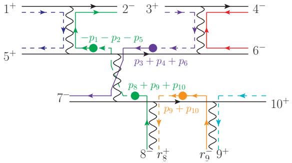
Calculating Feynman diagrams using Chirality Flow

Andrew Lifson

In collaboration with Joakim Alnefjord, Christian Reuschle & Malin Sjödahl

Partikeldagarna 2020

Based on *Eur.Phys.J.C* 80 (2020) 11, 1006, hep-ph:2003.05877 (Massless QED & QCD) & hep-ph:2011.10075 (Full tree-level Standard Model)



Lund University

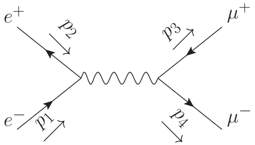
November 23rd 2020

Outline

1. Calculating Feynman diagrams in QFT
2. Calculating Feynman diagrams using spinor helicity
3. Calculating Feynman diagrams using chirality flow

Traditional Matrix Element Calculations

- Keep all particles unpolarised
- Obtain amplitude as matrix
- Square the matrix amplitude
  - Spin states are orthogonal
- Move components around
- Use spin sums
- Take trace of fermionic structure
- Requires identities of γ^μ
- Simplify



$$\sim [\bar{v}_r(p_2)\gamma^\mu u_s(p_1)][\bar{u}_t(p_4)\gamma_\mu v_w(p_3)]$$

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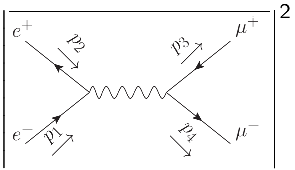

$$\sim \sum_{r,s,t,w} [\bar{v}_r(p_2)\gamma^\mu u_s(p_1)][\bar{u}_t(p_4)\gamma_\mu v_w(p_3)]$$
$$\times [\bar{u}_s(p_1)\gamma^\nu v_r(p_2)][\bar{v}_w(p_3)\gamma_\nu u_t(p_4)]$$

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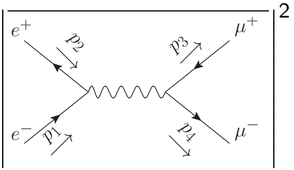

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$$\sim \sum_{r,s,t,w} [\gamma^\nu v_r(p_2)\bar{v}_r(p_2)\gamma^\mu u_s(p_1)\bar{u}_s(p_1)]$$

$$\times [\gamma_\nu u_t(p_4)\bar{u}_t(p_4)\gamma_\mu v_w(p_3)\bar{v}_w(p_3)]$$

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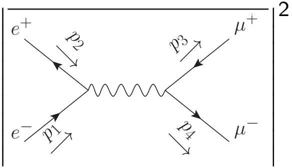


$$\sim \mathrm{Tr}\left[\gamma^\nu(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\right]$$
$$\times\, \mathrm{Tr}\left[\gamma_\nu(\not{p}_4 + m_\mu)\gamma_\mu(\not{p}_3 + m_\mu)\right]$$

$$\mathrm{Tr}\left[\gamma^{\mu_1}\gamma^{\mu_2}\right] = 4g^{\mu_1\mu_2}$$
$$\mathrm{Tr}\left[\gamma^{\mu_1}\ldots\gamma^{\mu_4}\right] = 4(g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_3\mu_2})$$
$$\mathrm{Tr}\left[\gamma^{\mu_1}\ldots\gamma^{\mu_{2n+1}}\right] = 0$$

Matrix Element Calculations: the Spinor-Helicity Method

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles for now: chirality $\sim$ helicity

Spinors:

$$u^+(p) = v^-(p) = \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix} \qquad u^-(p) = v^+(p) = \begin{pmatrix} |p] \\ 0 \end{pmatrix}$$

$$\bar{u}^+(p) = \bar{v}^-(p) = \begin{pmatrix} [p| & 0 \end{pmatrix} \qquad \bar{u}^-(p) = \bar{v}^+(p) = \begin{pmatrix} 0 & \langle p| \end{pmatrix}$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^\mu \\ \sqrt{2}\bar{\tau}^\mu & 0 \end{pmatrix} \qquad \sqrt{2}\tau^\mu = (1, \vec{\sigma}), \quad \sqrt{2}\bar{\tau}^\mu = (1, -\vec{\sigma}),$$

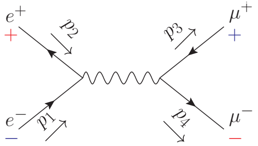
Polarisation vectors: in backup slides

- Amplitude written in terms of Lorentz-invariant spinor inner products $\langle ij \rangle \equiv \langle i||j\rangle$ and $[ij] \equiv [i||j]$
  - These are well known complex numbers, $\langle ij \rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$
- Remove $\tau/\bar{\tau}$ matrices with $\langle i|\bar{\tau}^\mu|j][k|\tau_\mu|l\rangle = \langle il\rangle[kj]$

Matrix Element Calculations: Our Simple Example

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles for now: chirality $\sim$ helicity

- Explicit helicities for external particles
- Now diagram is a complex number
  - Easy to square
- Square first, then sum over helicities
  - Some helicity configurations vanish
  - CP-invariance relates helicity configurations



$$\sim \langle p_2|\bar{\tau}^\mu|p_1]\langle p_4|\bar{\tau}_\mu|p_3]$$
$$= [p_1|\tau^\mu|p_2\rangle\langle p_4|\bar{\tau}_\mu|p_3]$$
$$= \langle p_4 p_2\rangle[p_1 p_3]$$

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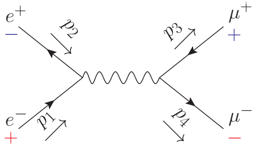

$$\sim [p_2|\tau^\mu|p_1\rangle\langle p_4|\bar{\tau}_\mu|p_3]$$
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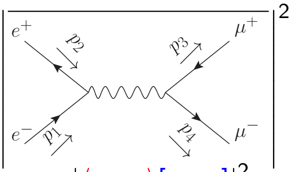


$$\sim |\langle p_4 p_2\rangle [p_1 p_3]|^2 + |\langle p_4 p_1\rangle [p_2 p_3]|^2$$

Conclusion: Much better than before

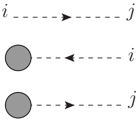
- 2×2 $\tau/\bar{\tau}$ far better than 4×4 γ
- Not intuitive which inner products we obtain
- Still requires $\tau/\bar{\tau}$ algebra and removal

Matrix Element Calculations: the Chirality-flow Method

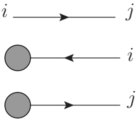
Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles for now (massive particles easily incorporated)

- $su(n)$ has a known pictorial flow method
  - Can we use it to calculate the Lorentz algebra?
- Define spinors and their inner products:

$[ij] = -[ji] =$, $[i| =$, $|j] =$

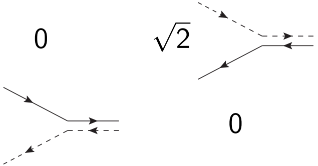


$\langle ij\rangle = -\langle ji\rangle =$, $\langle i| =$, $|j\rangle =$



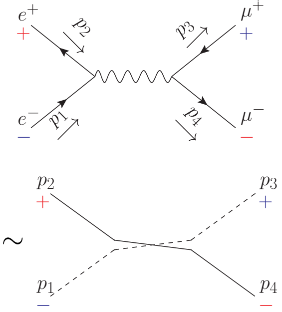
- Define Dirac (Pauli) Matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^\mu \\ \sqrt{2}\bar{\tau}^\mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{2}\,\cdot \\ \sqrt{2}\,\cdot & 0 \end{pmatrix}$$



Our Simple Example with the Chirality-Flow Method

- Explicit helicities for external particles
- Draw flow lines in only way possible
- Choose arrow direction
- Immediately read off inner products (complex numbers)



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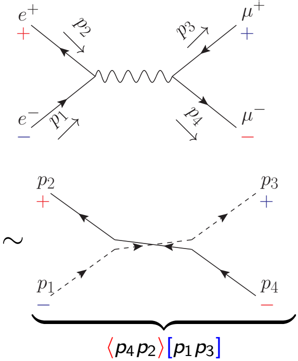

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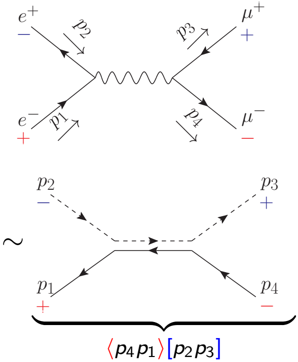

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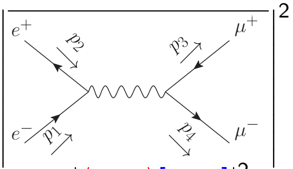
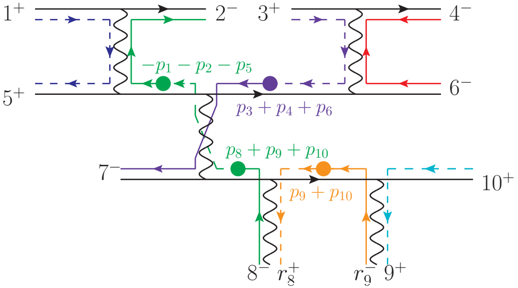


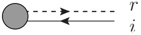
$$\sim |\langle p_4 p_2 \rangle [p_1 p_3]|^2 + |\langle p_4 p_1 \rangle [p_2 p_3]|^2$$

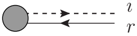
Conclusion: Even better than before

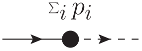
- Intuitive which inner products we obtain
- No matrix structure!
- Removes $\tau/\bar{\tau}/\gamma$ algebra

A More Complicated Example with Chirality Flow



$$(\varepsilon^\mu_-)^* = \frac{1}{[ir]}$$


$$(\varepsilon^\mu_+)^* = \frac{1}{\langle ri\rangle}$$




$$= \sum_i |i\rangle[i|$$

$$= \sum_i |i]\langle i|$$

$$\langle ij\rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$$

$$= \underbrace{(\sqrt{2}ei)^8}_{\text{vertices}} \underbrace{\frac{(-i)^3}{s_{1\,2}\; s_{3\,4}\; s_{8\,9\,10}}}_{\text{photon propagators}} \underbrace{\frac{(i)^4}{s_{1\,2\,5}\; s_{3\,4\,6}\; s_{8\,9\,10}\; s_{9\,10}}}_{\text{fermion propagators}} \underbrace{\frac{1}{[8r_8]\langle r_9 9\rangle}}_{\text{polarization vectors}} \quad [15]\langle 64\rangle[10\;9]$$

$$\times \Big(\langle r_9 9\rangle[9 r_8] + \langle r_9 10\rangle[10 r_8]\Big)\Big(\underbrace{[33]}_{0}\langle 37\rangle + [34]\langle 47\rangle + [36]\langle 67\rangle\Big)$$

$$\times \Big(-\langle 89\rangle[91]\langle 12\rangle - \langle 89\rangle[95]\langle 52\rangle - \langle 8\,10\rangle[10\;1]\langle 12\rangle - \langle 8\,10\rangle[10\;5]\langle 52\rangle\Big)$$

Summary of Chirality-Flow and Conclusions

- Spinor helicity simplifies traditional ME calculations
  - Amplitude is a complex number
  - $4 \times 4\ \gamma \to 2 \times 2\ \tau/\bar{\tau}$
- Chirality-flow simplifies further by removing all matrices
  - Can immediately read off inner products $\sim \sqrt{2p_i \cdot p_j}$
- Can be used for any Standard Model process at tree-level , i.e.
  - Massive spinors and polarisation vectors
  - Non-Abelian vertices
  - R_ξ or axial gauges
  - Scalars (which are trivial)
- Also for any tree-level process whose Feynman rules involve p^μ, (Minkowski) $g_{\mu\nu}, \varepsilon^\mu, \gamma^\mu$, and Dirac or Weyl spinors
- See hep-ph:2003.05877, hep-ph:2011.10075 & online seminar in January for details

Massive Chirality Flow: Spinors

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Massive momentum $p^\mu = p^{\flat,\mu} + \alpha q^\mu$, $(p^\flat)^2 = q^2 = 0$
Spin measured relative to axis $s^\mu = (p^{\flat,\mu} - \alpha q^\mu)/m$

$$u^+(p) = \begin{pmatrix} -e^{-i\varphi}\sqrt{\alpha}|q] \\ |p^\flat\rangle \end{pmatrix} \qquad u^-(p) = \begin{pmatrix} |p^\flat] \\ e^{i\varphi}\sqrt{\alpha}|q\rangle \end{pmatrix}$$

$$v^-(p) = \begin{pmatrix} e^{-i\varphi}\sqrt{\alpha}|q] \\ |p^\flat\rangle \end{pmatrix} \qquad v^+(p) = \begin{pmatrix} |p^\flat] \\ -e^{i\varphi}\sqrt{\alpha}|q\rangle \end{pmatrix}$$

$$e^{i\varphi}\sqrt{\alpha} = \frac{m}{\langle p^\flat q\rangle}, \qquad e^{-i\varphi}\sqrt{\alpha} = \frac{m}{[qp^\flat]} \tag{1}$$

- Recall in chirality flow ($\langle ij\rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$):

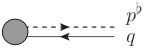
$\langle ij\rangle = -\langle ji\rangle =$  , $[ij] = -[ji] =$  ,

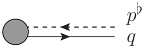
$|j\rangle =$  , $|j] =$  ,

$\langle i| =$  , $[i| =$  ,

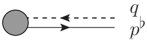
Massive Chirality Flow: Incoming Polarisation Vectors

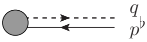
Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Massive momentum $p^\mu = p^{\flat,\mu} + \alpha q^\mu$, $(p^\flat)^2 = q^2 = 0$
Spin measured relative to axis $s^\mu = (p^{\flat,\mu} - \alpha q^\mu)/m$

$\varepsilon^\mu_-(p) = \frac{1}{\langle q p^\flat \rangle}$ 

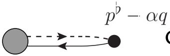
or $\frac{1}{\langle q p^\flat \rangle}$ 

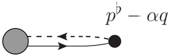
,

$\varepsilon^\mu_+(p) = \frac{1}{[p^\flat q]}$ 

or $\frac{1}{[p^\flat q]}$ 

,

$\varepsilon^\mu_0(p) = \frac{1}{m\sqrt{2}}$ 

or $\frac{1}{m\sqrt{2}}$ 

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, $[ij] = -[ji] =$ 

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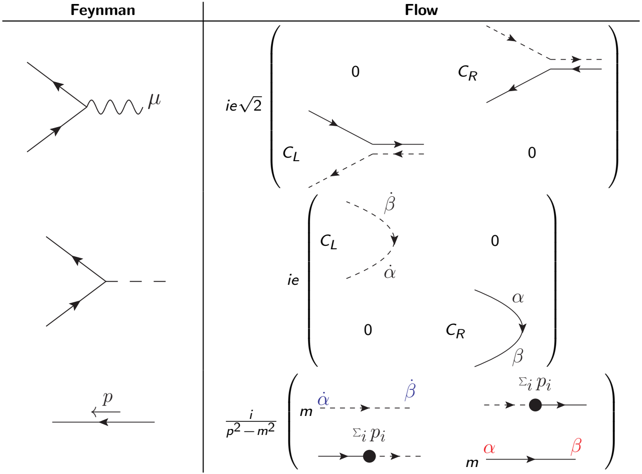
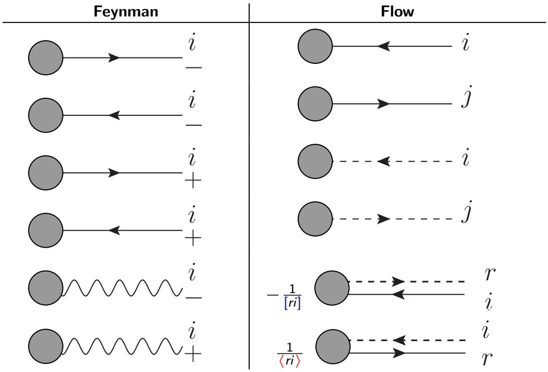
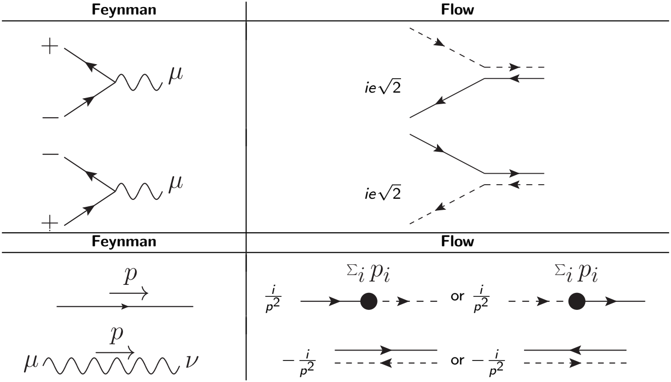
Massive EW Flow Rules: Fermion Vertices & Propagator



The Massless QED Flow Rules: External Particles

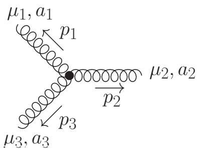
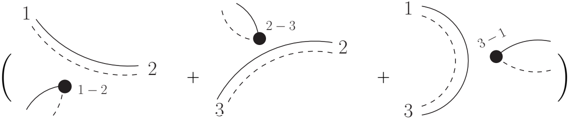
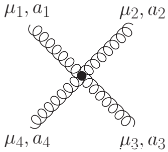
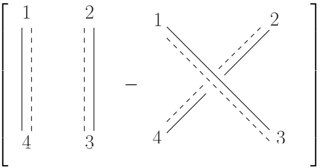


The massless QED Flow Rules: Vertices and Propagators



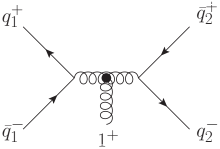
The Non-Abelian Flow Vertices

Here using QCD couplings (for EW flow rules sub EW couplings)

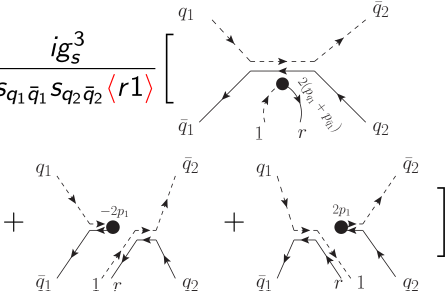
Feynman	Flow
	$-\frac{g_s f^{abc}}{2}$ 
	$ig_s^2 \sum_{Z(2,3,4)} f^{a_1 a_2 b} f^{b a_4 a_3}$ 

Massless QCD Example: $q_1\bar{q}_1 \to q_2\bar{q}_2 g$

- Triple-gluon vertex provides new structures

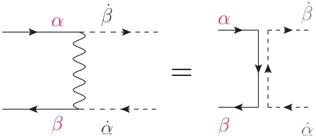


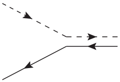
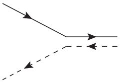
$$= \frac{i g_s^3}{2 s_{q_1\bar{q}_1} s_{q_2\bar{q}_2} \langle r1 \rangle} \Bigg[\cdots \Bigg]$$



Motivating QED Chirality-Flow: Vertices and Propagators

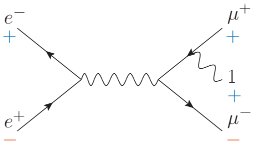
- vertices $\frac{\gamma^\mu}{\sqrt{2}} \to \tau^\mu, \bar{\tau}^\mu$ contracted with vector propagator $g_{\mu\nu}$
- Fierz identity with indices: $\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tau_{\mu}^{\dot{\alpha}\beta} = \delta_{\alpha}^{\ \beta}\delta^{\dot{\alpha}}_{\ \dot{\beta}}$
- Fierz identity with flow:



- $\Rightarrow \tau^{\mu,\dot{\alpha}\beta} =$  , $\bar{\tau}^{\mu}_{\alpha\dot{\beta}} =$ 
- $\Rightarrow g_{\mu\nu} =$  , or 
- Fierz identity already utilised in flow rule

Massless QED: Simple Example Spinor Helicity

- Regular spinor-helicity $\equiv$ easy



$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}} \left(\tilde{\lambda}_{e^-,\dot{\alpha}}\tau_\mu^{\dot{\alpha}\beta}\lambda_{e^+,\beta}\right)\left(\lambda^{\alpha}_{\mu^-}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}(\not{p}_1+\not{p}_{\mu^+})^{\dot{\beta}\delta}\not{\epsilon}^+_{\delta\dot{\gamma}}(1,r)\tilde{\lambda}^{\dot{\gamma}}_{\mu^+}\right)$$

$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}\left(\tilde{\lambda}_{e^-,\dot{\alpha}}\tau_\mu^{\dot{\alpha}\beta}\lambda_{e^+,\beta}\right)\tilde{\lambda}_{1,\dot{\delta}}\tilde{\lambda}^{\dot{\delta}}_{\mu^+}$$

$$\times\left(\lambda^{\alpha}_{\mu^-}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tilde{\lambda}_1^{\dot{\beta}}\lambda_1^{\delta}\lambda_{r,\delta}+\lambda^{\alpha}_{\mu^-}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tilde{\lambda}^{\dot{\beta}}_{\mu^+}\lambda^{\delta}_{\mu^+}\lambda_{r,\delta}\right)$$

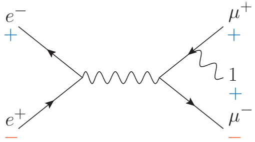
$$\sim \lambda^{\beta}_{\mu^-}\lambda_{e^+,\beta}\left(\tilde{\lambda}_{e^-,\dot{\alpha}}\tilde{\lambda}_1^{\dot{\alpha}}\lambda_1^{\delta}\lambda_{r,\delta}+\tilde{\lambda}_{e^-,\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}_{\mu^+}\lambda^{\delta}_{\mu^+}\lambda_{r,\delta}\right)\tilde{\lambda}_{1,\dot{\delta}}\tilde{\lambda}^{\dot{\delta}}_{\mu^+}$$

Correct Answer

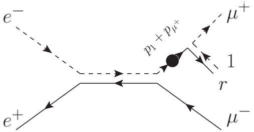
$$\frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}\left([e^-1]\langle 1r\rangle+[e^-\mu^+]\langle\mu^+r\rangle\right)[1\mu^+]\langle\mu^-e^+\rangle$$

Massless QED: Simple Example Chirality Flow

- Helicity flow $\equiv$ super easy and intuitive



$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}$$



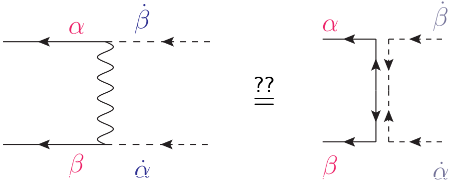
- Immediately read off inner products

Correct Answer

$$\frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}\Big([e^-1]\langle 1r\rangle + [e^-\mu^+]\langle \mu^+ r\rangle\Big)[1\mu^+]\langle \mu^- e^+\rangle$$

Fun with Arrows and the Fierz Identity

- Sometimes have to contract $\tau^\mu \tau_\mu$ or $\bar{\tau}^\mu \bar{\tau}_\mu$
- This would lead to arrows pointing towards each other, e.g.



- To fix, use charge conservation of a current:
  - $\lambda_i^{\alpha} \bar{\tau}^{\mu}_{\alpha\dot{\beta}} \tilde{\lambda}_j^{\dot{\beta}} = \tilde{\lambda}_{j,\dot{\alpha}} \tau^{\mu,\dot{\alpha}\beta} \lambda_{i,\beta}$
- Or in pictures:
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