Simplified Differential Equations for Master Integrals @ N3LO

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In collaboration with: C. Papadopoulos and N. Syrrakos.

Based on: DC and N. Syrrakos, 2010.06947 [hep-ph] (to appear in JHEP)

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Massless 3-loop 2→2 families with up to one external off-shell leg

HL-LHC and possible future upgrades/colliders will require high-precision theoretical predictions, which for $2 \to 2$ scatterings means reaching N3LO computations. These computations demand the calculation of 3-loop Feynman Integrals!

For all the external particles on-shell (relevant for di-jet or di-photon productions) there exist 9 (2) families of MIs, all of whom have been recently calculated in the literature

- *V. A. Smirnov, Phys. Lett.* ***B567*** *(2003) 193–199.*
- *J. M. Henn, A. V. Smirnov and V. A. Smirnov, JHEP* ***07*** *(2013) 128.*
- *J. M. Henn, A. V. Smirnov and V. A. Smirnov, JHEP* ***03*** *(2014) 088.*
- *J. Henn, B. Mistlberger, V. A. Smirnov and P. Wasser, JHEP* ***04****, 167 (2020).*
- *DC and N. Syrrakos, 2010.06947 [hep-ph].*

Keeping one external leg off-shell (relevant for Higgs−jet in gluon fusion production) things become more complicated and we have 18 (3) families of MIs, of whom only one has been computed

- *S. Di Vita, P. Mastrolia, U. Schubert, and V. Yundin, JHEP* ***09****, 148 (2014)*
- *DC and N. Syrrakos, 2010.06947 [hep-ph].*

All these families together with the families with 2 external legs off-shell (di-boson productions) need to be calculated for future comparisons with the experiments.

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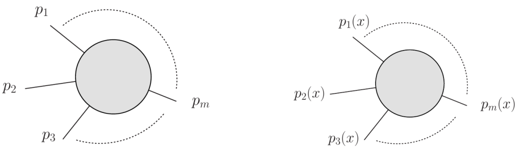
Quick review of SDE

For any family of master integrals (MIs), **G**, one applies the following procedure [C. G. Papadopoulos, JHEP **07** (2014), 088]:

1) Parametrize the external momenta in terms of an dimensionless parameter, x, in such a way that captures the off-shellness of an external leg.



2) Take derivatives of the MIs with respect to x and create, using integration-by-parts identities (IBPs) a system of differential equations (DE) in one independent variable

$$\partial_x \mathbf{G}(\{s_{ij}\}, x, \varepsilon) = \mathbf{H}(\{s_{ij}\}, x, \varepsilon)\mathbf{G}(\{s_{ij}\}, x, \varepsilon)$$

3) Find boundary conditions at $x \to 0$ and solve the differential equation.

The application of this method has plenty of advantages compared to the standard method of differential equations!

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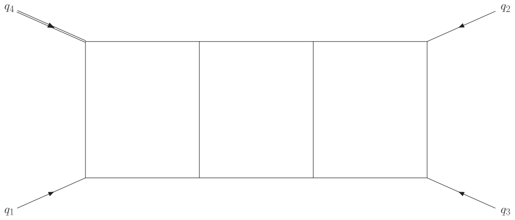
3-loop ladder-box with one external massive leg

We adopt the basis of universal transcendental MIs[1] (UT basis) and the notation for the kinematics from [S. Di Vita, et al, JHEP **09**, 148 (2014)], where this family was first studied



The external momenta can be expressed in Mandelstam variables

$$q_1^2 = q_2^2 = q_3^2 = 0\,, \quad q_4^2 = m^2\,, \quad q_2{\cdot}q_3 = s/2\,, \quad q_1{\cdot}q_3 = t/2\,, \quad q_1{\cdot}q_2 = (m^2 - s - t)/2\,.$$

For this family we obtained a set of 83 master integrals in contrast with [S. Di Vita, et al, JHEP **09**, 148 (2014)], where a set of 85 was presented (we found via IBPs and analytic check of the solutions that $\mathcal{T}_7 = \mathcal{T}_8$ and $\mathcal{T}_{45} = \mathcal{T}_{46}$).

[1][J. M. Henn, Phys. Rev. Lett. **110** (2013), 251601].

The class of Feynman Integrals (FI) describing this family can be expressed as follows

$$G_{a_1,\ldots,a_{15}}(\{q_j\},\varepsilon) = \int \left(\prod_{r=1}^{3} \frac{d^d l_r}{i\pi^{d/2}} \right) \frac{e^{3\varepsilon\gamma_E}}{D_1^{a_1}\ldots D_{15}^{a_{15}}} \quad \text{with} \quad d = 4-2\varepsilon\,,$$

where $D_{11},\ldots,D_{15}$ are propagators coming from irreducible-scalar-products (ISPs) $(\{a_{11},a_{12},a_{13},a_{14},a_{15}\} \leq 0)$ and the chosen parametrization for the propagators is

$$D_1 = l_1^2\,,\quad D_2 = l_2^2\,,\quad D_3 = l_3^2\,,\quad D_4 = (l_1-l_2)^2\,,\quad D_5 = (l_2-l_3)^2\,,$$
$$D_6 = (l_3+q_2)^2\,,\quad D_7 = (l_1+q_{23})^2\,,\quad D_8 = (l_2+q_{23})^2\,,\quad D_9 = (l_3+q_{23})^2\,,$$
$$D_{10} = (l_1+q_{123})^2\,,\quad D_{11} = (l_1+q_2)^2\,,\quad D_{12} = (l_2+q_2)^2\,,\quad D_{13} = (l_2+q_{123})^2\,,$$
$$D_{14} = (l_3+q_{123})^2\,,\quad \text{and} \quad D_{15} = (l_1-l_3)^2\,.$$

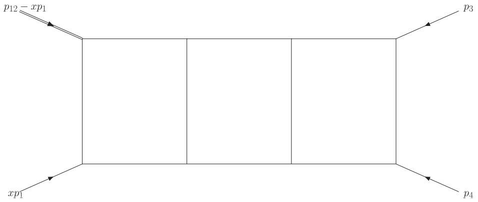
Moving to the SDE approach we choose the following parametrization

$$q_1 \to x p_1\,,\quad q_2 \to p_3\,,\quad q_3 \to -p_{123}\,,\quad q_4 \to p_{12} - x p_1 \quad \text{with} \quad p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0\,.$$

We express the Mandelstam variables and the external mass in terms of the parameter x and the new Mandelstam variables of the null momenta p_j

$$s = s_{12}\,,\quad t = x s_{23}\,,\quad m^2 = (1-x)s_{12}\,,$$

where $s_{12} = p_{12}^2$ and $s_{23} = p_{23}^2$.



After making the transformations ($l_1 \to k_1 - q_{23}$, $l_2 \to -k_2 - q_{23}$, $l_3 \to k_3 - q_{23}$) and applying the SDE approach to the propagators, they take the following form

$$D_1 = (k_1 + p_{12})^2\,, \quad D_2 = (k_2 - p_{12})^2\,, \quad D_3 = (k_3 + p_{12})^2\,, \quad D_4 = (k_1 + k_2)^2\,,$$
$$D_5 = (k_2 + k_3)^2\,, \quad D_6 = (k_3 + p_{123})^2\,, \quad D_7 = k_1^2\,, \quad D_8 = k_2^2\,, \quad D_9 = k_3^2\,,$$
$$D_{10} = (k_1 + xp_1)^2\,, \quad D_{11} = (k_1 + p_{123})^2\,, \quad D_{12} = (k_2 - p_{123})^2\,,$$
$$D_{13} = (k_2 - xp_1)^2\,, \quad D_{14} = (k_3 + xp_1)^2\,, \quad \text{and} \quad D_{15} = (k_1 - k_3)^2\,.$$

Having a UT basis we obtained a DE with respect to x, which is of canonical form

$$\partial_x \mathbf{g} = \varepsilon \left(\sum_{i=1}^{4} \frac{\mathbf{M}_i}{x - l_i} \right) \mathbf{g}$$

with $\mathbf{M}_i$ being purely numerical matrices and $l_i = \{0,\, 1,\, s_{12}/(s_{12} + s_{23}),\, -s_{12}/s_{23}\}$.

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We solve the DE in a Laurent expansion of the MIs up to weight six, in the Euclidean region of the invariants, which is

$$0 < x < 1\,, \quad s_{12} < 0\,, \quad s_{12} < s_{23} < 0\,.$$

The solution can be written in the compact form

$$\begin{aligned}\mathbf{g} = {} & \varepsilon^0 \mathbf{b}_0^{(0)} + \varepsilon \left(\sum \mathcal{G}_i \mathbf{M}_i \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)} \right) + \varepsilon^2 \left(\sum \mathcal{G}_{ij} \mathbf{M}_i \mathbf{M}_j \mathbf{b}_0^{(0)} + \sum \mathcal{G}_i \mathbf{M}_i \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)} \right) + \ldots \\ & + \varepsilon^6 \left(\mathbf{b}_0^{(6)} + \sum \mathcal{G}_{ijklmn} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{M}_m \mathbf{M}_n \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{ijklm} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{M}_m \mathbf{b}_0^{(1)} \right. \\ & \left. + \sum \mathcal{G}_{ijkl} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{b}_0^{(2)} + \sum \mathcal{G}_{ijk} \mathbf{M}_i \mathbf{M}_j \mathrm{M}_k \mathbf{b}_0^{(3)} + \sum \mathcal{G}_{ij} \mathbf{M}_i \mathbf{M}_j \mathbf{b}_0^{(4)} + \sum \mathcal{G}_i \mathbf{M}_i \mathbf{b}_0^{(5)} \right)\,,\end{aligned}$$

where the matrices $\mathbf{b}_0^{(i)}$ are the boundary terms and $\mathcal{G}_i, \ldots, \mathcal{G}_{ijklmn}$ are Goncharov poly-logarithms [A. B. Goncharov, Math. Res. Lett. **5** (1998), 497-516] of weight $1, \ldots, 6$, respectively, with argument x and letters from the set l_i.

Our results were numerically crossed-checked with the results from [S. Di Vita, et al, JHEP **09**, 148 (2014)] using `PolyLogTools` [C. Duhr and F. Dulat, JHEP **08** (2019) 135], and perfect agreement was found in all cases!

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Boundary conditions: known and zeroes

- Some integrals are known in close form and thus we can directly obtain boundary conditions for them

$$\{gb_1,\, gb_2,\, gb_3,\, gb_4,\, gb_5,\, gb_6,\, gb_7,\, gb_{17},\, gb_{18},\, gb_{19},\, gb_{44}\}\,.$$

- If a basis element has as an overall prefactor of x in such a power such as its leading regions contributing to its asymptotic limit $x \to 0$ (expansion-by-regions [M. Beneke and V. A. Smirnov, Nucl. Phys. B **522** (1998), 321-344]) are of the form $x^{\alpha+\beta\varepsilon}$ with $\alpha > 0$, then its boundary term should vanish

$$\{gb_{10},\, gb_{11},\, gb_{14},\, gb_{15},\, gb_{21},\, gb_{22},\, gb_{23},\, gb_{24},\, gb_{25},\, gb_{26},\, gb_{28},\, gb_{31},\, gb_{37},\, gb_{38},\, gb_{45},\, gb_{46},\, gb_{47},\, gb_{48},\, gb_{50},\, gb_{53},\, gb_{55},\, gb_{58},\, gb_{59},\, gb_{63},\, gb_{64},\, gb_{66},\, gb_{68},\, gb_{70},\, gb_{80},\, gb_{82},\, gb_{83}\} = 0\,.$$

Thus From 83 $\to$ 41 unknown boundaries!

Basis Element	Asymptotic Limit of Master Integral $x \to 0$
$g_{32} \equiv (s_{12}+s_{23}x)\varepsilon^5 F_{32}$	$F_{32} \equiv G_{1,0,0,1,1,2,0,1,0,1,0,0,0,0,0} \sim x^{-3\varepsilon},\, x^0$
$g_{41} \equiv (s_{12}+s_{23}x)\varepsilon^5 F_{41}$	$F_{41} \equiv G_{0,1,0,2,1,1,0,0,1,1,0,0,0,0,0} \sim x^{-3\varepsilon},\, x^0$
$g_{42} \equiv s_{12}s_{23}x\varepsilon^4 F_{42}$	$F_{42} \equiv G_{0,1,0,2,2,1,0,0,1,1,0,0,0,0,0} \sim x^{-1-3\varepsilon},\, x^{-3\varepsilon},\, x^0$
$g_{56} \equiv (s_{12}+s_{23}x)\varepsilon^6 F_{56}$	$F_{56} \equiv G_{1,1,0,1,1,1,0,0,1,1,0,0,0,0,0} \sim x^0$
$g_{71} \equiv s_{12}^2 s_{23}x\varepsilon^5 F_{71}$	$F_{71} \equiv G_{0,1,1,2,1,1,1,0,1,1,0,0,0,0,0} \sim x^{-1-3\varepsilon},\, x^{-3\varepsilon},\, x^0$
$g_{83} \equiv -s_{12}^3 x\varepsilon^6 F_{83}$	$F_{83} \equiv G_{1,1,1,1,1,1,1,1,1,1,0,-1,0,0,0} \sim x^{-3\varepsilon},\, x^0$

Boundary conditions: relations between boundaries

We define the resummation matrix at $x = 0$ through the Jordan-decomposition of $\mathbf{M}_0$

$$\mathbf{M}_0 = \mathbf{S}_0 \mathbf{D}_0 \mathbf{S}_0^{-1} \quad \longrightarrow \quad \mathbf{R}_0 = \mathbf{S}_0 e^{\varepsilon \mathbf{D}_0 \log(x)} \mathbf{S}_0^{-1} \, .$$

$\mathbf{R}_0$ correctly resumms the logarithms of x from the basis elements, meaning that we can write

$$\mathbf{g} = \mathbf{R}_0 \mathbf{g}_{\text{reg0}} \, ,$$

where $\mathbf{g}_{\text{reg0}}$ is the regular part of the basis element at $x = 0$, via which are defined the asymptotic boundaries

$$\mathbf{g}_{\text{bound}} = \mathbf{g}_{\text{reg0}} \Big|_{x=0} \, .$$

Multiplying $\mathbf{R}_0$ from the right with $\mathbf{g}_{\text{bound}}$ and from the left with $\mathbf{T}^{-1}$ (transformation U.T. basis elements $\rightarrow$ MIs), we obtain the asymptotic limit at $x \rightarrow 0$ of the MI

$$\mathbf{F}_{x \rightarrow 0} = \mathbf{T}^{-1} \mathbf{R}_0 \mathbf{g}_{\text{bound}} \, .$$

This should be equal to the asymptotic limit found for the MI by expansion-by-regions (found by asy [B. Jantzen, A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C **72** (2012), 2139]). Thus by comparing the regions found by asy with that found by the resummation matrix method we obtain relations between different boundaries. In fact, we obtain two kind of relations.

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1) We call *pure* the relations that contain only boundaries of UT basis elements, e.g.:

$$gb_{71} = \left(-12gb_2 + 4gb_{13} + 32gb_{16} + 48gb_{41} + 36gb_{42} - 45gb_{43}\right)/30\,.$$

2) We call *impure* the relations between boundaries and asymptotic limits, e.g.

$$gb_{41} = F_{41}^{\text{soft}} s_{12}\varepsilon^5 + gb_2/9 - gb_{13}/12 - 2gb_{16}/3\,.$$

As expected, in these *pure* relations between the boundaries the prefactors are just numbers $\longrightarrow$ Working perfectly even when a full analytic reduction is a bottleneck!!!

• By applying this method we obtain 28 *pure* relations and thus the problem of computing 41 boundaries is reduced to the calculation of the 13 asymptotic regions

$$\{F_8^{\text{hard}},\ F_9^{\text{hard}},\ F_{12}^{\text{hard}},\ F_{13}^{\text{hard}},\ F_{16}^{\text{hard}},\ F_{20}^{\text{hard}},\ F_{27}^{\text{hard}},\ F_{29}^{\text{hard}},\ F_{32}^{\text{soft}},\ F_{39}^{\text{soft}},\ F_{41}^{\text{soft}},\ F_{51}^{\text{hard}},\ F_{56}^{\text{hard}}\}$$

where with *hard* we denote the x^0 region and with *soft* the $x^{-3\varepsilon}$.

• We calculated the *hard* limits with the use of the method of expansion-by-regions in the momentum space (significantly easier in SDE) and IBP reduction.

• The soft limits were calculated using standard expansion-by-region approach, meaning computing their Feynman-parameter representation provided by asy.

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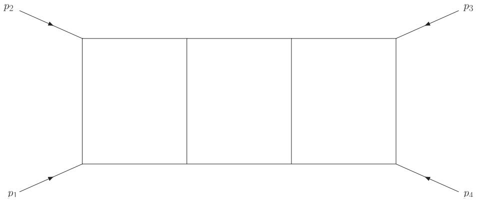
$x \to 1$ *limit: Massless problem*

The $x \to 1$ limit yields the solution for a canonical basis of the massless ladder-box:



The chosen normalisation of the FI is

$$G_{a_1,\ldots,a_{15}}\left(\{p_j\},\varepsilon\right) = (-s_{12})^{3\varepsilon}\int\left(\prod_{l=1}^{3}\frac{d^d k_l}{i\pi^{d/2}}\right)\frac{e^{3\varepsilon\gamma_E}}{D_1^{a_1}\ldots D_{15}^{a_{15}}} \quad \text{with} \quad d = 4-2\varepsilon$$

and the propagators being

$$\begin{aligned}
&D_1 = (k_1+p_{12})^2\,, \quad D_2 = (k_2-p_{12})^2\,, \quad D_3 = (k_3+p_{12})^2\,, \quad D_4 = (k_1+k_2)^2\,,\\
&D_5 = (k_2+k_3)^2\,, \quad D_6 = (k_3+p_{123})^2\,, \quad D_7 = k_1^2, \quad D_8 = k_2^2, \quad D_9 = k_3^2,\\
&D_{10} = (k_1+p_1)^2\,, \quad D_{11} = (k_1+p_{123})^2\,, \quad D_{12} = (k_2-p_{123})^2\,,\\
&D_{13} = (k_2-p_1)^2\,, \quad D_{14} = (k_3+p_1)^2\,, \quad \text{and} \quad D_{15} = (k_1-k_3)^2\,.
\end{aligned}$$

We compared our results numerically with pySecDec [S. Borowka et al, Comput. Phys. Commun. **222** (2018), 313-326] and perfect agreement was found in all cases!

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Briefly the procedure for taking the $x \to 1$ limit is:

1) Rewrite the solution as an expansion in $\log(1-x)$:

$$\mathbf{g} = \sum_{n\geq 0} \epsilon^n \sum_{i=0}^{n} \frac{1}{i!} \mathbf{c}_i^{(n)} \log^i(1-x)$$

2) Define the regular part of $\mathbf{g}$ at $x = 1$ and from it the truncated part:

$$\mathbf{g}_{reg} = \sum \epsilon^n \mathbf{c}_0^{(n)} \quad \text{and} \quad \mathbf{g}_{trunc} = \mathbf{g}_{reg}\Big|_{x=1}$$

3) Define the resummation matrix $\mathbf{R}_1$ and from it the purely numerical matrix $\mathbf{R}_{10}$:

$$\mathbf{R}_1 = e^{\epsilon \mathbf{M}_1 \log(1-x)} = \mathbf{S}_1 e^{\epsilon \mathbf{D}_1 \log(1-x)} \mathbf{S}_1^{-1} \quad \text{and} \quad \mathbf{R}_1 \xrightarrow{(1-x)^{a_i \epsilon} \to 0} \mathbf{R}_{10}$$

4) Find the $x \to 1$ limit by acting $\mathbf{R}_{10}$ to $\mathbf{g}_{trunc}$:

$$\mathbf{g}_{x\to 1} = \mathbf{R}_{10} \mathbf{g}_{trunc}$$

5) Reduce the number of the basis elements to the number of the MI of the massless problem using the property $\mathbf{R}_{10}^2 = \mathbf{R}_{10} \Rightarrow \mathbf{R}_{10}\mathbf{g}_{x\to 1} = \mathbf{g}_{x\to 1}$ and/or IBPs.

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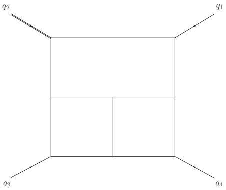
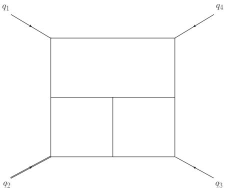
4-point 3-loop planar families with 1 off-shell leg

Ongoing work with F. Gasparotto and L. Mattiazzi (Padova University).

To complete the set of all planar families one needs to solve the two tennis-courts:





- The first contains 117 MI and the second 166 MI.
- Finding of a UT basis using:
  - *One-loop and two-loop building blocks [P. Wasser, MSc thesis (2016)].*
  - *Magnus exponential [M. Argeri, et al, JHEP **1403** (2014) 082].*
  - *`DlogBasis` to find integrands of d-log form [J. Henn, et al, JHEP **04**, 167 (2020)].*
- Boundaries: methods described herein (many MI in common with ladderbox).
- Analytical solutions for the 3 physical regions (Fibration Basis) for fast evaluations.

Thank you!

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