

Analytic representation of all planar two-loop five-point Master Integrals with one off-shell leg

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based on work in collaboration with D. Canko and C. Papadopoulos
arXiv:2009.13917 [hep-ph] (to appear in JHEP)

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Motivation

- Indications for the need of physics Beyond the Standard Model mostly from Cosmology (e.g. is dark matter a new particle/ particles?).
- Absence of a clear signal for BSM physics from the LHC → Collider Physics at the Precision Frontier¹.
- Measured cross sections are being compared to improved theoretical predictions looking for deviations from the SM → constraints on BSM physics.
- LHC Run 3 and HL-LHC Run will require at least NNLO corrections.
- NNLO → two-loop Feynman Diagrams → two-loop Feynman Integrals (FI).
- Numerical approaches not efficient (physical kinematics) → need for analytical results.



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¹G. Heinrich, arXiv:2009.00516 [hep-ph]



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Computing Feynman Integrals

Introduce a family of two-loop FI for a specific scattering process (e.g. $2 \rightarrow 2, 2 \rightarrow 3$):

$$F_{a_1, \dots, a_N}(\{p_j\}, \epsilon) = \int \left(\prod_{r=1}^2 \frac{d^d k_r}{i\pi^{d/2}} \right) \frac{e^{2\epsilon\gamma_E}}{D_1^{a_1} \dots D_N^{a_N}}, \quad (1)$$

$$D_i = (c_{ij}k_j + f_{ij}p_j)^2, \quad d = 4 - 2\epsilon$$

Two pillars

- 1 Integrals of total derivatives wrt loop momenta vanish within DR. (IBP identities)
- 2 FI satisfy differential equations (DE) derived wrt kinematic invariants.

Computing Master Integrals

- IBP reduction identifies a minimal set of Master Integrals, called basis of MI \mathbf{G} .
- Use DE to compute them.

$$\frac{\partial}{\partial s_{ij}} \mathbf{G} = \mathbf{A}(\{s_{ij}\}, \epsilon) \mathbf{G} \quad (2)$$

- Simplified DE (SDE)²: introduce an external parameter x and differentiate wrt it.

$$\partial_x \mathbf{G} = \mathbf{A}(\{s_{ij}\}, x, \epsilon) \mathbf{G} \quad (3)$$

- In general the matrix \mathbf{A} can be very complicated.
- Instead of \mathbf{G} use a special basis³ $\mathbf{g} = \mathbf{T}\mathbf{G}$



G. Papadopoulos, JHEP **07** (2014), 088

J. M. Henn, Phys. Rev. Lett. **110** (2013), 251601



Computing Master Integrals

- Canonical SDE

$$\partial_x \mathbf{g} = \epsilon \mathbf{M}(\{s_{ij}\}, x) \mathbf{g} \quad (4)$$

- ϵ is fully factorised
- Differential matrix $\mathbf{M}(\{s_{ij}\}, x)$ is Fuchsian, i.e. it has simple poles in x (for all cases considered in this talk).
- Solved with recursive iterations in terms of Goncharov PLs (GPLs)

$$\mathcal{G}(a_1, a_2, \dots, a_n; x) = \int_0^x \frac{dt}{t - a_1} \mathcal{G}(a_2, \dots, a_n; t) \quad (5)$$

$$\mathcal{G}(0, \dots, 0; x) = \frac{1}{n!} \log^n(x) \quad (6)$$



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2 \rightarrow 3 with one massive leg (e.g. $W+2$ jets)

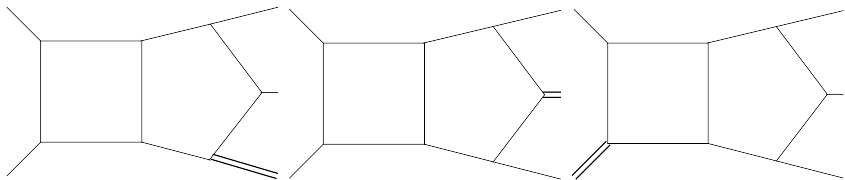


Figure: The two-loop diagrams representing the top-sector of the planar pentabox family P_1 (74 MI), P_2 (75 MI) and P_3 (86 MI). All external momenta are incoming.

- IBP tools: FIRE6, KIRA2

Traditional approach

Semi-numerical approach

S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow and M. Zeng, JHEP **11** (2020), 117

Kinematics

- External momenta q_i , $i = 1 \dots 5$
- $\sum_1^5 q_i = 0$, $q_1^2 \equiv p_{1s}$, $q_i^2 = 0$, $i = 2 \dots 5$
- $\{q_1^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\}$, with $s_{ij} := (q_i + q_j)^2$

$$d\mathbf{g} = \epsilon \sum_a d \log(W_a) \tilde{\mathbf{M}}_{ag} \quad (7)$$

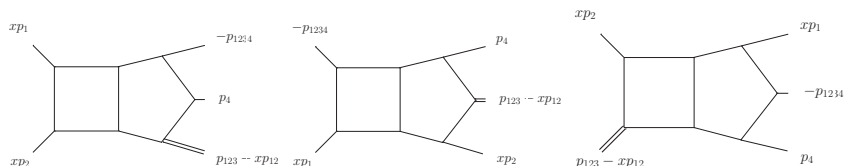
Very difficult to solve analytically!



SDE approach

Re-parametrize external momenta in terms of a dimensionless parameter x .

$$q_1 \rightarrow p_{123} - xp_{12}, \quad q_2 \rightarrow p_4, \quad q_3 \rightarrow -p_{1234}, \quad q_4 \rightarrow xp_1 \quad (8)$$



Kinematics

- Underline momenta p_i , $i = 1 \dots 5$

$$\sum_1^5 p_i = 0, \quad p_i^2 = 0, \quad i = 1 \dots 5, \quad \text{with } p_{i\dots j} := p_i + \dots + p_j$$

$$\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}, x\}, \quad \text{with } S_{ij} := (p_i + p_j)^2$$



SDE approach

- Much simpler canonical DE

$$\partial_x \mathbf{g} = \epsilon \sum_b \frac{1}{x - l_b} \mathbf{M}_b \mathbf{g} \quad (9)$$

- Naturally expressed in terms of GPLs.
- Boundary terms: need $x \rightarrow 0$ limit, Expansion-By-Regions⁴ (Chris Wever, Adam Kardos (Gsuite)).
- Re-write $\mathbf{g} = \mathbf{T} \mathbf{G}$
- Asymptotic expansion around $x \rightarrow 0$ for \mathbf{G} .



⁴B. Jantzen, A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C **72** (2012), 2139

Analytic solution up to $\mathcal{O}(\epsilon^4)$

$$\begin{aligned}
 \mathbf{g} = & \epsilon^0 \mathbf{b}_0^{(0)} + \epsilon \left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)} \right) \\
 & + \epsilon^2 \left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)} \right) \\
 & + \epsilon^3 \left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(2)} + \mathbf{b}_0^{(3)} \right) \quad (10) \\
 & + \epsilon^4 \left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(1)} \right. \\
 & \left. + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(3)} + \mathbf{b}_0^{(4)} \right)
 \end{aligned}$$

$$\mathcal{G}_{ab\dots} := \mathcal{G}(l_a, l_b, \dots; x)$$



Pure solution: residue matrices \mathbf{M}_i are independent of the kinematics.

Universally transcendental.

Transcendental weight \sim number of integrations.



Explicit expressions up to $\mathcal{O}(\epsilon^1)$

$$gb_{72}^{P1} = \frac{3}{2} + \epsilon \left[\mathcal{G} \left(\frac{S_{45}}{S_{12}}, x \right) - \mathcal{G} \left(\frac{S_{12} - S_{34}}{S_{12}}, x \right) - 3\mathcal{G}(0, x) + \mathcal{G}(1, x) - \log(S_{12} - S_{34}) - 2 \log(-S_{51}) \right] \quad (11)$$

$$gb_{73}^{P2} = \epsilon \left[3\mathcal{G} \left(\frac{S_{12} + S_{23}}{S_{12}}, x \right) - 3\mathcal{G} \left(\frac{S_{12} - S_{34}}{S_{12}}, x \right) - 3\mathcal{G} \left(\frac{S_{45}}{S_{12}}, x \right) + 3\mathcal{G}(1, x) - 3 \log(S_{12} - S_{34}) + 3 \log(-S_{51}) \right] \quad (12)$$

$$g_{84}^{P3} = \frac{1}{2} + \epsilon \left[\frac{5}{2} \mathcal{G} \left(\frac{S_{45}}{S_{12}}, x \right) - \frac{3}{2} \mathcal{G} \left(\frac{S_{12} - S_{34}}{S_{12}}, x \right) - \frac{5}{2} \mathcal{G} \left(-\frac{S_{45}}{S_{23} - S_{45}}, x \right) - 2\mathcal{G}(0, x) + \frac{5}{2} \mathcal{G}(1, x) - \log(-S_{12}) - \frac{3}{2} \log(S_{12} - S_{34}) + \frac{3}{2} \log(-S_{51}) \right]$$



Numerics (N=32 digits, 1.9, 3.3, 2 sec) in GiNaC

P_1	g_{72}	ϵ^0 : 3/2 ϵ^1 : -2.2514604753379400332169314784961 ϵ^2 : -17.910593443812320786572184851867 ϵ^3 : -26.429770706459534336624681550003 ϵ^4 : 21.437938934510558345847354772412
P_2	g_{73}	ϵ^1 : 2.8124788185742741402751457351382 ϵ^2 : 5.4813042746593704203645729908938 ϵ^3 : 11.590234540689191439870956817546 ϵ^4 : -5.9962816226829136730734255754596
P_3	g_{84}	ϵ^0 : 1/2 ϵ^1 : 3.2780415861887284967738281876762 ϵ^2 : 0.11455863130537720411162743574627 ϵ^3 : -16.979642659429606120982671925458 ϵ^4 : -48.101985355625914648042310964575



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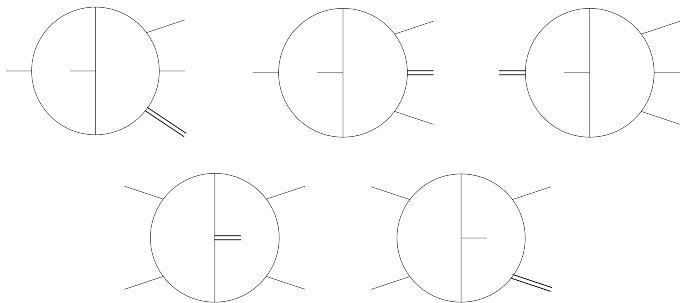
Summary

- Lots of progress on multi-loop multi-scale FI calculations.
- Ability to do high precision phenomenology.
- Insights from many areas, e.g. formal theory (N=4 SYM), collider physics, special functions, mathematical physics, programming.
- Push to compute all MI for $2 \rightarrow 3$ with up to one massive leg at two loops.
- Automated package for NNLO predictions following the NLO paradigm.



Future work

Compute remaining non-planar MI for two-loop $2 \rightarrow 3$ with one massive leg⁵,



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C. G. Papadopoulos and C. Wever, JHEP **02** (2020), 112



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Thank you for your attention!



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Automated tools

- Ginac⁶ for the numerical calculation of GPLs.
- PolyLogTools⁷ for the algebraic manipulation of GPLs.
- FIRE6⁸ and KIRA2⁹ for the IBP reduction.
- FIESTA4¹⁰ for Expansion-By-Regions.
- pySecDec¹¹ and FIESTA4 for numerical computation of FI, used for cross-checking our results.

⁶J. Vollinga and S. Weinzierl, *Comput. Phys. Commun.* **167** (2005), 177

⁷C. Duhr and F. Dulat, *JHEP* **08** (2019), 135

⁸A. V. Smirnov and F. S. Chuharev, arXiv:1901.07808 [hep-ph]

⁹J. Klappert, F. Lange, P. Maierhöfer and J. Usovitsch, arXiv:2008.06494 [hep-ph]

¹⁰A. V. Smirnov, *Comput. Phys. Commun.* **204** (2016), 189-199

¹¹S. Borowka, G. Heinrich, S. Jahn, S. P. Jones, M. Kerner, J. Schlenk and T.

Comput. Phys. Commun. **222** (2018), 313-326



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Scattering kinematics

- Results in Euclidean region (all initial kinematic invariants are negative).
- GPLs and solutions are real there.
- Analytic continuation to get results in physical regions for phenomenology.
- Fibration basis techniques (exploit symbol algebra and coproduct to analytically continue GPLs).
- Numerically: $\{S_{ij}, x\} \rightarrow \{S_{ij} + i\delta_{ij}\eta, x + i\delta\eta\}$ for $\eta \rightarrow 0$.
- Constraints on δ_{ij} , δ_x from one-scale integrals and second graph polynomial of top sector FI.



Accumulated results

- Fully analytical results for all MI for $2 \rightarrow 2$ with up to one mass in all kinematic regions.
- Analytical results for all MI for $2 \rightarrow 2$ with two masses in Euclidean region (numerical analytic continuation for physical regions)¹²¹³¹⁴.
- All planar 2-loop MI for $2 \rightarrow 3$ with up to one mass.
- Two planar families for $2 \rightarrow 2$ with up to one mass at three loops¹⁵.
- Pentagon with up to one massive leg to all orders¹⁶.

¹²C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **01** (2015), 072

¹³J. M. Henn, K. Melnikov and V. A. Smirnov, JHEP **05** (2014), 090

¹⁴F. Caola, J. M. Henn, K. Melnikov and V. A. Smirnov, JHEP **09** (2014), 043

¹⁵D. D. Canko and NS, [arXiv:2010.06947 [hep-ph]]

¹⁶NS, [arXiv:2012.10635 [hep-ph]]

