Science & Technology Facilities Council
ISIS

Accelerator Physics

Lecture 10: Longitudinal Dynamics

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Contents

Acceleration and Energy Gain, 1

- To accelerate we require a force **in the direction of motion!**
- Newton-Lorentz force on a charged particle:

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = q\left(\vec{E} + \vec{B} \times \vec{v}\right) \tag{1}$$

- Second term is always perpendicular to motion: **no acceleration**
- Hence to accelerate along the direction of motion we need an electric field in that direction.

$$\frac{\mathrm{d}p}{\mathrm{d}t} = qE_z \tag{2}$$

Acceleration and Energy Gain, 2

- In relativistic dynamics energy and momentum are linked,

$$E^2 = E_0^2 + p^2c^2, \qquad \mathrm{d}E = v\,\mathrm{d}p \tag{3}$$

- The rate of energy gain per unit length of acceleration is therefore,

$$\frac{\mathrm{d}E}{\mathrm{d}z} = v\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{\mathrm{d}p}{\mathrm{d}t} = qE_z \tag{4}$$

- And the **kinetic energy gained** from the electric field along z is,

$$\begin{aligned} \mathrm{d}W &= \mathrm{d}E = qE_z\,\mathrm{d}z \\ \therefore \qquad W &= q\int E_z\,\mathrm{d}z = qV \end{aligned} \tag{5}$$

Units of Energy

- Accelerator physics typically uses units of **electron volts** for energy.
- ***1 eV (electron volt)*** *is the kinetic energy lost (or gained) by a particle of unit charge when accelerated from rest through a potential difference of one volt in vacuum.*
- Some useful conversions:

1 eV	1.602×10^{-19} J
1 eV/c^2	1.783×10^{-36} kg

electron	9.109×10^{-31} kg	0.511 MeV/c^2
proton	1.673×10^{-27} kg	938.272 MeV/c^2

Methods of Acceleration

- Electrostatic fields are limited by insulation problems and magnetic fields don't accelerate
- **For circular machines DC acceleration is impossible** as $\oint \vec{E} \cdot \mathrm{d}\vec{s} = 0$
- From Maxwells equations,

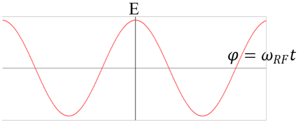
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \qquad \oint \vec{E} \cdot \mathrm{d}\vec{s} = -\iint \frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{A} \tag{6}$$

a **time varying magnetic field generates an electric field**.

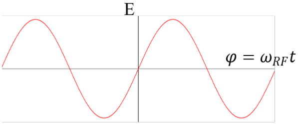
- Therefore, use time varying fields which for most accelerator applications are at RF frequencies.

Phase Conventions

1. For linear accelerators, the origin of time is taken as the positive crest of the RF voltage.
2. For circular accelerators, the origin of time is taken as the positive gradient zero crossing of the RF voltage.



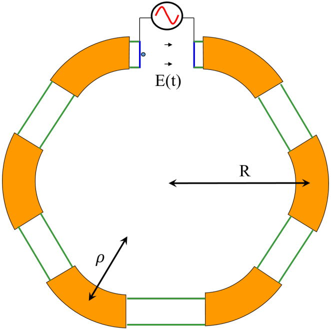
$E(t) = E_0 \cos(\omega_{RF} t)$



$E(t) = E_0 \sin(\omega_{RF} t)$

3. We will stick to the circular accelerator convention.

The Synchrotron, 1



- Constant orbit during acceleration
- Revolution frequency increases with energy
- RF cavity frequency increases with energy
- Magnetic field strength increases to maintain orbit radius
- Synchronism condition:

$$T = hT_{RF} = \frac{2\pi R}{v} \tag{7}$$

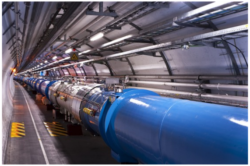
The Synchrotron, 2

The **synchrotron** is so called because the accelerating RF cavities and the magnetic fields all have to work in synchronism in order for it to work. There is a **synchronous RF phase** for which the energy gain is precisely what is required to match the increase in magnetic field each turn. This implies the following conditions:

- Energy gain per turn, $\Delta E_{\mathrm{turn}} = eV \sin \phi_s$
- Synchronous particle
- RF synchronism $\omega_{\mathrm{RF}} = h\omega$
- Constant orbit
- $B\rho = p/e$, implying a varying magnetic field

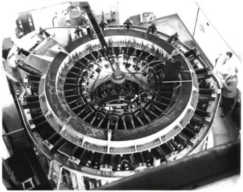
Examples of Synchrotrons

ISIS Spallation Source, UK



LHC, CERN, Switzerland

Diamond Light Sources, UK



Glasgow Synchrotron, UK

Energy Ramping

The momentum and magnetic field must increase following the magnetic rigidity equation

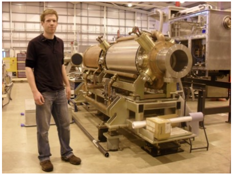
$$p = eB\rho \quad \Rightarrow \quad \frac{\mathrm{d}p}{\mathrm{d}t} = e\rho\frac{\mathrm{d}B}{\mathrm{d}t} \tag{8}$$

$$\begin{aligned} \Delta p_{\mathrm{turn}} &= e\rho\frac{\mathrm{d}B}{\mathrm{d}t}T \\ &= e\rho\frac{\mathrm{d}B}{\mathrm{d}t}\frac{2\pi R}{v} \end{aligned} \tag{9}$$

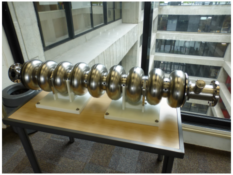
From equation 3 we have $\mathrm{d}E = v\,\mathrm{d}p$, therefore

$$\begin{aligned} \Delta E_{\mathrm{turn}} &= v\Delta p_{\mathrm{turn}} \\ &= 2\pi Re\rho\frac{\mathrm{d}B}{\mathrm{d}t} \end{aligned} \tag{10}$$

RF Acceleration



ISIS RF cavity, h=2



9 cell SC cavity for ILC

The energy gain is provided by the RF voltage,

$$\Delta E_{\text{turn}} = 2\pi Re\rho\frac{\mathrm{d}B}{\mathrm{d}t} = eV\sin\phi_s \tag{11}$$

$$\phi_s = \arcsin\left(2\pi R\rho\frac{\dot{B}}{V}\right) \tag{12}$$

where ϕ_s = **synchronous phase**. Each **synchronous particle** satisfies the rigidity equation (eqn 8). They have the nominal energy and follow the nominal trajectory.

Frequency Change

- **Acceleration increases the revolution frequency**, so the RF frequency has to follow

$$f = \frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R} = \frac{1}{2\pi R}\frac{p(t)c^2}{E(t)} = \frac{1}{2\pi R}\frac{ec^2\rho B(t)}{E(t)} \tag{13}$$

- Using the relativistic equation $E^2 = (m_0c^2)^2 + p^2c^2$ we find the RF frequency must **follow the magnetic field** with

$$\frac{f_{RF}(t)}{h} = \frac{c}{2\pi R}\sqrt{\frac{B(t)^2}{B(t)^2 + (m_0c^2/ec\rho)^2}} \tag{14}$$

- When B becomes large in comparison to $m_0c^2/ec\rho$ (corresponding to $v \to c$) the frequency tends to $c/2\pi R$

Revolution Frequency Increase

We've seen that the revolution and RF frequency change during acceleration depending on the particle type and the magnetic field ramp. This is more **important at lower energies and for heavier particles**.

PSB	50 MeV - 1.4 GeV	602 kHz - 1746 kHz	190%
PS	1.4 GeV - 25.4 GeV	437 kHz - 477 kHz	9%
SPS	25.4 GeV - 450 GeV	43.45 kHz - 43.478 kHz	0.06%
LHC	450 GeV - 7 TeV	11.245 kHz	2×10^{-6}

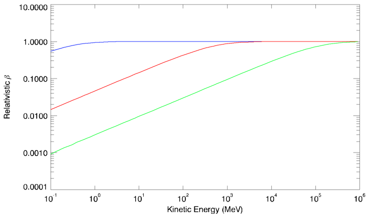
In lower energy circular accelerators the RF system needs more flexibility.

Particle Types and Acceleration

The specific accelerating technology depends upon the evolution of the particle velocity

- **Electrons** reach a constant velocity ($\sim c$) at low energy
- **Protons** and heavy ions require much more energy to reach a constant velocity
- RF resonators will be optimised for different velocities/frequencies
- Magnetic field follows the momentum increase

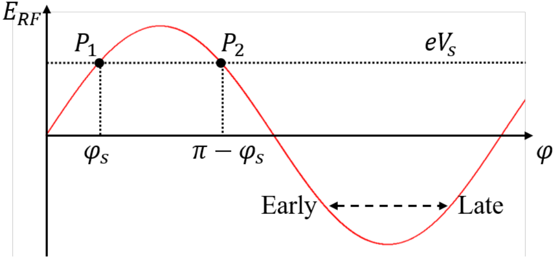
$$E = \gamma m_0 c^2 \, , \gamma = \frac{E}{E_0} = \frac{1}{\sqrt{1-\beta^2}}$$



Electron, 0.511 MeV; Proton, 938 MeV; Uranium-238, 222 GeV

Phase stability in a Linac - 1

- Consider a series of gaps, operating in the 2π mode
- 2π mode implies $\vec{E}$ is the same in all gaps at any given time
- $eV_s = eV \sin \phi_s$, the energy gain required for a particle to reach the next gap with the same RF phase: P_1, P_2



Phase stability in a Linac - 2

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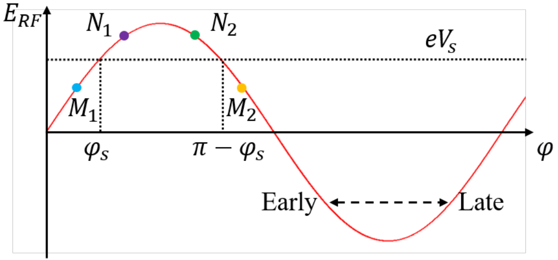
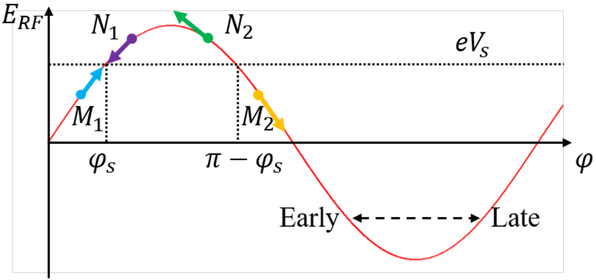


Phase stability in a Linac - 3



- With increasing energy comes an increase in velocity
- M_1 and N_1 move toward the synchronism $\Rightarrow$ **STABLE**
- M_2 and N_2 move away from synchronism $\Rightarrow$ **UNSTABLE**
- **N.B. Ultra-relativistic particles no longer gain velocity**

Off-Energy Particles

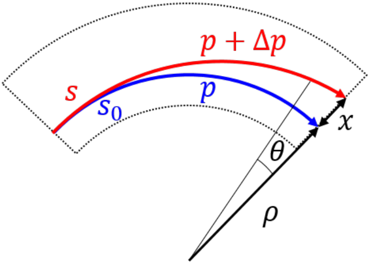
If a particle is slightly **off the design momentum** it will have a **different orbit**.

- Path length of an orbit displaced by x

$$\mathrm{d}s_0 = \rho \mathrm{d}\theta \qquad \mathrm{d}s = (\rho + x)\mathrm{d}\theta$$

- Relative difference in path length (D_x **= dispersion**)

$$\frac{\mathrm{d}l}{\mathrm{d}s_0} = \frac{\mathrm{d}s - \mathrm{d}s_0}{\mathrm{d}s_0} = \frac{x}{\rho} = \frac{D_x}{\rho}\frac{\mathrm{d}p}{p}$$



Momentum Compaction

- Integrating leads to the total path length change

$$\Delta C = \oint \mathrm{d}l = \oint \frac{x}{\rho(s_0)} \mathrm{d}s_0 = \oint \frac{D_x(s_0)}{\rho(s_0)} \frac{\mathrm{d}p}{p} \mathrm{d}s_0 \tag{15}$$

  note that since D_x is usually positive the total path length increases for higher energy particles.

- **Momentum compaction factor,** α_c is defined as

$$\alpha_c \equiv \frac{\mathrm{d}L/L}{\mathrm{d}p/p} = \frac{1}{L} \oint \frac{D_x(s_0)}{\rho(s_0)} \mathrm{d}s_0 \approx \frac{1}{C} \sum_i \langle D_x \rangle_i \, \theta_i \tag{16}$$

  where $\langle D_x \rangle_i$ and θ_i are the average dispersion and the bending angle of the i^{th} dipole.

Transition Energy, 1

- **Off-momentum particles** have different revolution frequencies to on-momentum particles due to different orbit lengths and velocities

$$f_r = \frac{\beta c}{2\pi R} \qquad \Rightarrow \qquad \frac{\mathrm{d}f_r}{f_r} = \frac{\mathrm{d}\beta}{\beta} - \frac{\mathrm{d}R}{R} = \frac{\mathrm{d}\beta}{\beta} - \alpha_c \frac{\mathrm{d}p}{p} \tag{17}$$

- Calculate $\mathrm{d}\beta/\beta$ as a function of $\mathrm{d}p/p$

$$p = \gamma m_0 \beta c \quad \Rightarrow \quad \frac{\mathrm{d}p}{p} = \frac{\mathrm{d}\beta}{\beta} + \frac{\mathrm{d}\gamma}{\gamma} = \left(1 - \beta^2\right)^{-1} \frac{\mathrm{d}\beta}{\beta} = \gamma^2 \frac{\mathrm{d}\beta}{\beta} \tag{18}$$

Transition Energy, 2

- Putting these two equations together (eqns 17 and 18) we get the relative change in revolution frequency

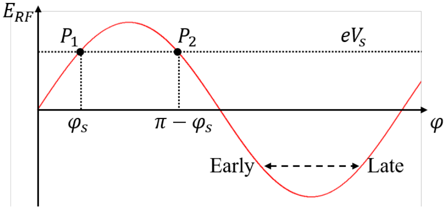
$$\frac{\mathrm{d}f_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha_c\right)\frac{\mathrm{d}p}{p} = \eta\frac{\mathrm{d}p}{p} \tag{19}$$

  where $\eta = \gamma^{-2} - \alpha_c = \gamma^{-2} - \gamma_t^{-2}$ is the **slip factor**
- Annoyingly in some references η is defined with a minus sign so **be careful!**
- **Transition energy** is when $\gamma = \gamma_t = \alpha_c^{-1/2}$ and $\eta = 0$. At this energy the revolution frequency is independent of momentum deviation.
- **Below transition** a higher momentum particle has a higher f_r than the synchronous particle, **above transition** the converse is true.

Phase stability in a Synchrotron - 1

The definition of the slip factor, η (equation 19), an increase in momentum:

- **Below transition** ($\eta > 0 \Rightarrow \gamma < \gamma_t$) gives a **higher revolution frequency** (increase in velocity dominated)
- **Above transition** ($\eta < 0 \Rightarrow \gamma > \gamma_t$) gives a **lower revolution frequency** as $v \approx c$ and a longer path (momentum compaction dominated)



Phase stability in a Synchrotron - 2

The definition of the slip factor, η (equation 19), an increase in momentum:

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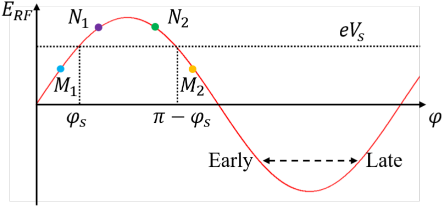
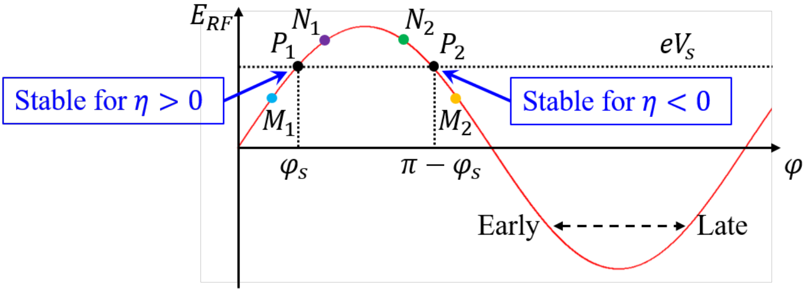


Phase stability in a Synchrotron - 3

$$\eta = \frac{1}{\gamma^2} - \alpha_c = \frac{1}{\gamma^2} - \frac{1}{\gamma_t^2}$$



Longitudinal Dynamics

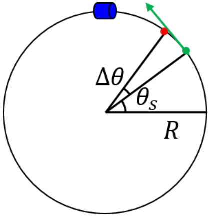
- The acceleration of charged particles in circular machines involves the coupled variables of energy and phase. The dynamics is often referred to as **synchrotron motion**.
- As there is a well defined **synchronous particle** (ϕ_s, E_s) it is best to consider particle coordinates with respect to that particle.
- Therefore we introduce a series of reduced variables:

$$
\begin{array}{ll}
\Delta E = E - E_s, & \text{particle energy} \\
\Delta p = p - p_s, & \text{particle momentum} \\
\Delta \phi = \phi - \phi_s, & \text{particle RF phase} \\
\Delta \theta = \theta - \theta_s, & \text{azimuthal angle} \\
\Delta f_r = f_r - f_{r,s}, & \text{revolution frequency}
\end{array}
$$

First Energy-Phase Equation

The RF phase coordinate is related to the azimuth by $\Delta\phi = \phi - \phi_s = -h\Delta\theta$, or

$$\Delta\omega = \frac{\mathrm{d}\Delta\theta}{\mathrm{d}t} = -\frac{1}{h}\frac{\mathrm{d}\Delta\phi}{\mathrm{d}t} = -\frac{1}{h}\frac{\mathrm{d}\phi}{\mathrm{d}t} \qquad (20)$$



From the definition of the slip factor (equation 19) and the relation between energy and momentum (equation 3) we get the **first energy phase equation**:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = h\omega_r\eta\frac{\mathrm{d}p}{p} = \frac{h\omega_r^2\eta}{\beta^2 E}\left(\frac{\Delta E}{\omega_r}\right) \qquad (21)$$

Second Energy-Phase Equation - 1

- The energy gain per turn has already been defined as $\Delta E_{\text{turn}} = eV \sin \phi_s$ (equation 11).
- So the rate of energy gain is $\dot{E} = f_r eV \sin \phi$
- The rate of relative energy change with respect to the synchronous particle is

$$\Delta \left(\frac{\dot{E}}{\omega_r} \right) = \frac{eV}{2\pi} (\sin \phi - \sin \phi_s) \tag{22}$$

- Expanding the L.H.S. to first order

$$\Delta(\dot{E}T_r) \cong \dot{E}\Delta T_r + T_{r,s}\Delta\dot{E} = \Delta E\dot{T}_r + T_{r,s}\Delta\dot{E} = \frac{\mathrm{d}(T_{r,s}\Delta E)}{\mathrm{d}t} \tag{23}$$

Second Energy-Phase Equation - 2

- This leads to the **second energy-phase equation**

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta E}{\omega_r}\right) = \frac{eV}{2\pi}\left(\sin\phi - \sin\phi_s\right) \tag{24}$$

- Combining these two equations leads to the **longitudinal equation of motion**

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\beta^2 E}{h\eta\omega_r^2}\frac{\mathrm{d}\phi}{\mathrm{d}t}\right) + \frac{eV}{2\pi}\left(\sin\phi - \sin\phi_s\right) = 0 \tag{25}$$

- This second order differential equation is non-linear. Also, the parameters within the bracket (in general) vary slowly in time

Longitudinal Hamiltonian

- The two energy-phase equations can also be derived from a **Hamiltonian, $\mathcal{H}$** (the total energy in the system) in canonical variables (ϕ, W) $(W = \Delta E/\omega_r)$

$$\mathcal{H}(\phi, W) = \frac{h\omega_0^2\eta}{2\beta^2 E_s}W^2 + \frac{q}{2\pi}U(\phi) \tag{26}$$

where $U(\phi) = \int_{\phi s}^{\phi} [V(\phi') - V(\phi_s)]\, d\phi'$ is the potential energy

- The two energy-phase equations are then derived from the Hamiltonian by

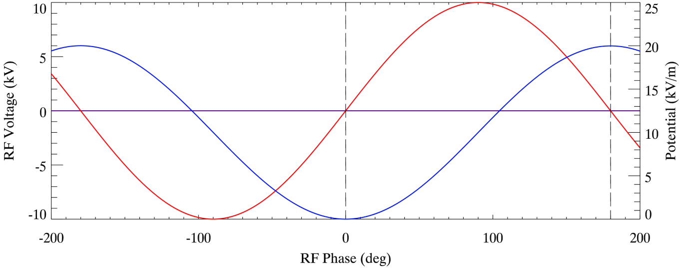
$$\frac{dW}{dt} = -\frac{\partial\mathcal{H}(\phi, W)}{\partial\phi} \qquad \frac{d\phi}{dt} = \frac{\partial\mathcal{H}(\phi, W)}{\partial W} \tag{27}$$

Single Harmonic RF - 1

Let's take the simple example we've been working with, single harmonic RF with $V(\phi) = V_1 \sin\phi$

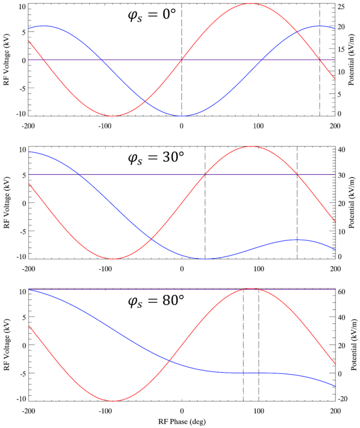
- The potential is:

$$U(\phi) = V_1 \left[\cos\phi_s - \cos\phi - (\phi - \phi_s)\sin\phi_s\right] \tag{28}$$



Single Harmonic RF - 2

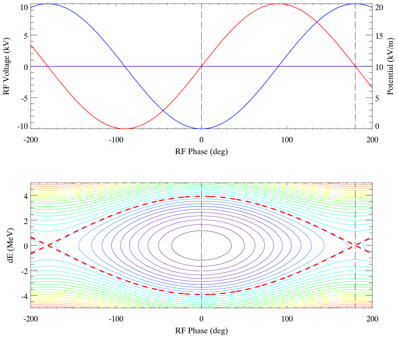
- What we have is a **potential well** created by the RF cavity voltage
- As the synchronous phase changes, or the amount of acceleration required to maintain synchronism changes, the shape of the well changes
- What does the Hamiltonian look like?



Single Harmonic RF - 3

$$\mathcal{H}(\phi, W) = \frac{h\omega_0^2\eta}{2\beta^2 E_s} W^2 + \frac{qV_1}{2\pi}\left[\cos\phi_s - \cos\phi - (\phi - \phi_s)\sin\phi_s\right] \tag{29}$$

- How does this help?
- Contours of constant $\mathcal{H}$ are **particle trajectories**, $\mathcal{H}$ is conserved
- Let's consider some particles near to ϕ_s ...



Small Amplitude Oscillations - 1

Rearranging the longitudinal EOM (eqn 25) assuming constant β, E, ω_r and η:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}(\sin\phi - \sin\phi_s) = 0 \quad \text{with} \quad \Omega_s^2 = \frac{h\eta\omega_r^2 eV\cos\phi_s}{2\pi\beta^2 E_s} \tag{30}$$

- Consider **small deviations** in phase from reference

$$\begin{aligned} \sin\phi - \sin\phi_s &= \sin(\phi_s + \Delta\phi) - \sin\phi_s \\ &\cong \Delta\phi\cos\phi_s \end{aligned} \tag{31}$$

- Thereby reducing the motion to a **harmonic oscillation**

$$\ddot{\phi} + \Omega_s^2\Delta\phi = 0 \tag{32}$$

where Ω_s is the **synchrotron angular frequency**

Small Amplitude Oscillations - 2

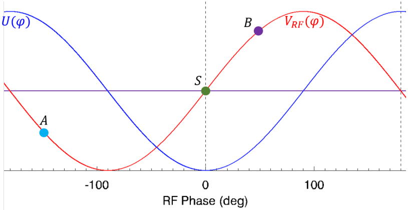
- The **synchrotron tune** $Q_s = \Omega_s/\omega_r$ is the number of synchrotron oscillations per revolution
- Typical values are $\ll 1$, 10^{-3} for proton synchrotrons and 10^{-1} for electron storage rings
- It also reveals a **stability condition** for ϕ_s as

$$\Omega_s^2 > 0 \qquad \Rightarrow \qquad \eta \cos\phi_s > 0 \tag{33}$$

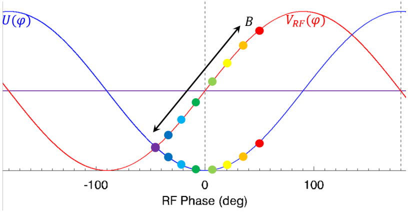
$$\begin{array}{lll} \gamma < \gamma_t & \eta > 0 & 0 < \phi_s < \pi/2 \\ \gamma > \gamma_t & \eta < 0 & \pi/2 < \phi_s < \pi \end{array}$$

Synchrotron Oscillations - 1

- Consider the simple case of no acceleration ($\phi_s = 0$), below transition ($\gamma < \gamma_t$)
- **Particle** S is synchronous
- **Particle** A is decelerated, f_r decreases so it arrives later (i.e. moves toward S)
- **Particle** B is accelerated, f_r increases so it arrives earlier (moves toward S)



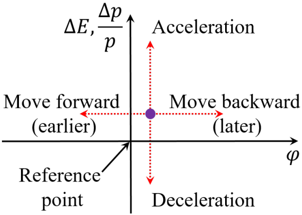
Synchrotron Oscillations - 2

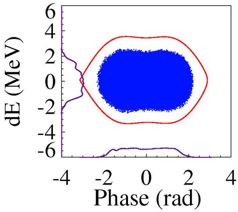


- The particle oscillates around the synchronous phase, so-called **synchrotron oscillations**
- The amplitude depends on the initial phase and energy
- **Synchrotron frequency** is much slower than the transverse (usually multiple revolutions per oscillation)
- The restoring force from the RF electric field is much smaller than the quadrupolar magnetic field

Longitudinal Phase Space

The energy-phase oscillations can also be observed in the longitudinal phase space we saw with the Hamiltonian

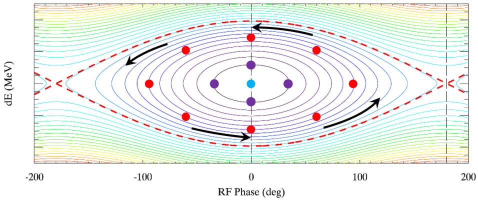




The particle trajectory in phase space describes the longitudinal motion.

Longitudinal emittance is the phase space area including all the particles

Longitudinal Phase Space Oscillations



- Particles follow Hamiltonian contours oscillating around the synchronous point (ϕ_s, E_s)
- Energy is exchanged for RF phase like exchanges between kinetic and potential energy
- This is called **synchrotron motion**

Examples

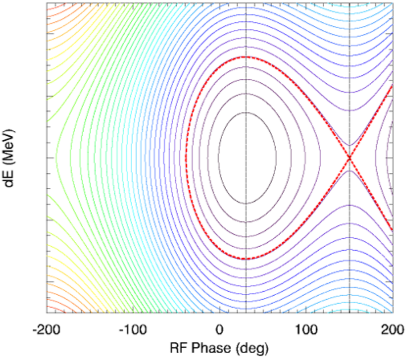
These are all for below transition

$\phi_s = 0°$, phase distribution

$\phi_s = 0°$, energy distribution

Large Amplitude Oscillations

- When $\Delta\phi$ is large the EOM is non-linear
- Move from elliptical orbits to hyperbolic close to UFP
- Can use Hamiltonian to calculate the separatrix



Separatrix

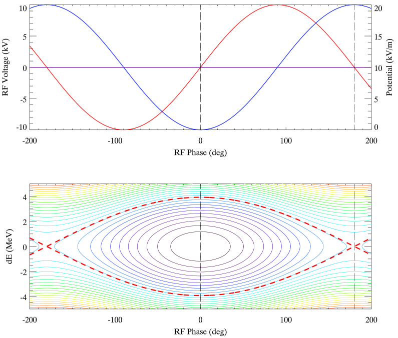
- First find the co-ordinate of UFP from $\frac{\mathrm{d}U}{\mathrm{d}\phi}$, $\phi = \pi - \phi_s$
- Calculate Hamiltonian of the separatrix from equation (29)

$$\begin{aligned} \mathcal{H}_{\text{sep}} &= \frac{qV_1}{2\pi}\left[\cos\phi_s - \cos(\pi - \phi_s) - (\pi - 2\phi_s)\sin\phi_s\right] \\ &= \frac{qV_1}{2\pi}\left[2\cos\phi_s - (\pi - 2\phi_s)\sin\phi_s\right] \end{aligned} \tag{34}$$

- Put back into the Hamiltonian to get separatrix equation

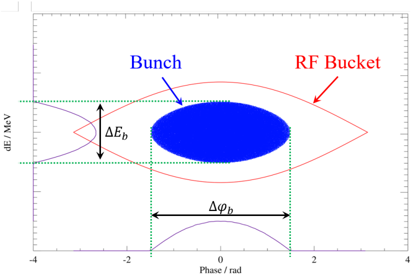
$$\Delta E_{\text{sep}} = \sqrt{\frac{qV\beta^2 E_s}{\pi h\eta}\left[\cos\phi_s + \cos\phi + (\phi - \pi + \phi_s)\sin\phi_s\right]} \tag{35}$$

RF Buckets



- Restoring force is non-linear $\Rightarrow$ speed depends on $(\phi, \Delta E)$
- Two **fixed points**, unstable and stable
- Two clear regions (**libration and rotation**) separated by the **separatrix** passing through the **UFP**, at the maximum of $U(\phi)$
- Oscillatory motion around the **SFP**, at the minimum of $U(\phi)$
- Rotary motion beyond the separatrix, the **RF bucket**

Terminology



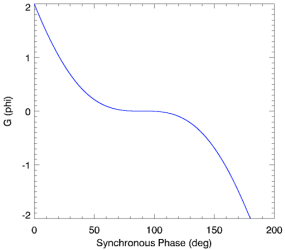
- Bunches of particles fill only a portion of the bucket area
- RF bucket area = **longitudinal acceptance** in units of **eVs**
- Bunch area = **longitudinal emittance** $= 4\pi\sigma_{\Delta E}\sigma_{\Delta t}$
- **N.B. References can use different definitions for emittances!**

Energy Acceptance

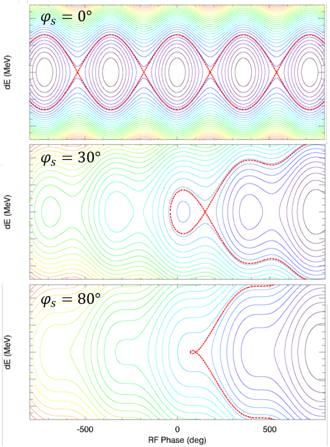
- It is clear the separatrix has a maximum at $\phi = \phi_s$
- RF bucket height also referred to as **energy acceptance**

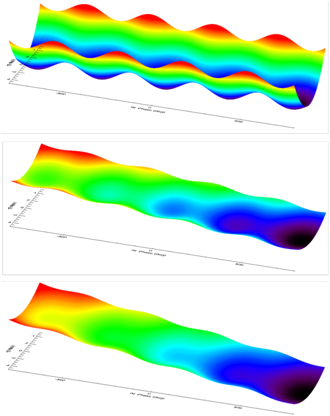
$$\left(\frac{\Delta E}{E_s}\right)_{\mathrm{max}} = \sqrt{\frac{qV\beta^2}{\pi h \eta E_s}\left[2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s\right]} \qquad (36)$$

- It depends strongly on ϕ_s
- It becomes smaller when ϕ_s is changing during acceleration
- A **higher voltage** $\Rightarrow$ **larger acceptance**
- For **higher** h the same voltage produces a **smaller acceptance**

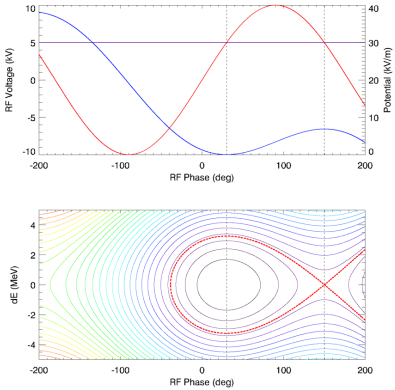


Accelerating Bucket - 1





Accelerating Bucket - 2



Examples (all below transition)

$\phi_s = 30°$, phase distribution

$\phi_s = 30°$, energy distribution

- Motion still divided into two clear regions
- Stable area (RF bucket) reduces in size

Summary

- How to accelerate charged particles ...
- What makes a synchrotron a synchrotron ...
- Momentum compaction, slip factors, dispersion, ...
- Transition, phase stability, synchronous phase, ...
- Deriving the equation of motion, ...
- $\mathcal{H}$amiltonians, potentials, fixed points, ...
- RF buckets, separatrices, emittance, synchrotron tune, ...
- Longitudinal acceptance, energy acceptance, ...
- What next?

References

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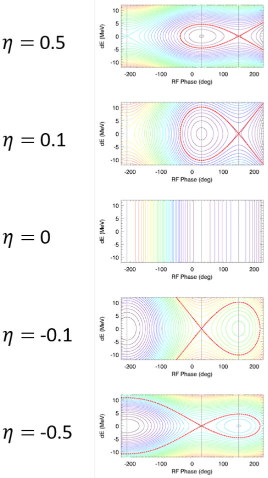
Spare Slides

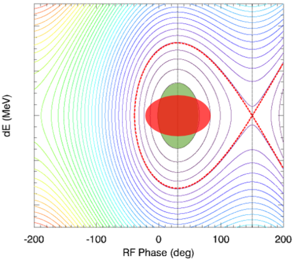
Crossing Transition

- Transition ⇒ velocity change and path length change compensate
- f_r is independent from the momentum offset
- Crossing transition makes the previous ϕ_s unstable
- RF needs to rapidly change its phase called a **phase jump**
- For example, PS (1.4 - 25.4 GeV) crosses transition at 6 GeV



Beam Matching - 1

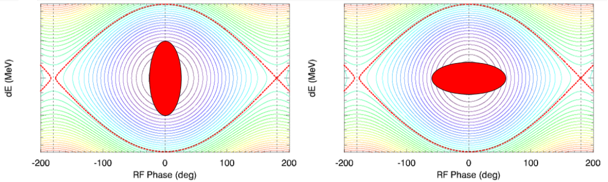
- How well can we confine and control particles?
- Want to put them within a given $\mathcal{H}$ contour and keep them there
- Make the density a function of $\mathcal{H}$ and it will conform to the contours in phase space
- It forms a **stationary distribution** (time independent) assuming $\mathcal{H}$ is time independent or adiabatically varying



- Matched
- Unmatched

Effect of a Mismatch

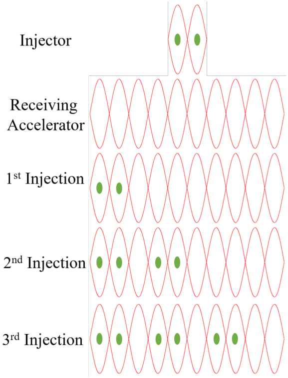
- Consider a short bunch with a large energy spread
- After a quarter synchrotron oscillation: a long bunch with a small energy spread



- For larger amplitudes the synchrotron motion is slower which leads to filamentation and emittance growth and possible beam loss
- Matched example, Mismatched example, Phase error

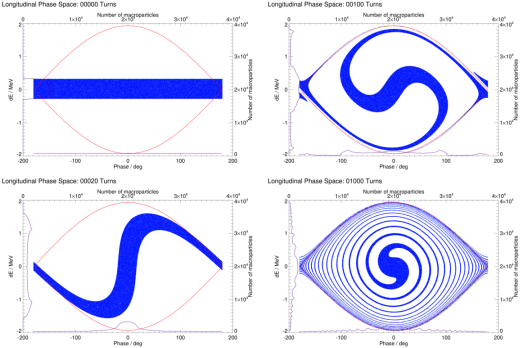
Injection

- Where are you injecting from?
- **Bunch to bucket transfer** ⇒ match bunch from bucket of previous accelerator to next
- Particles always subject to longitudinal focusing
- Time structure of beam preserved
- No need for bunch capture, adiabatic or otherwise



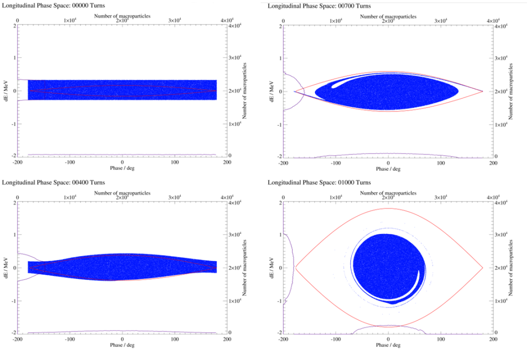
RF Capture - 1

- A debunched beam can be captured by the synchrotron RF
- With constant RF volts during injection the beam filaments to fill the bucket



RF Capture - 2

- Increase the RF volts **adiabatically** (i.e. slowly with respect to the synchrotron motion)
- Capture a large portion in a relatively small emittance



Space Charge

- Another important factor for intense low/medium energy hadron beams is the effect of **space charge**
- The self field of the beam then becomes significant compared to the RF focusing field
- Depends on the beam distribution and, in general, is a time varying addition to the potential
- Makes analytical solutions, and understanding intense beams difficult!
- Look for stationary distributions **WITH space charge**
- Useful stationary distribution found by Hofmann and Pedersen