# Scheme-invariant evolution of deep-inelastic structure functions in N-space

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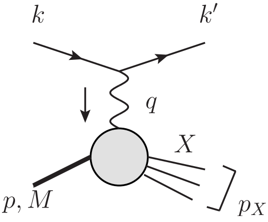
# Motivation

- Goal: to perform a measurement of $\alpha_s$ from scaling violation of the DIS structure functions, which are directly measured at the starting scale $Q_0^2$.
- The effects of heavy quarks are considered.
- Non-singlet evolution is studied to $N^3LO$ and singlet evolution to $N^2LO$.

# Deep inelastic scattering




- Kinematic invariants:

$$Q^2 = -q^2, \qquad x = \frac{Q^2}{2p.q}$$

- The cross section factorizes into leptonic and hadronic tensor:

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}Q^2\mathrm{d}x} \sim L_{\mu\nu} W^{\mu\nu}$$

- The hadronic tensor can be expressed through two structure functions:

$$W_{\mu\nu}(x, Q^2) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1(x, Q^2) + \left(p_\mu - \frac{p \cdot q}{q^2} q_\mu\right)\left(p_\nu - \frac{p \cdot q}{q^2} q_\nu\right) \frac{F_2(x, Q^2)}{p \cdot q}$$

- The structure functions $F_i(x, Q^2)$ are observables. They receive contributions from both light and heavy $(c, b)$ quarks.

- The structure functions factorize in Mellin space

$$M\left[F_{(2,L)}(x,Q^2)\right](N)=\int_0^1 dx\, x^{N-1}F_{(2,L)}(x,Q^2)=\sum_j \mathcal{C}^{(N)}_{(2,L),j}\left(\frac{Q^2}{\mu^2},\frac{m_c^2}{\mu^2},\frac{m_b^2}{\mu^2}\right)\cdot f_j^{(N)}(\mu^2)$$

  into perturbative Wilson coefficients $\mathcal{C}_j$ and nonperturbative parton distribution functions $f_j$.
- The whole mass dependence is encoded in the Wilson coefficients.
- The Wilson coefficients can be split

$$\mathcal{C}^{(N)}_{(2,L),j}\left(\frac{Q^2}{\mu^2},\frac{m_c^2}{\mu^2},\frac{m_b^2}{\mu^2}\right)=C^{(N)}_{(2,L),j}\left(\frac{Q^2}{\mu^2}\right)+h^{(N)}_{(2,L),j}\left(\frac{Q^2}{\mu^2},\frac{m_c^2}{\mu^2},\frac{m_b^2}{\mu^2}\right)$$

  into a light and a heavy flavor part.

$$\mathcal{C}^{(N)}_{(2,L),j}\left(\frac{Q^2}{\mu^2},\frac{m_c^2}{\mu^2},\frac{m_b^2}{\mu^2}\right) = C^{(N)}_{(2,L),j}\left(\frac{Q^2}{\mu^2}\right) + h^{(N)}_{(2,L),j}\left(\frac{Q^2}{\mu^2},\frac{m_c^2}{\mu^2},\frac{m_b^2}{\mu^2}\right)$$

- In the limit $Q^2 \gg m^2$, the heavy flavor part factorizes

$$h^{(N)}_{(2,L),j}\left(\frac{Q^2}{\mu^2},\frac{m_c^2}{\mu^2},\frac{m_b^2}{\mu^2}\right) = \sum_i C^{(N)}_{(2,L),j}\left(\frac{Q^2}{\mu^2}\right) \cdot A^{(N)}_{ij}\left(\frac{m_c^2}{\mu^2},\frac{m_b^2}{\mu^2}\right)$$

  [M. Buza et al., Nucl. Phys. B472 (1996) 611–658; M. Buza et al., Nucl. Phys. B485 (1997) 420–456.]

  into the light flavor Wilson coefficients and the massive operator matrix elements (OMEs) of local twist-2 operators $O_i$ between partonic states $j$,

$$A^{(N)}_{ij}\left(\frac{m^2}{\mu^2}\right) = \langle j|\, O_i \,|j\rangle \ .$$

- By this method, contributions for $Q^2 \gg m^2$ in $h^{(N)}_{(2,L),j}$ have been computed in the literature.

  [J. Blümlein et al., Nucl. Phys. B755 (2006) 272–285; I. Bierenbaum et al., Nucl. Phys. B780 (2007) 40–75; I. Bierenbaum et al., Nucl. Phys. B803 (2008) 1–41; I. Bierenbaum, J. Blümlein and S. Klein, Nucl. Phys. B820 (2009) 417–482; S. W. G. Klein, PhD thesis (2009); A. Behring et al., Eur. Phys. J. C74 (2014) no.9, 3033; J. Ablinger et al., Nucl. Phys. B886 (2014) 733–823; J. Ablinger et al., Nucl. Phys. B890 (2014) 48–151; K. Schönwald, PhD thesis (2018).]

# The project

- Goal: measurement of $a_s$ from scaling violation starting from an input $F_2(x, Q_0^2)$.
- We include the known logarithmic terms in $h_{(2,L),j}^{(N)}$.
- Scaling proceeds through the Altarelli-Parisi equations
  [D. Gross and F. Wilczek, Phys. Rev. D 8 (1973) 3633–3652,
  G. Altarelli and G. Parisi, Nucl. Phys. B 126 (1977) 298–318.]

$$C^{(N)}(Q^2) = c_1^{(N)} a_s(Q^2) + c_2^{(N)} a_s^2(Q^2) + \cdots$$

$$\frac{\mathrm{d}}{\mathrm{d}\log Q^2} f^{NS\,(N)}(Q^2) = P^{NS,(N)}(Q^2)\, f^{NS\,(N)}(Q^2)$$

$$\frac{d}{d\log Q^2}\left(f^{S\,(N)}(Q^2)\right)_i = \left(P^{S\,(N)}(Q^2)\right)_{ij}\left(f^{S\,(N)}(Q^2\right)_j$$

$$\frac{d}{d\log Q^2} a_s(Q^2) = -\sum_{k=0}^{\infty}\beta_k a_s^{k+2}(Q^2), \qquad a_s = \alpha_s/(4\pi)$$

# The approach

- We want to solve analytically
  [W. Furmanski, R. Petronzio, Z. Phys. C11 (1982) 293–314,
  E. B. Zijlstra, W. L. Van Neerven, Nucl. Phys. B383 (1992) 525–574,
  J. Blümlein, V. Ravindran, W. L. Van Neerven, Nucl. Phys. B586 (2000) 349–381,

  J. Blümlein and A. Guffanti, Nucl. Phys. Proc. Suppl. 152 (2006) 87.]

$$F_2^{NS}(x, Q^2) = E^{NS}(Q^2, Q_0^2)\, F_2^{NS}(Q_0^2)$$

  and

$$\begin{pmatrix} F_2^S(x,t) \\ \partial_t F_2^S(x,t) \end{pmatrix} = \mathbf{E}^S(t,0) \begin{pmatrix} F_2^S(x,0) \\ \partial_t F_2^S(x,0) \end{pmatrix}$$

$$t = -\frac{2}{\beta_0} \log \frac{a_s(Q^2)}{a_s(Q_0^2)}$$

- $E^{NS}(Q^2, Q_0^2)$ and $\mathbf{E}^S(t,0)$ are physical (*scheme-invariant*) evolution operators.

# The non-singlet sector

- We aim to include the contributions due to heavy quarks.
- In the limit $Q^2 \gg m^2$ the heavy flavour terms have the following factorized form:

$$h^{(N)}_{(2,L),j}\left(\frac{Q^2}{\mu^2}, \frac{m_c^2}{\mu^2}, \frac{m_b^2}{\mu^2}\right) = \sum_i C^{(N)}_{(2,L),j}\left(\frac{Q^2}{\mu^2}\right) \cdot A^{(N)}_{ij}\left(\frac{m_c^2}{\mu^2}, \frac{m_b^2}{\mu^2}\right).$$

- One obtains for the heavy quark Wilson coefficients at $Q^2 = \mu^2$

$$h^{NS}_{2,q} = -\frac{\beta_{0,Q}}{4} \ln^2 \frac{Q^2}{m^2} + \frac{1}{2} \hat{P}^{(1),NS}_{qq} \ln \frac{Q^2}{m^2} + a^{(2),NS}_{qq} + \frac{\beta_{0,Q}}{4} \zeta_2 P^{(0)}_{qq} + \hat{C}^{(2),NS}_q$$

at two loops and similar expressions for the three-loop heavy quark Wilson coefficient $h_3$ and the two-mass contribution $\hat{\hat{h}}_3$.
[M. Buza et al., Nucl. Phys. B472 (1996) 611–658; M. Buza et al., Eur. Phys. J. C1 (1998) 301–320; I. Bierenbaum, J. Blümlein and S. Klein, Nucl. Phys. B820 (2009) 417–482; S. W. G. Klein, PhD thesis (2009); A. Behring et al., Eur. Phys. J. C74 (2014) no.9, 3033; J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, C. Schneider and F. Wißbrock, Nucl. Phys. B921 (2017), 585–688.]

# The non-singlet sector

- The evolution operator is obtained by

$$
\begin{aligned}
\ln E^{NS}(Q^2, Q_0^2) &= \ln F_2^{NS}(Q^2) - \ln F^{NS}(Q_0^2) \\
&= \ln \mathcal{C}^{NS}\left(a, Q^2, m_c^2, m_b^2\right) + \int_{a_0}^{a} da \left[\frac{P_0 a + P_1 a^2 + P_2 a^3 + P_3 a^4}{-\beta_0 a^2 - \beta_1 a^3 - \beta_2 a^4 - \beta_3 a^5}\right] \\
&\quad - \ln \mathcal{C}^{NS}\left(a_0, Q_0^2, m_c^2, m_b^2\right)
\end{aligned}
$$

- After a perturbative expansion in $a = a_s(Q^2)$,

$$
\begin{aligned}
E^{NS}(Q^2, Q_0^2) = & \left(\frac{a}{a_0}\right)^{-\frac{P_0}{\beta_0}} \Bigg\{ 1 + (a - a_0)\left(c_1 + \frac{\beta_1 P_0}{\beta_0^2} - \frac{P_1}{\beta_0}\right) + \frac{1}{2}(a - a_0)^2 \left(c_1 + \frac{\beta_1 P_0}{\beta_0^2} - \frac{P_1}{\beta_0}\right)^2 \\
& + \frac{1}{2}\left(a^2 - a_0^2\right)\left[-c_1^2 + 2c_2 + \frac{\beta_1 P_1}{\beta_0^2} + \left(\frac{\beta_2}{\beta_0^2} - \frac{\beta_1^2}{\beta_0^3}\right) P_0 - \frac{P_2}{\beta_0}\right] \\
& + (a - a_0)\left(a^2 - a_0^2\right)\left(c_1 + \frac{\beta_1 P_0}{\beta_0^2} - \frac{P_1}{\beta_0}\right)\left(-\frac{c_1^2}{2} + c_2 + \frac{\beta_1 P_1}{2\beta_0^2} + \left(\frac{\beta_2}{2\beta_0^2} - \frac{\beta_1^2}{2\beta_0^3}\right) P_0 - \frac{P_2}{2\beta_0}\right) \\
& + \left(a^3 - a_0^3\right)\left(\frac{c_1^3}{3} - c_2 c_1 + c_3 + \frac{\beta_1 P_2}{3\beta_0^2} + \left(\frac{\beta_2}{3\beta_0^2} - \frac{\beta_1^2}{3\beta_0^3}\right) P_1 + \left(\frac{\beta_1^3}{3\beta_0^4} - \frac{2\beta_2\beta_1}{3\beta_0^3} + \frac{\beta_3}{3\beta_0^2}\right) P_0 - \frac{P_3}{3\beta_0}\right) \\
& + a^2\left(h_2(Q^2, m_c^2) + h_2(Q^2, m_b^2)\right) - a_0^2\left(h_2(Q_0^2, m_c^2) + h_2(Q_0^2, m_b^2)\right) \\
& + a^3\left(h_3(Q^2, m_c^2) + h_3(Q^2, m_b^2) + \hat{\hat{h}}_3(Q^2, m_b^2, m_c^2)\right) - a_0^3\left(h_3(Q_0^2, m_c^2) + h_3(Q_0^2, m_b^2) + \hat{\hat{h}}_3(Q_0^2, m_b^2, m_c^2)\right) \\
& - a_0 c_1\left[a_0(a - a_0)\left(h_2(Q_0^2, m_c^2) + h_2(Q_0^2, m_b^2)\right) + a^2\left(h_2(Q^2, m_c^2) + h_2(Q^2, m_b^2)\right) - a_0^2\left(h_2(Q_0^2, m_c^2) + h_2(Q_0^2, m_b^2)\right)\right] \\
& + (a - a_0)\left(\frac{\beta_1 P_0}{\beta_0^2} - \frac{P_1}{\beta_0}\right)\left[a^2\left(h_2(Q^2, m_c^2) + h_2(Q^2, m_b^2)\right) - a_0^2\left(h_2(Q_0^2, m_c^2) + h_2(Q_0^2, m_b^2)\right)\right] \Bigg\}
\end{aligned}
$$

# The singlet sector

- The singlet sector requires two input measurements: $F_2^S(x, t=0)$ and $\frac{\partial}{\partial t} F_2^S(x, t=0)$.
- Their scaling in $t$ is given by

$$\frac{\partial}{\partial t} \begin{pmatrix} F_2^S(t) \\ \frac{\partial}{\partial t} F_2^S(t) \end{pmatrix} = \left[ \sum_{i=0}^{\infty} a^i \mathbf{K}_i \right] \begin{pmatrix} F_2^S(t) \\ \frac{\partial}{\partial t} F_2^S(t) \end{pmatrix}$$

- The components $K_{IJ}$ of the matrix $\mathbf{K}_i$ are, in the massless case, given by

$$K_{IJ} = \frac{\partial C_{I,m}}{\partial t} C_{m,J}^{-1} + \frac{\beta_0 a}{\beta(a)} C_{I,m} \gamma_{mn} C_{n,J}^{-1}$$

- In the massive case $C_{I,j}$ also contains the massive contributions.
- $\mathbf{K}_i$ can be computed in perturbation theory. They depend on anomalous dimensions, $\beta_i$, massless and massive Wilson coefficients.

# The singlet sector

- The equation is solved by [J. Blümlein, A. Vogt, Phys.Rev. D58 (1998) 014020]

$$\begin{pmatrix} F_2^S(t) \\ \frac{\partial}{\partial t} F_2^S(t) \end{pmatrix} = \left[1 + \sum_{k=1}^{\infty} a^k \mathbf{U}_k\right] \mathbf{L}(a, a_0) \left[1 + \sum_{k=1}^{\infty} a_0^k \mathbf{U}_k\right]^{-1} \begin{pmatrix} F_2^S(0) \\ \frac{\partial}{\partial t} F_2^S(0) \end{pmatrix}$$

$$\mathbf{L}(a, a_0) = \left(\frac{a}{a_0}\right)^{-\frac{2}{\beta_0}} \left[\exp(r_-)\mathbf{e}_- + \exp(r_+)\mathbf{e}_+\right]$$

$$r_\pm = \frac{1}{8}\left[-\gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} \pm \sqrt{(\gamma_{qq}^{(0)} - \gamma_{gg}^{(0)})^2 + 4\gamma_{qq}^{(0)}\gamma_{gg}^{(0)}}\right] \quad \text{eigenvalues of } \mathbf{K}_0$$

$$\mathbf{e}_\pm = \frac{1}{r_\pm - r_\mp}[\mathbf{K}_0 - r_\mp \mathbf{1}]$$

$$\widetilde{\mathbf{K}}_k = \mathbf{K}_k + \sum_{i=1}^{k-1} \mathbf{K}_{k-i}\mathbf{U}_i$$

$$\mathbf{U}_k = -\frac{2}{\beta_0 k}\left[\mathbf{e}_-\widetilde{\mathbf{K}}_k\mathbf{e}_- + \mathbf{e}_+\widetilde{\mathbf{K}}_k\mathbf{e}_+\right] + \frac{\mathbf{e}_+\widetilde{\mathbf{K}}_k\mathbf{e}_-}{r_- - r_+ - \frac{\beta_0}{2}k} + \frac{\mathbf{e}_+\widetilde{\mathbf{K}}_k\mathbf{e}_-}{r_+ - r_- - \frac{\beta_0}{2}k}.$$

# Numerical Implementation

- The splitting functions $P_{ij}$ are known to $N^3LO$.
  [S. Moch, J. Vermaseren and A. Vogt, Nucl. Phys. B688 (2004) 101–134.]
  $P_{qq}^{(3),NS}$ can be approximated by a Padé approximant.
  [J. Blümlein, H. Böttcher and A. Guffanti, Nucl. Phys. B774 (2007) 182–207.]
- $C_{i,j}$ are known to 3 loops. Numerical representation of $C_{2,q}^{(3)}$ to be used.
  [W. A. Bardeen, A. J. Buras, D. W. Duke, and T. Muta, Phys. Rev. D18 (1978) 3998–4017;
  W.L. van Neerven and E.B. Zijlstra, Phys. Lett. B272 (1991) 127–133;
  J.A.M. Vermaseren, A. Vogt, S. Moch, Nucl. Phys. B724 (2005) 3–182.]
- $A_{ij}$ are known to 2 loops and a number of results to 3 loops.
  [I. Bierenbaum, J. Blümlein, S. Klein, Nucl. Phys. B820 (2009) 417–482;
  J. Ablinger, J. Blümlein, S. Klein, C. Schneider and F. Wissbrock, Nucl. Phys. B844 (2011) 26–54;
  A. Behring et al., Eur. Phys. J. C74 (2014) no.9, 3033.]
- Sufficient for $N^3LO$ analysis of the non-singlet and $NNLO$ of the singlet.
- All quantities depend on harmonic sums up to weight 5 and $S_6(N)$.
  [J. Vermaseren, Int. J. Mod. Phys. A 14 (1999) 2037–2076;
  J. Blümlein and S. Kurth, Phys. Rev. D 60 (1999) 014018]

# Numerical Implementation

- The evolution kernels are given by harmonic sums

$$S_{k_1,k_2,\ldots,k_m}(N) = \sum_{n_1=1}^{N} \frac{(\mathrm{sign}(k_1))^{n_1}}{n_1^{|k_1|}} \sum_{n_2=1}^{n_1} \frac{(\mathrm{sign}(k_2))^{n_2}}{n_2^{|k_2|}} \cdots \sum_{n_m=1}^{n_{m-1}} \frac{(\mathrm{sign}(k_m))^{n_m}}{n_m^{|k_m|}}$$

- Need systematic way to compute harmonic sums, e.g.

$$S_{-2,3}(N) = \sum_{n_1=1}^{N} \frac{(-1)^{n_1}}{n_1^2} \sum_{n_2=1}^{n_1} \frac{1}{n_2^3}$$

  and their analytic continuation for complex $N$.

  [J. Blümlein, Comput. Phys. Commun. 180 (2009) 2218.]

- Use factorial series for $|N| \to \infty$ and recurrences for $N \to N-1$. Algorithms are available in the Mathematica package HarmonicSums

  [J. Ablinger, PhD thesis (2012); J. Ablinger, PoS (LL2014) 019; J. Blümlein, Clay Math. Proc. **12** (2010) 167-188; J. Ablinger, J. Blümlein and C. Schneider, J. Math. Phys. 54 (2013), 082301; ]

- Factors of $(-1)^N \to$ project on even $N$ (unpolarized case) or odd $N$ (polarized case).
- Perform inverse Mellin transformation numerically as last step (one contour integral).

# Conclusions

- A framework of the scheme independent study of the scaling violations of deep-inelastic structure functions has been provided.
- As the next step, the framework is implemented in a Fortran library.
- Numerical illustrations will be provided.
- The effect of the heavy flavour contribution is studied.