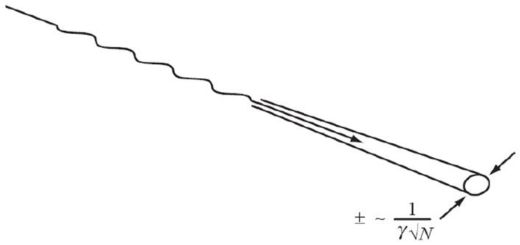
A more efficient use of space is to place a single device in the ring that causes the electron beam to oscillate many times.

Single dipole



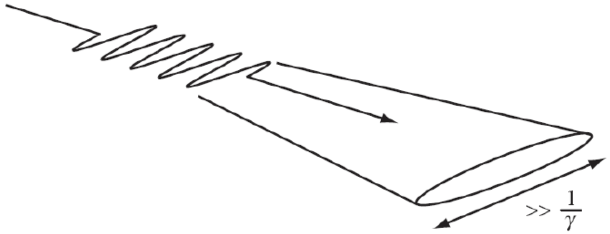
The radiation sweeps out a single arc

‘Undulator’



The oscillation amplitude is small with respect to the radiation opening angle, giving rise to interference effects

‘Multipole Wiggler’



The oscillation amplitude is large with respect to the radiation opening angle, producing a broad spectrum

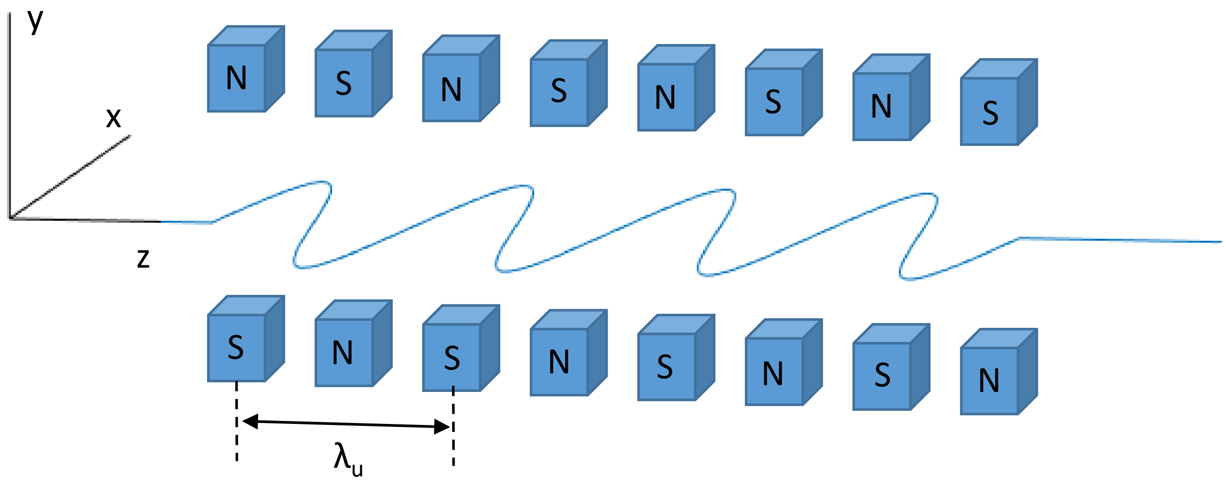
Trajectory of Motion

In order to understand the properties of the radiation emitted by an insertion device, we have to first calculate the electron beam trajectory. We consider a planar device with mid-plane symmetry. The magnetic field is vertical, with sinusoidal variation along the device:

$$B_y = B_0 \sin(k_u z)\,, \qquad k_u = \frac{2\pi}{\lambda_u}$$

Considering the Lorentz force $\mathbf{F} = \gamma m_e \frac{d\mathbf{v}}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ and relativistic energies ($v_z \sim c \gg v_x, v_y$), the equation of motion in the horizontal plane is:

$$\gamma m_e \frac{dv_x}{dt} = e v_z B_y \qquad \Rightarrow \qquad \ddot{x}(z) = \frac{d^2 x}{dz^2} = -\frac{e}{\gamma m_e c} B_y$$



Trajectory of Motion

$$\ddot{x}(z) = -\frac{e}{\gamma m_e c} B_0 \sin(k_u z)$$

The transverse velocity is simply found by integration:

$$\dot{x}(z) = \frac{eB_0}{\gamma m_e c}\frac{\cos(k_u z)}{k_u} = \frac{K}{\gamma}\cos(k_u z)\,, \qquad K = \frac{eB_0\lambda_u}{2\pi m_e c}$$

where we have defined the *dimensionless* undulator parameter *K.*

To get the horizontal position, we integrate a second time to get

$$x(z) = \frac{K}{\gamma k_u}\sin(k_u z)$$

Or in the time domain,

$$x(t) = \frac{K}{\gamma k_u}\sin(\omega_u t)$$

where we have defined $\omega_u = \overline{\beta_z} c k_u$ and used $z \approx \overline{\beta_z} ct$

Trajectory of Motion

Given the total velocity of the particle is constant, the horizontal oscillations cause the velocity in the longitudinal plane to vary, where

$$\beta^2 = \beta_x^2 + \beta_z^2$$

Substituting $\beta_x = \dot{x}(z) = \frac{K}{\gamma}\cos(k_u z)$ and using the identity $2\cos^2 a = 1 + \cos 2a$,

$$\beta_z \approx \beta^2\left(1 - \frac{K^2}{4\beta^2\gamma^2} - \frac{K^2}{4\beta^2\gamma^2}\cos(2k_u z)\right)$$

the average velocity in the longitudinal plane is therefore

$$\overline{\beta_z} \approx \beta^2\left(1 - \frac{K^2}{4\beta^2\gamma^2}\right)$$

and we can write

$$\dot{z} = \overline{\beta_z} - \frac{K^2}{4\gamma^2}\cos(2k_u z)$$

And after integration and moving to the time domain we have

$$x(t) = \frac{K}{\gamma k_u}\sin(\omega_u t) \qquad z(t) = \overline{\beta_z}ct - \frac{K^2}{8\gamma^2 k_u}\sin(2\omega_u t)$$

Trajectory of Motion

$$x(t) = \frac{K}{\gamma k_u}\sin(\omega_u t) \qquad z(t) = \overline{\beta_z}ct - \frac{K^2}{8\gamma^2 k_u}\sin(2\omega_u t)$$

The actual amplitude of motion can be very small. If we take for example a 6 GeV electron beam, and an undulator with period length λ_u= 50 mm and K=1, then then maximum amplitude of oscillation is only 0.7 µm.

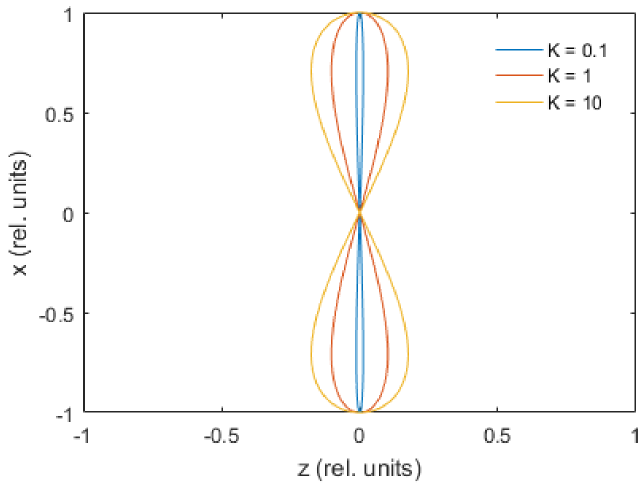
The maximum angle is similarly small. In this example, θ_{max} is 85 µrad.

$$\theta_{max} = \left(\frac{dx}{dz}\right)_{max} = \frac{K}{\gamma}$$

Trajectory of Motion

$$x(t) = \frac{K}{\gamma k_u}\sin(\omega_u t) \qquad z(t) = \overline{\beta_z} ct - \frac{K^2}{8\gamma^2 k_u}\sin(2\omega_u t)$$

If we view the particle motion in a reference frame that moves with the average velocity of the particle, the amplitude of the motion appears as a figure-of-eight pattern.



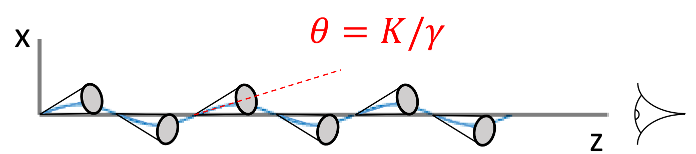
At small K values, the motion is approximately a simple harmonic oscillation in the horizontal plane and an emission spectrum consisting of a single harmonic.

As K increases, the figure-of-eight pattern becomes more pronounced, leading to increased emission at higher harmonics.

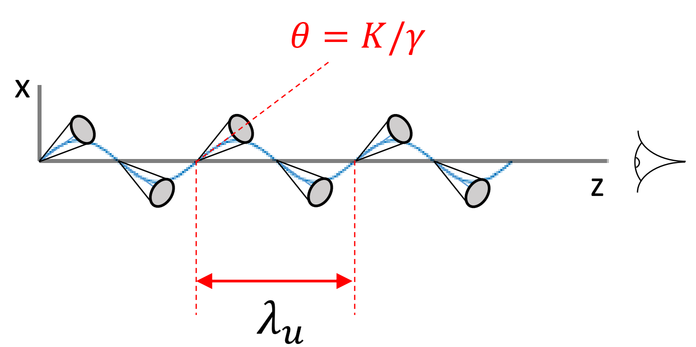
Radiation Spectrum: Qualitative Treatment

From the first lecture, we have seen that the natural opening angle of synchrotron radiation is $\theta \sim 1/\gamma$. We have also seen that the maximum angle of deflection for the electron beam when passing through an insertion device is $\theta_{max} = K/\gamma$. This allows us to identify two regimes:

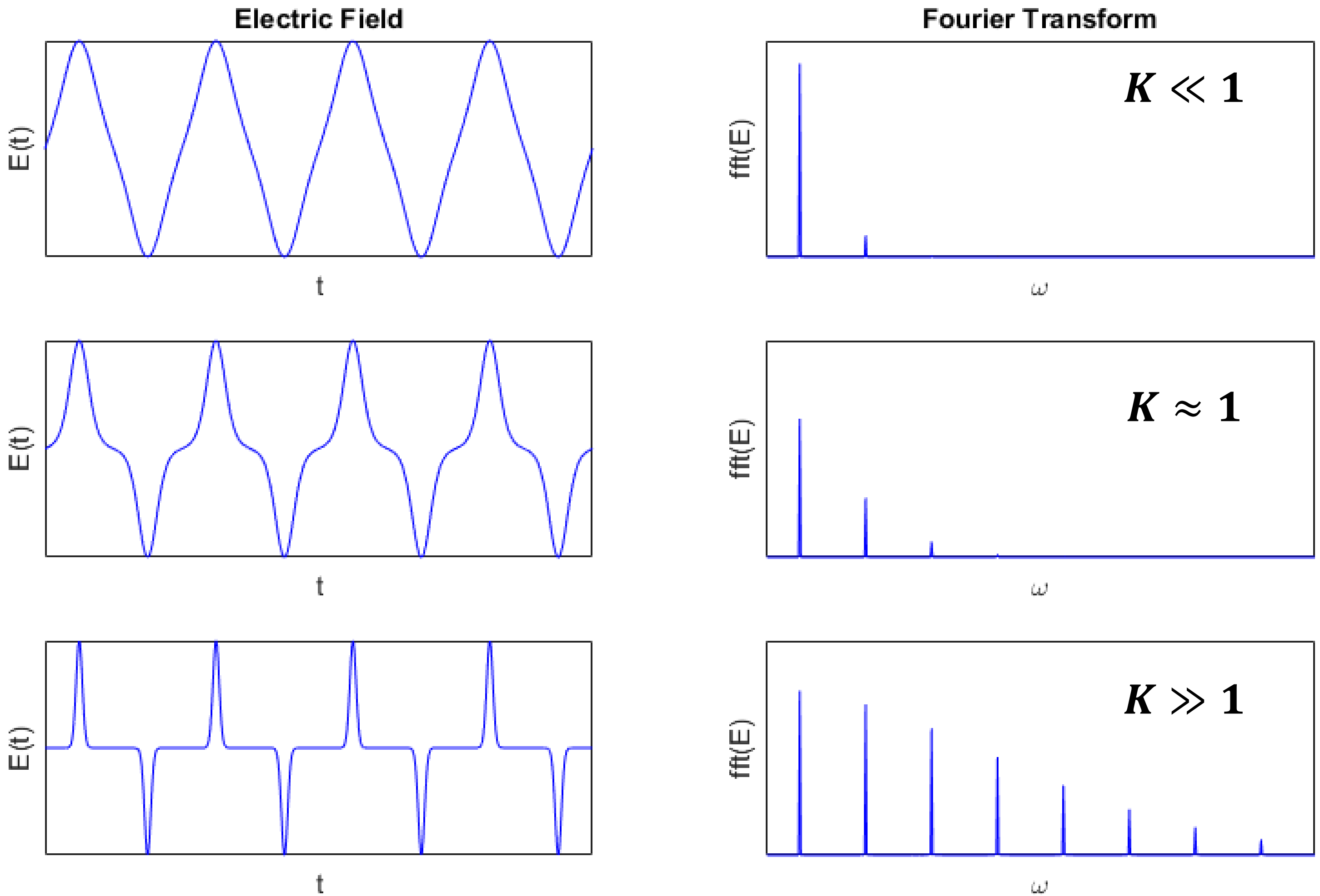
$\boldsymbol{K \lesssim 1}$: The angular deflection is less than the opening angle of the radiation. The observer see the radiation continuously along the length of the device. Such devices are commonly described as **undulators**.



$\boldsymbol{K \gtrsim 1}$: The angular deflection is larger than the opening angle of the radiation. The observer only sees flashes of radiation as the electron beam sweeps from side to side. These devices are usually termed **wigglers**.



Radiation Spectrum: Qualitative Treatment



Resonant wavelength

The properties of the radiation emitted by an electron travelling through an insertion device can be understood as interference of the wavefronts emitted by the same electron at different points along the axis of the device.

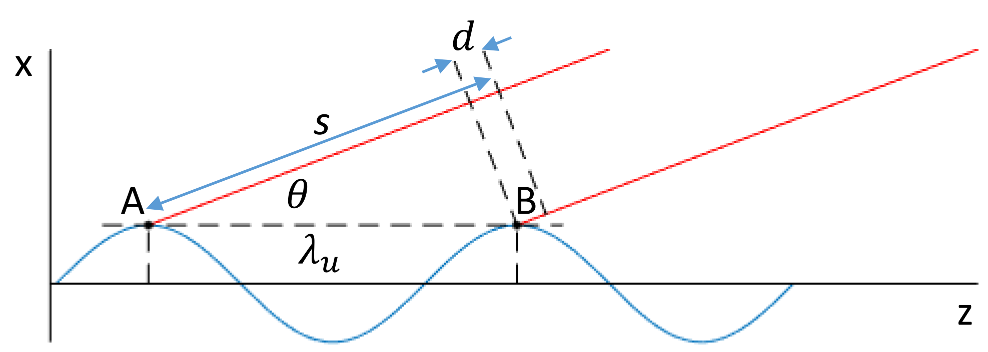
The time taken for the electron to travel from point A to point B is $\tau = \frac{\lambda_u}{\bar{\beta}_z c}$.

In the same time, the radiation emitted at point A will have travelled a distance $s = c\tau$. That is, the radiation will have advanced a distance *d* ahead of the electron

$$d = c\tau - \lambda_u \cos\theta = \frac{\lambda_u}{\bar{\beta}_z} - \lambda_u \cos\theta$$

For constructive interference, the distance *d* must be an integer number of wavelengths

$$n\lambda_r = \lambda_u \left(\frac{1}{\bar{\beta}_z} - \cos\theta \right)$$



Resonant wavelength

From the analysis of the particle motion in the insertion device, we have

$$\overline{\beta_z} \approx \beta^2\left(1 - \frac{K^2}{4\beta^2\gamma^2}\right)$$

So

$$\overline{\beta_z} \approx 1 - \frac{1}{2\gamma^2} - \frac{K^2}{4\gamma^2} \approx 1 - \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

Substituting back into the resonance condition, and making use of the small angle approximation $\cos\theta \approx 1 - \theta^2/2$ we now have

$$n\lambda_r = \lambda_u\left(\left[1 + \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)\right] - \left[1 - \frac{\theta^2}{2}\right]\right)$$

$$\boxed{\lambda_r = \frac{\lambda_u}{2n\gamma^2}\left(1 + \frac{K^2}{2} + \theta^2\gamma^2\right)}$$

This gives the condition for constructive interference in an insertion device.

Resonant wavelength

$$\lambda_r = \frac{\lambda_u}{2n\gamma^2}\left(1 + \frac{K^2}{2} + \theta^2\gamma^2\right)$$

Properties of radiation emitted by insertion devices:

- The fundamental wavelength of the radiation is shorter than the period of the device by a factor $2\gamma^2$. To put this in context, a 3 GeV electron beam passing through an insertion device with period of 25 mm would emit radiation in the region ~3.6 Å
- The radiation wavelength can be changed either by changing the energy of the electrons (via the γ^2 term), or by changing the magnetic field of the device (via K)
- The resonant wavelength increases with increasing observation angle

Radiation from a Planar Undulator

From previous lecture, the energy radiated by an electron, per unit frequency, per unit solid angle is found from the Fourier transform of the electric field as seen by the observer

$$\frac{d^2W}{d\Omega d\omega} = \frac{2}{2\pi\mu_0 c}\left|\int_{-\infty}^{\infty}\big(r\mathbf{E}(t)\big)e^{i\omega t}dt\right|^2$$

where the electric field is found for the far-field (i.e. considering only the acceleration term)

$$\mathbf{E}(t) = \frac{e}{4\pi\varepsilon_0 c}\left(\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{r(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right)_{ret}$$

The subscript 'ret' means the field is related to the electron acceleration at the retarded time of emission, not the time of observation.

Substituting the electric field into the above expression, and integrating by parts to simplify leads to the expression

$$\frac{d^2W}{d\Omega d\omega} = \frac{e^2}{16\pi^3\epsilon_0 c}\omega^2\left|\int_{-\infty}^{\infty}(\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}))e^{i\omega\left(t'+\frac{r(t')}{c}\right)}dt'\right|^2$$

Radiation from a Planar Undulator

In the case of a array of magnets consisting of N periods (as is the case of an undulator), we can split the integral up into an integration over one period of the device, multiplied by a series of terms containing the phase information, i.e.

$$\frac{d^2W}{d\Omega d\omega} = \frac{e^2}{16\pi^3\epsilon_0 c}\omega^2 \left| \int_{-\lambda_u/2\overline{\beta_z}c}^{\lambda_u/2\overline{\beta_z}c} (\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta}))e^{i\omega\left(t' + \frac{r(t')}{c}\right)} dt' \right|^2 \times \left| 1 + e^{i\omega d/c} + e^{i2\omega d/c} + \cdots + e^{i(N-1)\omega d/c} \right|^2$$

Here, d is the distance between successive wavefronts found when calculating the resonant wavelength of an undulator

$$d = \frac{\lambda_u}{\overline{\beta_z}} - \lambda_u \cos\theta$$

The distance between successive wavefronts can also be re-expressed in terms of the fundament frequency

$$d = \lambda_1 = \frac{2\pi c}{\omega_1}$$

Radiation from a Planar Undulator

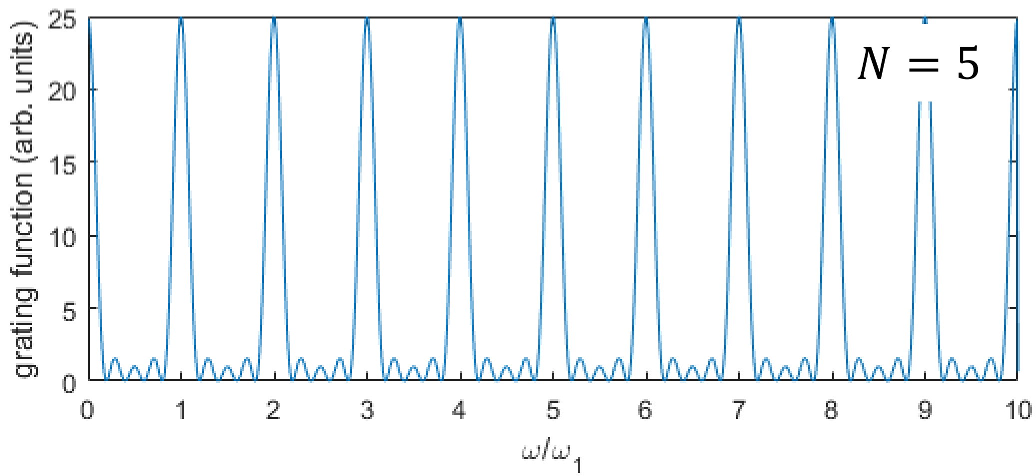
The series of phase factors in the radiation integral can now be simplified using the identity

$$\left|1 + e^{i\delta} + \cdots + e^{i(N-1)\delta}\right|^2 = \frac{\sin^2 N\delta/2}{\sin^2 \delta/2}$$

giving the so-called 'grating function', in analogy to diffraction gratings

$$\left|1 + e^{i\omega d/c} + e^{i2\omega d/c} + \cdots + e^{i(N-1)\omega d/c}\right|^2 = \frac{\sin^2 N\pi\omega/\omega_1}{\sin^2 \pi\omega/\omega_1}$$

The grating function represents the interference between successive periods, selecting a narrow range of frequencies close to each harmonic (i.e. $n = \omega/\omega_1 = 1, 2, 3\,..$).



Radiation from a Planar Undulator

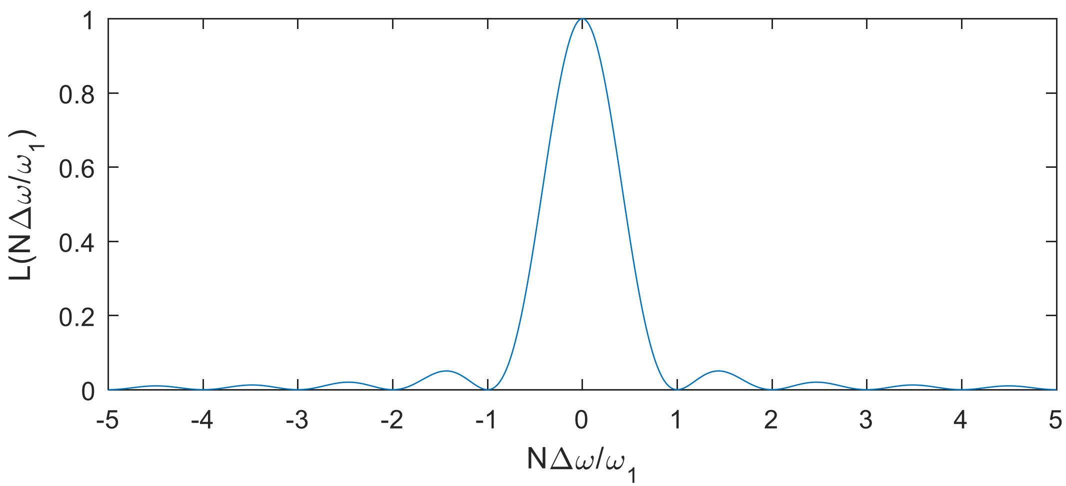
The next step is to investigate the shape of the function close to the harmonics. We start by normalising the grating function to unit amplitude, and focussing on the region close to each harmonic ($\Delta\omega = \omega - n\omega_1$) leads to the 'line-shape function'

$$L\left(\frac{N\Delta\omega}{\omega_1}\right) = \frac{1}{N^2}\frac{\sin^2 N\pi\Delta\omega/\omega_1}{\sin^2 \pi\Delta\omega/\omega_1}$$

For large N ($\gtrsim 10$), the shape of the function becomes independent of the number of periods. The full-width half maximum of this peak occurs at $\frac{\mathrm{N}\Delta\omega}{\omega_1} = \pm 0.5$, i.e. the bandwidth of the nth harmonic $\omega_n = n\omega_1$ is

$$\frac{\Delta\omega}{\omega_n} = \frac{1}{nN}$$

which decreases with increasing harmonic and increasing number of periods.



Radiation from a Planar Undulator

The radiation integral can now be expressed in the form

$$\frac{d^2W}{d\Omega d\omega} = \frac{e^2\gamma^2N^2}{4\pi\epsilon_0 c} F_n(K,\theta,\phi) L\left(\frac{N\Delta\omega}{\omega_1(\theta)}\right)$$

where

$$F_n(K,\theta,\phi) \propto \omega^2 \left| \int_{-\lambda_u/2\overline{\beta_z}c}^{\lambda_u/2\overline{\beta_z}c} (\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})) e^{i\omega\left(t'+\frac{r(t')}{c}\right)} dt' \right|^2$$

This integral can in general be solved numerically. However, analytic solutions also exist for some trajectories such as purely sinusoidal motion [1, 2]. In this case,

σ-mode (H-polarisation)

$$F_n(K,\theta,\phi) = \frac{n^2}{A^2}\left|A_x, A_y\right|^2$$

π-mode (V-polarisation)

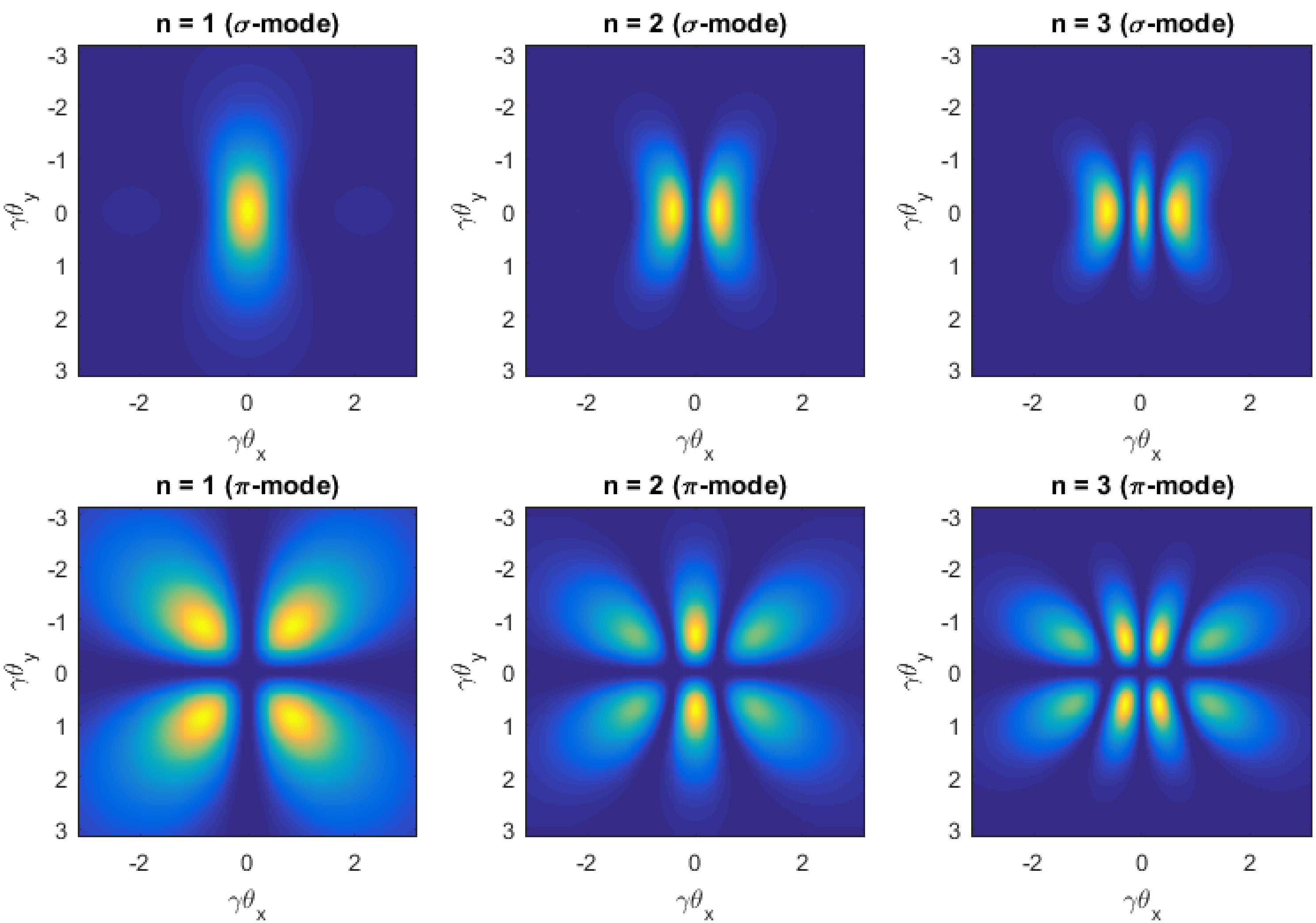
$$A = 1 + K^2/2 + \gamma^2\theta^2$$

$$A_x = 2\gamma\theta\cos(\phi)S_0 + K(S_1 + S_{-1}), \qquad A_y = 2\gamma\theta\sin(\phi)\,S_0$$

$$S_q = \sum_{p=-\infty}^{\infty} J_p(Y)\,J_{n+2p+q}(X)$$

$$X = 2n\gamma\theta K\cos(\phi)\,/A, \qquad Y = nK^2/4A$$

Relative Angular Flux Density



Radiation from a Planar Undulator

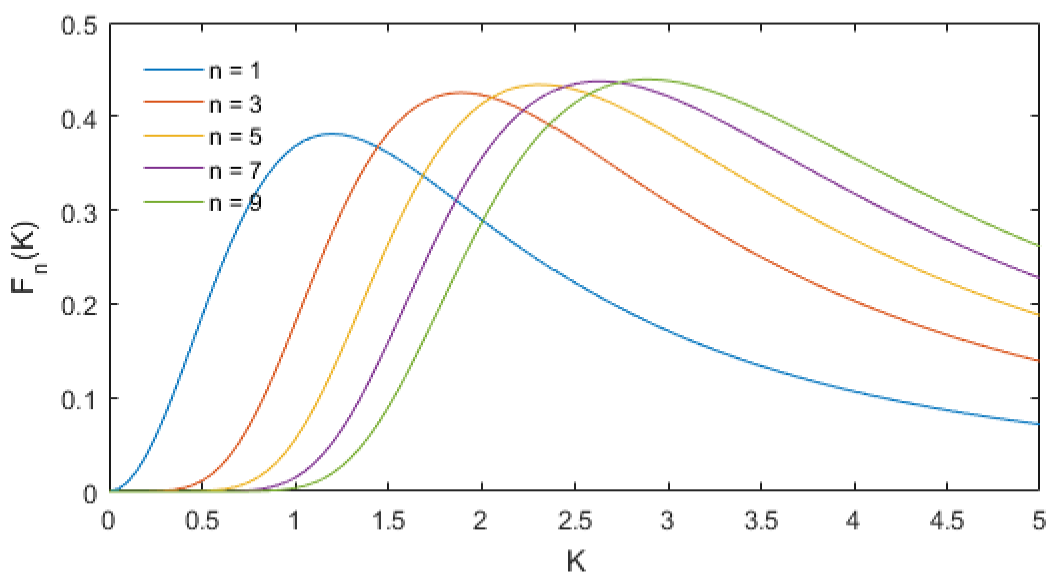
On axis, the expression can be simplified:

$$F_n(K,0,0) = F_n(K) = \frac{n^2K^2}{(1+K^2/2)^2}\left[J_{n+\frac{1}{2}}(Z) - J_{n-\frac{1}{2}}(Z)\right]^2$$

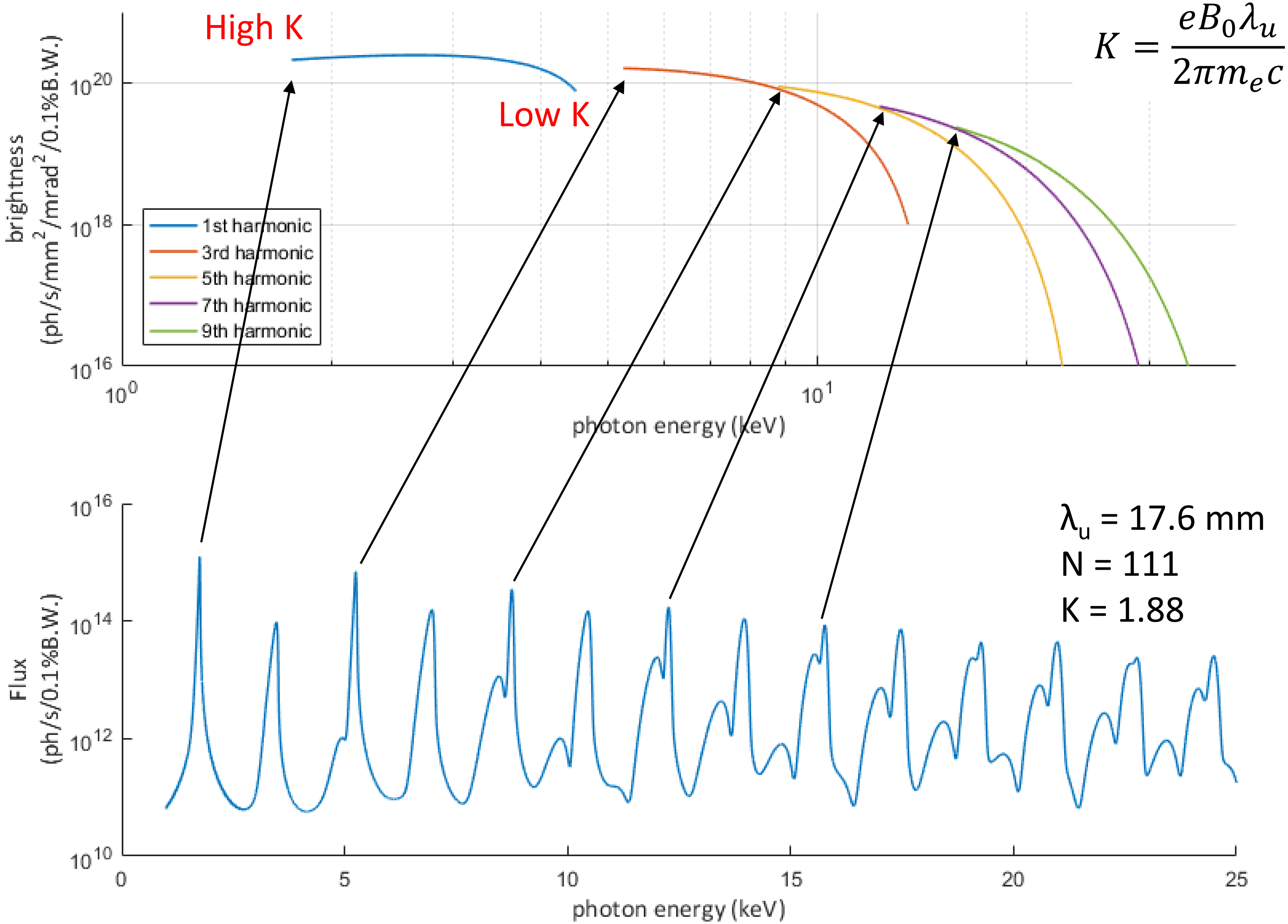
with

$$Z = \frac{nK^2}{4(1+K^2/2)}$$

This equation has the property that only odd harmonics are non-zero (no even harmonics observed on-axis). As K increases, the higher harmonics become stronger as a result of the longitudinal modulation.

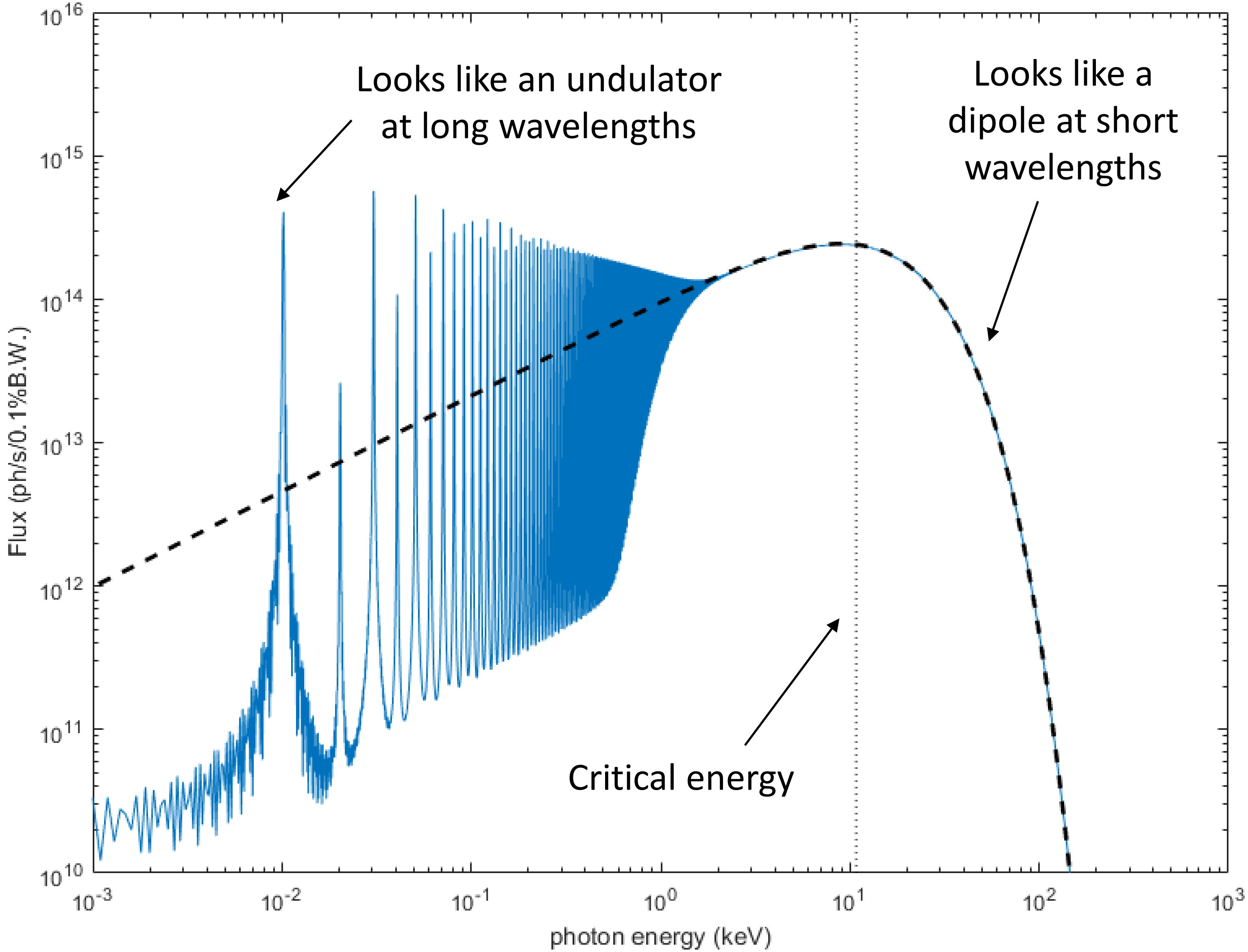


Spectrum for a real undulator



The photon energy is varied by changing the K parameter (the B-field)

Spectrum for a wiggler



Summary of properties

Undulators

- $K \lesssim 1$ (weaker field, shorter period)
- Amplitude of motion less than opening angle of radiation
- Observer sees light from the whole device
- Discrete lines appear in the spectrum
- Power scales with N_{period}^2
- Line width shrinks with nN_{period}

Multipole Wigglers

- $K \gtrsim 1$
- Amplitude of motion greater than opening angle of radiation
- Observer misses part of the radiation as the beam sweeps from side to side
- Continuous spectrum at short wavelengths (more like a dipole)
- Power scales with $2N_{period}$
- Spectrum looks like an undulator at long wavelengths

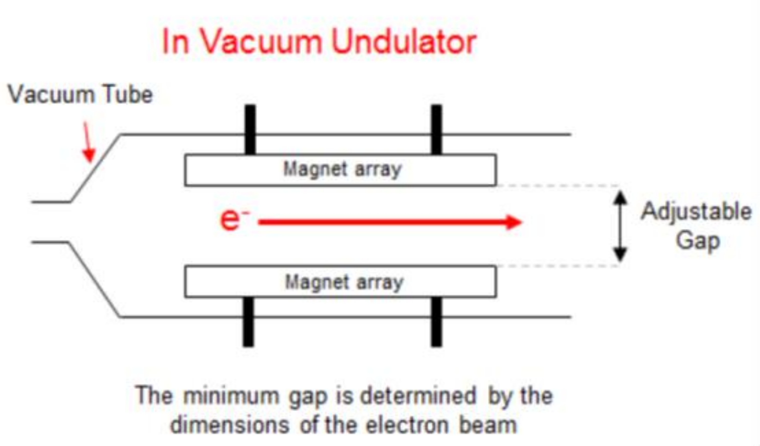
Types of Undulators and Wigglers

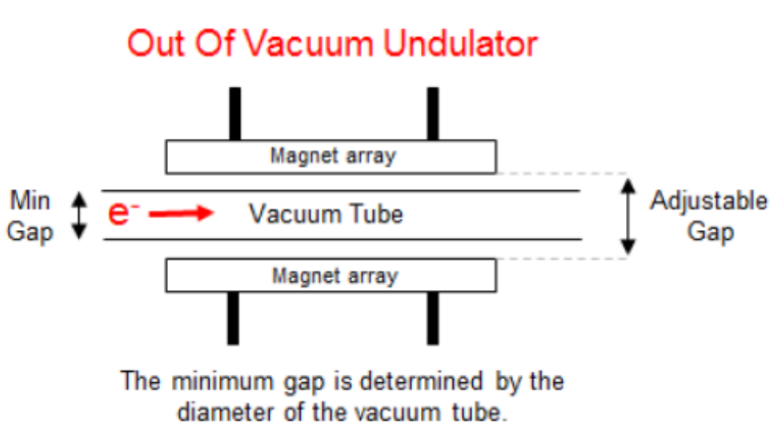
Electromagnetic undulators: The field is generated by current-carrying coils. They may have iron poles

Permanent magnet undulators: The field is generated by permanent magnets such as Samarium Cobolt (SmCo; 1T) or Neodymium Iron Boron (NdFeB; 1.4T). They may also have iron poles (hybrid undulators)

APPLE-II undulators: Arrays of permanent magnets which can slide longitudinally in order to change the polarisation of the magnetic field to generate horizontal, vertical or circularly polarised radiation

In-vacuum undulators: permanent magnet arrays that sit within the vacuum pipe. This allows the gap to be closed to small values (< 5 mm!) and hence high fields



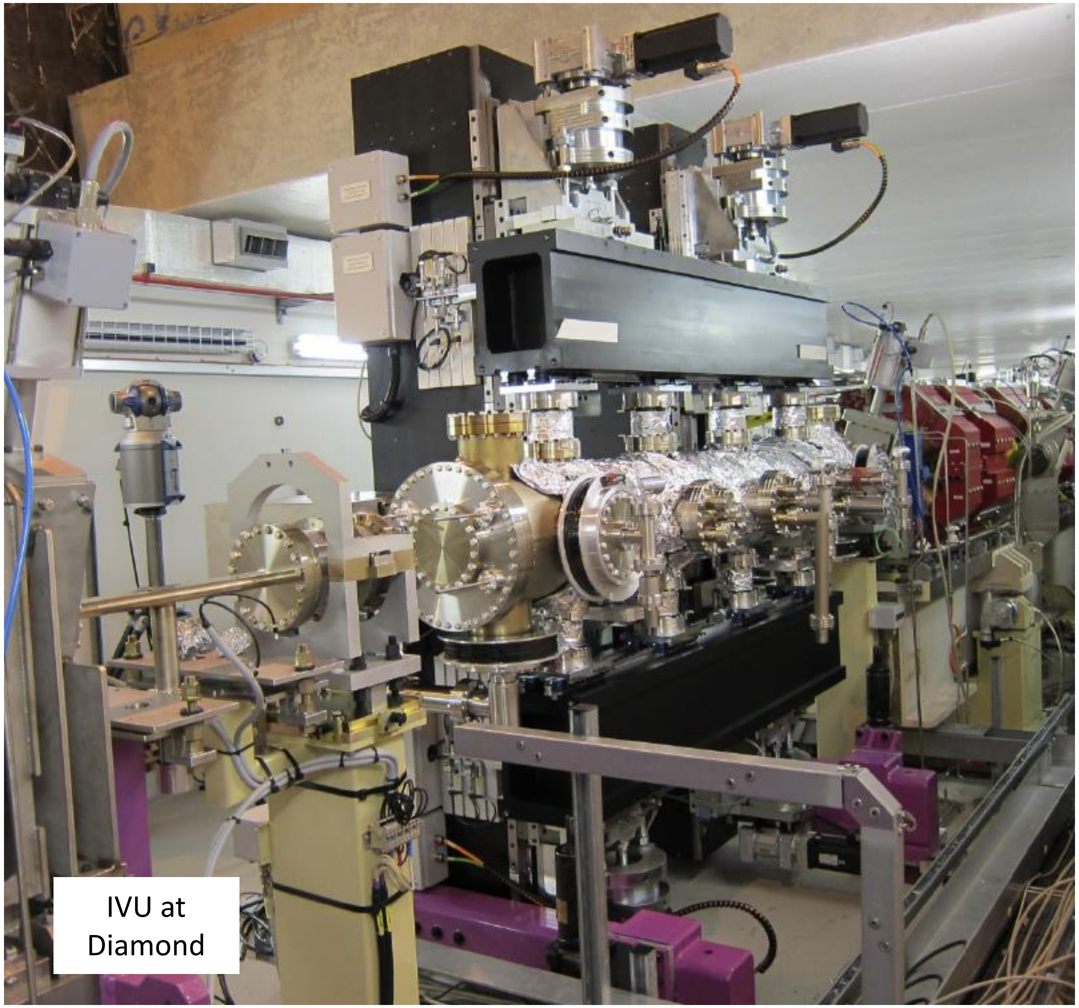


Electromagnetic Undulators

- Out of vacuum
- Field controlled by varying the current in the coils
- Possible to generate variable polarisations
- Space for coils becomes an issue for short periods and high fields
- Generally used only for long period devices

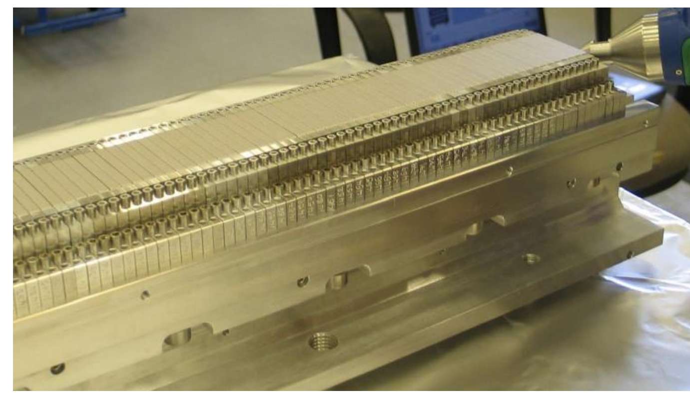
HU64 at Soleil

In-vacuum Undulators

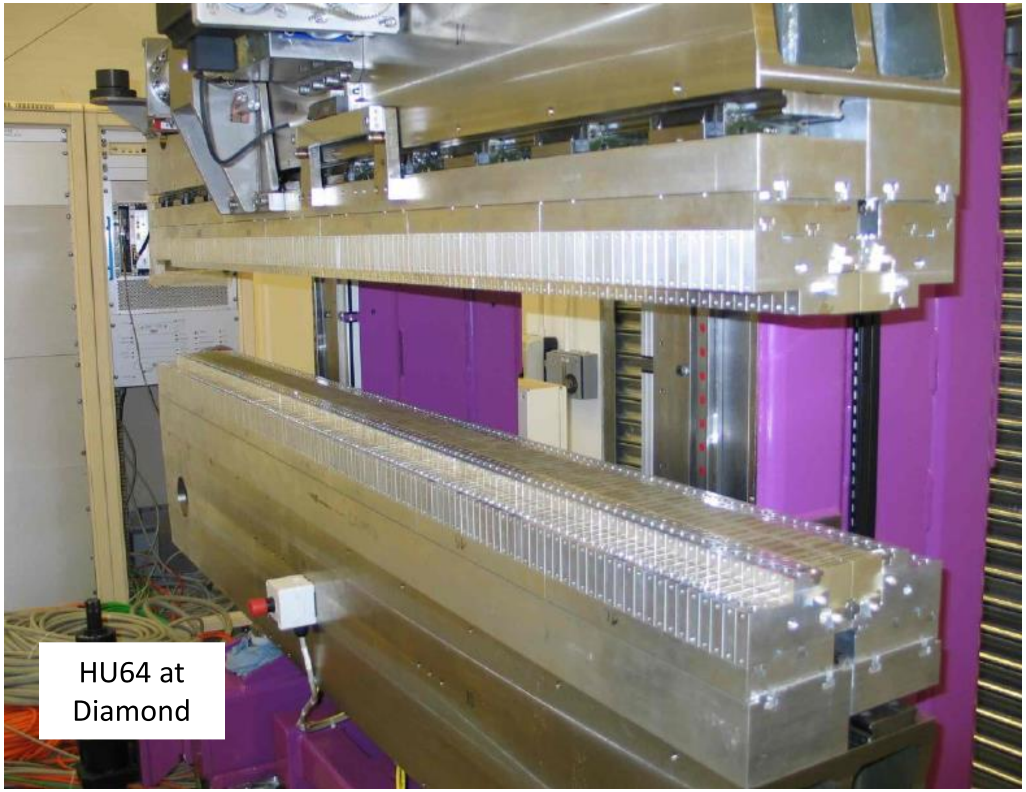


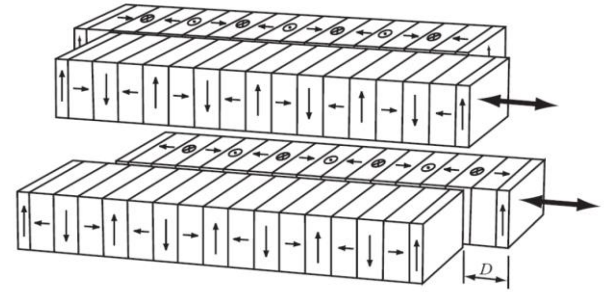
IVU at Diamond

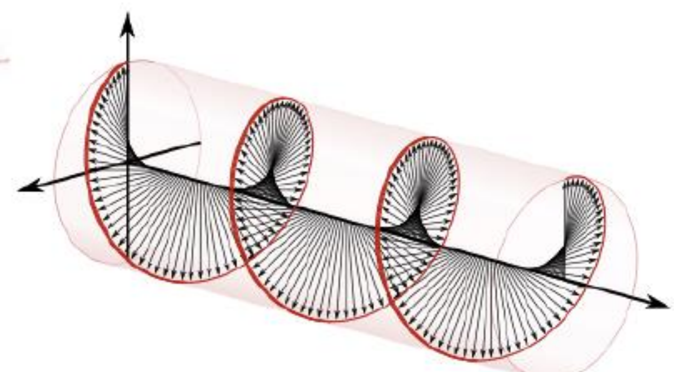
- Permanent magnet array
- Field aligned vertically
- Change field by varying the gap
- Period length typically 20-30 mm



Advanced Planar Polarised Light Emitter Undulators (APPLE-II)



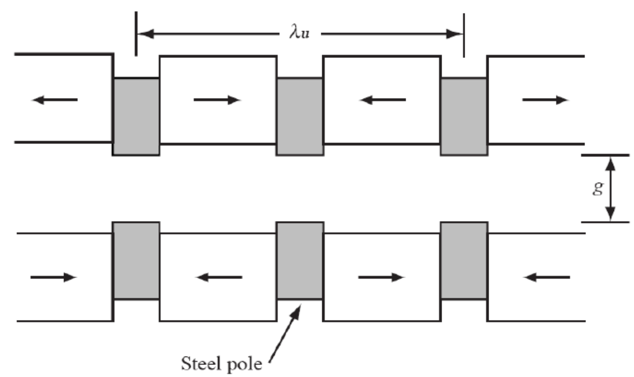




- Helical field controlled by adjusting axis phase
- Typically out of vacuum
- Gap controls field strength
- Generate light with arbitrary polarisation
- Period length typically 50-150 mm

Hybrid Devices

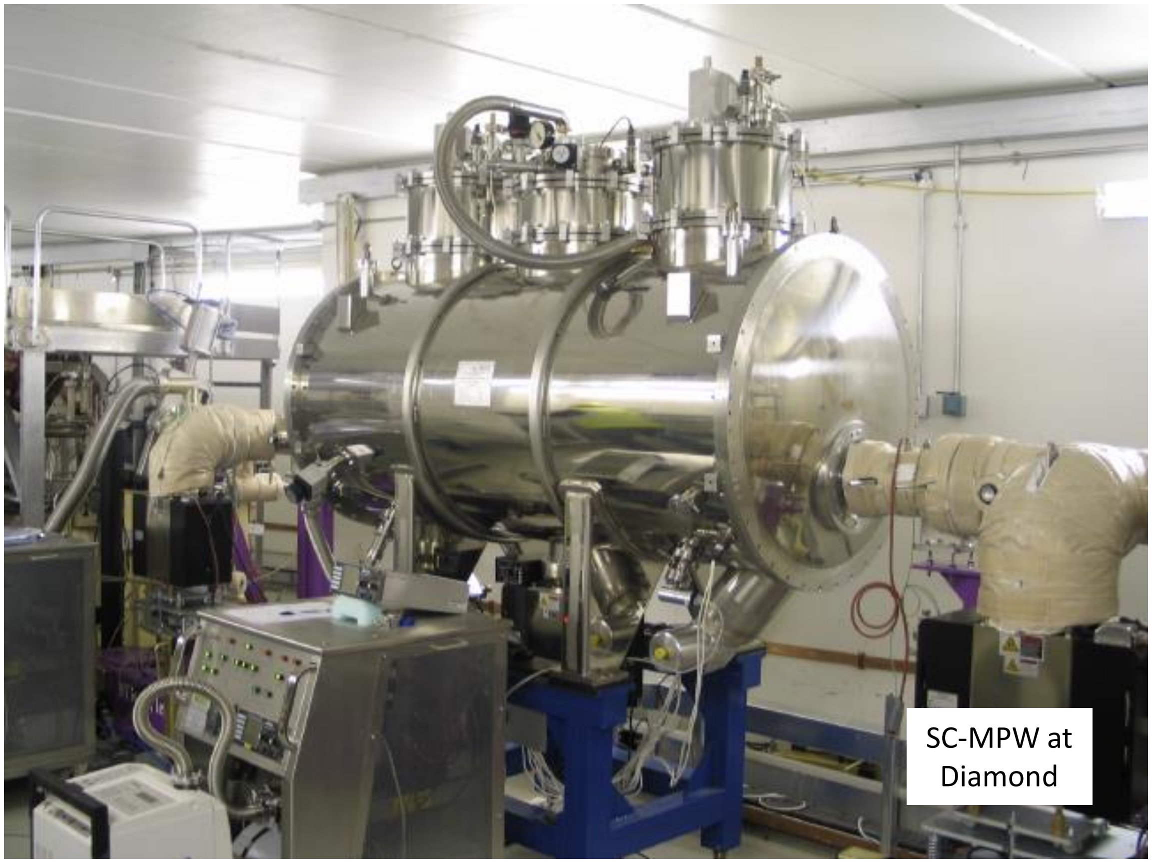
- Permanent magnets with steel poles
- Allow higher fields to be reached
- Field strength again controlled via gap
- Both undulator and wiggler configurations



Cryogenic Permanent Magnet Undulators (CPMU)

- Permanent magnet array cryogenically cooled (e.g. NdFeB cooled to 148 K)
- Hybrid configuration to get higher fields with shorter periods (λ_u ~15-20 mm)
- Exploits field enhancement in magnetic materials at low temperature
- Cooled with liquid nitrogen

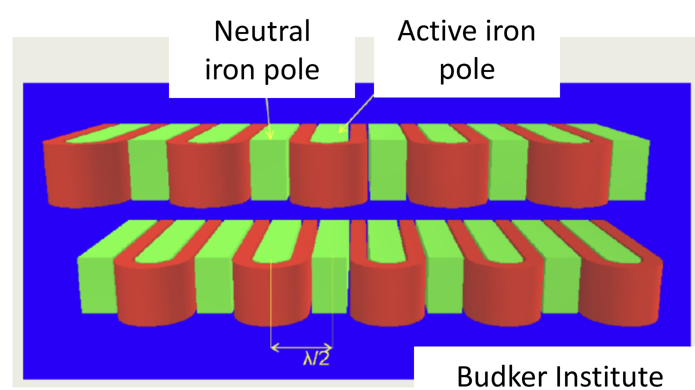
Super-conducting Wigglers



- Used when very high field is required (3-10 T)
- Need cryogenic system to keep the coil super-conducting
- Nb_3Sn and NbTi wires

Example at Diamond

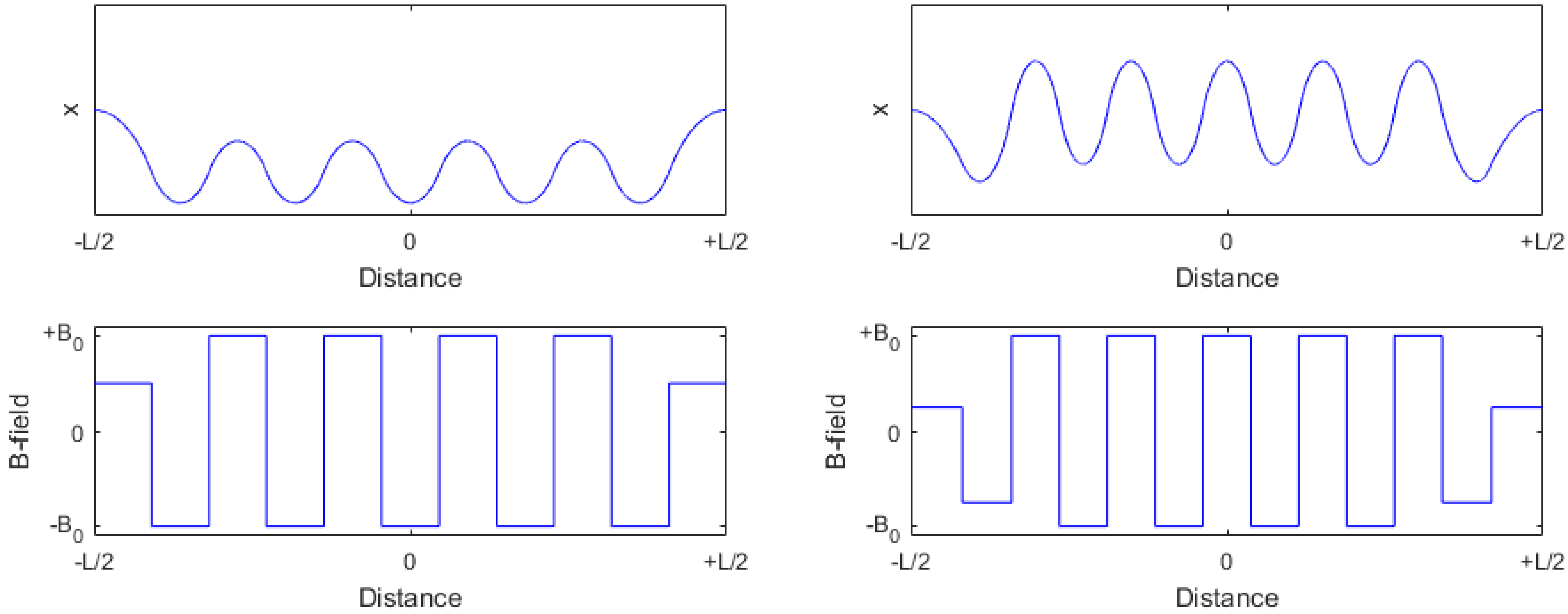
- SC-MPW60
- 3.5 T, Cooled to 4 K, 24 periods
- Fixed gap 10 mm
- K = 21



Trajectory through an ID

We want to ensure that the electron beam enters and exits the device on-axis. To do this requires a special configuration of poles.

One solution is to have poles of one-half strength at each end. Another solution requires the ratio $\left[\frac{B_0}{4}, \frac{3B_0}{4}, B_0, \dots, B_0, \frac{3B_0}{4}, \frac{B_0}{4}\right]$, and has the advantage that the average displacement through the device is zero.



Trajectory through an ID

In general, the magnetic fields will not be perfect, and the electron beam will exit the device with an overall change in displacement and angle. Assuming the real field profile is known, these can be found by direct integration of the equation of motion:

$$\ddot{x} = \frac{e}{\gamma m_e c} B_y(s)$$

To minimise the impact of the device, we need to ensure the first and second field integrals are zero:

$$\int_{-L/2}^{L/2} B_y(s)ds = 0 \qquad \text{(angle)}$$

$$\int_{-L/2}^{L/2} \int_{-L/2}^{s} B_y(s')ds'ds = 0 \qquad \text{(position)}$$

This can be achieved through careful shimming of the device, and any final corrections can be applied using dipole trim coils at the entrance and exit.

Focussing effects

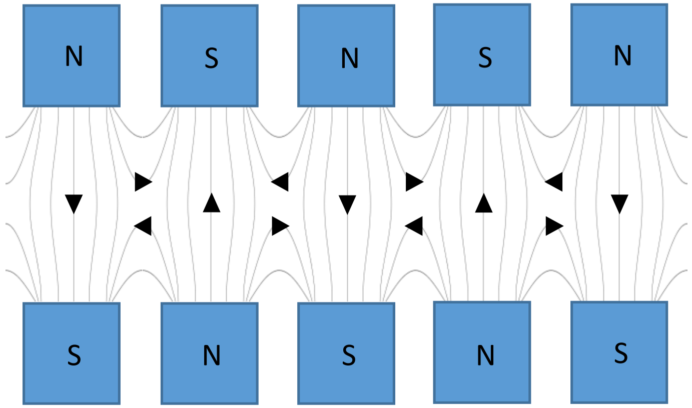
So far we have only considered what happens to electrons if they are travelling through the centre of the device and assumed an idealised purely vertical magnetic field. In fact, in order to satisfy Maxwell's equations, real insertion devices of finite width must have horizontal and longitudinal components as well. However, assuming the magnet pole width is large compared to the oscillation amplitude, the variation in the horizontal plane can in general be neglected.

In this situation, an improved approximation for the magnetic field for a planar device is:

$$B_x(x,y,s) = 0$$
$$B_y(x,y,s) = B_0 \cosh(k_u y) \cos(k_u s)$$
$$B_s(x,y,s) = -B_0 \sinh(k_u y) \sin(k_u s)$$

On axis, this reduces to the previous expression $B_y(0,0,s) = B_0 \cos(k_u s)$.

The introduction of a longitudinal field component leads to a focussing effect in the vertical plane.



Focussing effects

At the point half way between the poles, the longitudinal field component is at its strongest. This is also the location where the horizontal velocity is at its maximum, giving rise to a vertical force (Lorentz) acting towards the mid-plane.

The equations of motion are now

$$\ddot{x} = \frac{e}{\gamma m_e c}\left(B_y - \dot{y}B_s\right)$$
$$\ddot{y} = \frac{e}{\gamma m_e c}(\dot{x}B_s - B_x)$$

Close to the mid-plane we can write

$$B_s \approx \frac{dB_s}{dy}y = \frac{dB_y}{ds}y$$

So, substitution into the above equations (with $B_x = 0$ and $\dot{y} = 0$) gives

$$\ddot{y} = \left(\frac{e}{\gamma m_e c}\right)^2 \int B_y ds \frac{dB_y}{ds}y$$

Focussing effects

Comparing with Hill's equation

$$\ddot{y} + K_y y = 0$$

We can identify the focussing term K_y as

$$K_y = \left(\frac{e}{\gamma m_e c}\right)^2 \int B_y ds \frac{dB_y}{ds}$$

Averaging over the length of the device and integrating by parts, this can be rewritten as

$$\widehat{K}_y = \left(\frac{e}{\gamma m_e c}\right)^2 \frac{\int B_y^2 ds}{L}$$

Which for sinusoidal motion gives

$$\widehat{K}_y = \left(\frac{e}{\gamma m_e c}\right)^2 \frac{B_0^2}{2} = \frac{1}{2\rho_0^2}$$

An undulator acts as a focussing quadrupole in the vertical plane and as a drift in the horizontal plane. The vertical focussing is second order in both energy and B-field.

Focussing effects

The vertical focussing from the undulator will clearly have an impact on the optics of the storage ring [7].

1) An increase in the vertical tune:

$$\Delta Q_y = \frac{\widehat{K}_y L_u \hat{\beta}_y}{4\pi}\left(1 + \frac{L_u^2}{12\hat{\beta}_y{}^2}\right)$$

2) A relative change in the beta-functions (beta-beat):

$$\frac{\Delta\beta_y}{\beta_y} = -\frac{\widehat{K}_y L_u \hat{\beta}_y}{2\sin\left(2\pi Q_y\right)}\left(1 - \frac{L_u^2}{12\hat{\beta}_y{}^2}\right)$$

Summary

Insertion devices are placed in electron storage rings to provide tuneable, high-brightness sources of synchrotron radiation

Wavelength shifter: 3-pole device, high B-field to increase critical wavelength.
Undulator: $K \lesssim 1$; brightness scales with N_{period}^2. Lines appear in spectrum.
Wiggler: $K \gtrsim 1$; brightness scales with $2N_{period}$. Continuous spectrum.

Can produce linear, elliptical or circularly polarised light

Many designs exist (electro-magnet, permanent-magnet, super-conducting, cryo-cooled, planar or helical, in-vacuum or out-of-vacuum, …).

Selecting which one is most appropriate depends upon:

- scientific requirements of the beamline
- electron beam parameters
- practical constraints

Insertion devices perturb the storage ring optics. Field quality must be very high, but even so correction strategies may be required.

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