Gauge-free gyrokinetic models for hybrid-kinetic simulations of magnetized fusion plasmas

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Abstract

What if you are interested in performing a particle simulation with kinetic electrons and gyrokinetic ions?

In principle, the kinetic motion of electrons is described in terms of electric and magnetic fields $(\mathbf{E}, \mathbf{B})$, while the standard gyrokinetic motion of ions is described in terms of electric and magnetic potentials $(\Phi, \mathbf{A})$.

The dependence of standard gyrokinetic theory on perturbed potentials $(\Phi_1, \mathbf{A}_1)$, instead of perturbed fields $(\mathbf{E}_1, \mathbf{B}_1)$, introduces the requirement of specifying a choice of gauge.

Understandably, you might be worried that your nonlinear gyrokinetic simulation results might depend on the choice of gauge you made.

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The dependence of standard gyrokinetic theory on perturbed potentials $(\Phi_1, \mathbf{A}_1)$, instead of perturbed fields $(\mathbf{E}_1, \mathbf{B}_1)$, introduces the requirement of specifying a choice of gauge.

Understandably, you might be worried that your nonlinear gyrokinetic simulation results might depend on the choice of gauge you made. (Don't worry, standard nonlinear gyrokinetic is fine.)

Of course, you might decide to use the potential representations of the electromagnetic fields in your electron kinetic description, but that seems wrong to be moving away from physical fields.

In this tutorial talk, I will show how standard gyrokinetic theory can be transformed into a gauge-free gyrokinetic theory, which is entirely expressed in terms of the perturbed electric and magnetic fields.

A gauge-free gyrokinetic model can be used for hybrid-kinetic simulations of magnetized plasmas in which particle species can be represented in terms of either a Vlasov kinetic description or a gauge-free Vlasov gyrokinetic description.

Overview

- Hierarchy of Orbital Time Scales in Magnetized Plasmas
- Dynamical Reduction & Adiabatic Invariance
- Electromagnetic Potentials and Fields: Gauge Freedom
- Structure of a Gauge-free Reduced Vlasov-Maxwell Theory
- Reduced Vlasov-Maxwell Theory: Guiding-center Paradigm
- Gauge-free Gyrokinetic Vlasov-Maxwell Theory
- Summary and Ongoing Work

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Hierarchy of Orbital Time Scales in Magnetized Plasmas

Particle motion in uniform magnetic field $\mathbf{B}_0$

- Gyro-motion perpendicular to a magnetic-field line with constant gyro-frequency $\Omega = q\,B_0/mc$ and constant gyro-radius

$$\rho \;=\; |\mathbf{v}\times\widehat{\mathsf{b}}_0|/\Omega \;\equiv\; v_\perp/\Omega$$

- Parallel motion along a magnetic-field line with constant parallel velocity $v_\| = \mathbf{v}\cdot\widehat{\mathsf{b}}_0$.

Particle motion in nonuniform magnetic field $\mathbf{B} \;=\; \nabla\alpha\times\nabla\beta$

- $(\alpha,\beta) \;=$ Euler potentials (field-line labels)
- $s \;=$ parallel coordinate along a field line
- $\widehat{\mathsf{b}}\equiv\partial\mathbf{x}/\partial s \;=$ unit vector along field line.

- Magnitude $B = |\nabla\alpha\times\nabla\beta|$ and direction $\widehat{\mathsf{b}}$ are not constant and $v_\perp$ and $v_\|$ are not conserved.

- Magnetic-field inhomogeneity → Magnetic drifts

  - Perpendicular gradient $\widehat{\mathsf{b}} \times \nabla \ln B$
  - Magnetic curvature $\widehat{\mathsf{b}} \cdot \nabla \widehat{\mathsf{b}}$
  - Parallel gradient $\widehat{\mathsf{b}} \cdot \nabla \ln B$
  - Magnetic twist $\widehat{\mathsf{b}} \cdot \nabla \times \widehat{\mathsf{b}}$

Quasi-periodic Hierarchy of Orbital Time Scales

$$\tau_{\rm g} \ll \tau_{\rm b} \ll \tau_{\rm d}$$

- Rapid gyro-motion about single field line: $\tau_{\rm g} \rightarrow J_{\rm g}$

- Intermediate bounce (or transit) motion along field line (parallel gradient): $\tau_{\rm b} \rightarrow J_{\rm b}$

- Slow drift (bounce-averaged precession) motion across field lines (perpendicular gradient & magnetic curvature): $\tau_{\rm d} \rightarrow J_{\rm d}$

Example: 100 MeV proton orbiting in geomagnetic dipole field at $2\,r_{\rm E}$

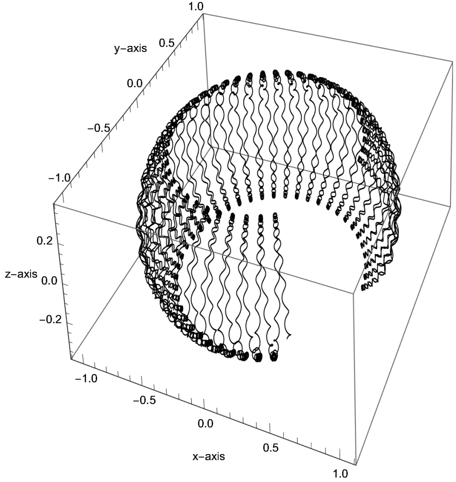

Figure: Normalized 3D Orbit ($\Omega_{\rm e}^{-1} d/dt = \epsilon\, d/d\tau$ with $\epsilon = 1/50$):
$\tau_{\rm g} \simeq \epsilon\,\tau_{\rm b} \ll \tau_{\rm b} \ll \tau_{\rm d} \simeq \tau_{\rm b}/\epsilon$.

Dynamical Reduction

$$8 \to 6\,(\text{guiding-center}) \to 4\,(\text{bounce-center}) \to 2\,(\text{drift-center})$$

$$(\mathbf{x}, \mathbf{p}; w, t) \xrightarrow{\rm g} \begin{cases} (\mathbf{X}, p_{\|}; W, t) \xrightarrow{\rm b} \begin{cases} (\alpha, \beta; k, t) \xrightarrow{\rm d} \begin{cases} (K, t) \\ (J_{\rm d}, \zeta_{\rm d}) \end{cases} \\ (J_{\rm b}, \zeta_{\rm b}) \end{cases} \\ (J_{\rm g}, \zeta_{\rm g}) \end{cases}$$

- Dynamical reduction by extended phase-space transformation (which includes the time-energy canonical pair (w, t)):
  $\Rightarrow$ asymptotic elimination of a fast orbital angle $\zeta_\ell = (\zeta_{\rm g}, \zeta_{\rm b}, \zeta_{\rm d})$
  $\Rightarrow$ construction of an adiabatic invariant $J_\ell = (J_{\rm g}, J_{\rm b}, J_{\rm d})$.

- Condition for adiabatic invariance $\langle dJ_\ell/dt \rangle_\ell \equiv 0$.

Lagrangian Reduction $\rightarrow$ Reduced Hamiltonian Dynamics

- Particle phase-space Lagrangian (σ = orbit parameter)

$$L(\mathbf{x}, t; \mathbf{p}, w; \Phi, \mathbf{A}) \;=\; \left(\frac{q}{c}\mathbf{A} + \mathbf{p}\right)\cdot\frac{d\mathbf{x}}{d\sigma} - w\,\frac{dt}{d\sigma} - \left(q\,\Phi \;+\; K \;-\; w\right)$$

  - Euler-Poincaré one-form: $L\,d\sigma \rightarrow \gamma = \gamma_\alpha\,\mathrm{d}z^\alpha - H\,\mathrm{d}\sigma$

  - Symplectic part $\gamma_\alpha\,\mathrm{d}z^\alpha \rightarrow$ Poisson bracket $\{\;,\;\}$

$$\boldsymbol{\omega} \equiv \mathrm{d}\gamma = \frac{1}{2}\,\omega_{\alpha\beta}\,\mathrm{d}z^\alpha \wedge \mathrm{d}z^\beta \;\;\rightarrow\;\; \mathbb{J} \equiv \boldsymbol{\omega}^{-1} : \mathbb{J}^{\alpha\beta} \equiv \left\{z^\alpha,\; z^\beta\right\}$$

  - Euler-Lagrange equations $\rightarrow$ Hamilton equations

$$\omega_{\alpha\beta}\,\frac{dz^\beta}{d\sigma} \;=\; \frac{\partial H}{\partial z^\alpha} \;\;\rightarrow\;\; \frac{dz^\alpha}{d\sigma} \;=\; \mathbb{J}^{\alpha\beta}\,\frac{\partial H}{\partial z^\beta} \;\equiv\; \left\{z^\alpha,\; H\right\}$$

- Dynamical reduction by phase-space transformation

$$\mathbf{z} \rightarrow \overline{\mathbf{z}} = \overline{\mathsf{T}}\mathbf{z} \;=\; (\overline{\mathbf{x}}, \overline{\mathbf{p}}, \overline{w}, t) \;\equiv\; \left(\overline{Z}^{a}; \overline{J}, \overline{\zeta}\right)$$

- ◦ Reduced Lagrangian: $\gamma \rightarrow \overline{\gamma} \equiv \overline{\mathsf{T}}^{-1}\gamma + \mathsf{d}S$

$$\begin{aligned} \overline{L}(\overline{\mathbf{z}}; \Phi, \mathbf{A}; \mathbf{E}, \mathbf{B}) &= \overline{\gamma}_{\alpha}\,\frac{d\overline{z}^{\alpha}}{d\sigma} \;-\; \overline{H}(\overline{\mathbf{p}}; \Phi, \mathbf{A}, \mathbf{E}, \mathbf{B}) \\ &\equiv \overline{P}_{a}\,\frac{d\overline{Z}^{a}}{d\sigma} \;+\; \overline{J}\,\frac{d\overline{\zeta}}{d\sigma} \;-\; \overline{H}(\overline{Z}^{a}, \overline{J}; \Phi, \mathbf{A}, \mathbf{E}, \mathbf{B}) \end{aligned}$$

- ◦ Reduced action invariance:

$$\overline{\zeta} + \delta\overline{\zeta} \;\rightarrow\; \frac{d\overline{J}}{d\sigma} \;=\; -\,\frac{\partial\overline{H}}{\partial\overline{\zeta}} \;\equiv\; 0$$

- ◦ Reduced Hamiltonian dynamics on $\overline{\mathbf{Z}}$-space ($\overline{\boldsymbol{\omega}} = \mathsf{d}\overline{\gamma} = \overline{\mathsf{T}}^{-1}\boldsymbol{\omega}$)

$$\overline{\boldsymbol{\omega}}^{-1} = \overline{\mathbb{J}} : \;\; \overline{\mathbb{J}}^{ab} \equiv \overline{\left\{\overline{Z}^{a},\; \overline{Z}^{b}\right\}} \;\rightarrow\; \frac{d\overline{Z}^{a}}{d\sigma} \;=\; \overline{\left\{\overline{Z}^{a},\; \overline{H}\right\}}$$

Electromagnetic Potentials and Fields: Gauge Freedom

- Gauge transformation

$$(\Phi, \mathbf{A}) \;\rightarrow\; \left(\Phi \;-\; \frac{1}{c}\,\frac{\partial\chi}{\partial t},\; \mathbf{A} \;+\; \nabla\chi\right)$$

- Gauge invariance of the Reduced Lagrangian density $\overline{\mathcal{L}}$

$$\frac{\partial}{\partial t}\left(\frac{\partial\overline{\mathcal{L}}}{\partial(\partial\chi/\partial t)}\right) \;+\; \nabla\cdot\left(\frac{\partial\overline{\mathcal{L}}}{\partial(\nabla\chi)}\right) \;=\; \frac{\partial\overline{\mathcal{L}}}{\partial\chi} \;\equiv\; 0$$

- Gauge freedom $\Leftrightarrow$ Charge conservation law

$$-\,\frac{1}{c}\,\frac{\partial}{\partial t}\left(\frac{\partial\overline{\mathcal{L}}}{\partial\Phi}\right) \;+\; \nabla\cdot\frac{\partial\overline{\mathcal{L}}}{\partial\mathbf{A}} \;=\; 0$$

- Minimal coupling in particle Lagrangian

$$\frac{q}{c}\,\mathbf{A}\cdot\frac{d\mathbf{x}}{dt} \;-\; q\,\Phi \;\;\Leftrightarrow\;\; L \rightarrow L + \frac{q}{c}\,\frac{d\chi}{dt}$$

Structure of a Gauge-free Reduced Vlasov-Maxwell Theory

Reduced canonical Hamiltonian formulation

$$\overline{H}(\overline{\mathbf{p}}, \overline{w}; \Phi, \mathbf{A}, \mathbf{E}, \mathbf{B}) \;\equiv\; m\,|\overline{\mathbf{v}}|^2/2 \;+\; q\,\Phi \;-\; \overline{w} \;+\; \overline{\Psi}(\overline{\mathbf{v}}; \mathbf{E}, \mathbf{B})$$

- ○ Gauge-free reduced velocity $\overline{\mathbf{v}} \equiv [\overline{\mathbf{p}} - (q/c)\mathbf{A}]/m$
- ○ Potentials and fields $(\Phi, \mathbf{A}, \mathbf{E}, \mathbf{B})$ are evaluated at $\overline{\mathbf{x}}$
- ○ Reduced ponderomotive Hamiltonian $\overline{\Psi}(\overline{\mathbf{v}}; \mathbf{E}, \mathbf{B})$.

- Canonical reduced Hamilton equations

$$d\overline{\mathbf{x}}/dt = \partial\overline{H}/\partial\overline{\mathbf{p}} \;\text{ and }\; d\overline{\mathbf{p}}/dt = -\,\overline{\nabla}H$$

- ○ Reduced force equation

$$m\,\frac{d\overline{\mathbf{v}}}{dt} \;=\; q\,\mathbf{E} \;+\; \frac{q}{c}\,\frac{d\overline{\mathbf{x}}}{dt} \times \mathbf{B} \;+\; \overline{\nabla}\mathbf{E}\cdot\overline{\boldsymbol{\pi}} + \overline{\nabla}\mathbf{B}\cdot\overline{\boldsymbol{\mu}}$$

- ○ Reduced electric and magnetic dipole moments

$$(\overline{\boldsymbol{\pi}}, \overline{\boldsymbol{\mu}}) \equiv (-\,\partial\overline{\Psi}/\partial\mathbf{E}, -\,\partial\overline{\Psi}/\partial\mathbf{B})$$

Reduced Vlasov-Maxwell Equations

- Reduced Vlasov equation

$$\frac{\partial \overline{f}}{\partial t} = -\frac{d\overline{\mathbf{x}}}{dt}\cdot\overline{\nabla f} - \frac{d\overline{\mathbf{p}}}{dt}\cdot\frac{\partial \overline{f}}{\partial \overline{\mathbf{p}}}$$

- Reduced Maxwell equations

$$\begin{aligned}
\nabla\cdot\mathbf{E} &= 4\pi\,\varrho \equiv 4\pi\left(\overline{\varrho} - \nabla\cdot\overline{\mathbb{P}}\right) \\
\nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} &= \frac{4\pi}{c}\,\mathbf{J} \equiv \frac{4\pi}{c}\left(\overline{\mathbf{J}} + \frac{\partial\overline{\mathbb{P}}}{\partial t} + c\,\nabla\times\overline{\mathbb{M}}\right)
\end{aligned}$$

- Reduced charge-current densities & Polarization-Magnetization

$$\left(\overline{\varrho}, \overline{\mathbf{J}}, \overline{\mathbb{P}}, \overline{\mathbb{M}}\right) \equiv \int_{\overline{\mathbf{p}}} \overline{f}\left(q,\; q\,\frac{d\overline{\mathbf{x}}}{dt},\; -\frac{\partial\overline{\Psi}}{\partial\mathbf{E}},\; -\frac{\partial\overline{\Psi}}{\partial\mathbf{B}}\right),$$

- Reduced Maxwell fields

$$\left.\begin{array}{r} \overline{\mathbb{D}} = \mathbf{E} + 4\pi\,\overline{\mathbb{P}} \\ \overline{\mathbb{H}} = \mathbf{B} - 4\pi\,\overline{\mathbb{M}} \end{array}\right\} \rightarrow \left\{ \begin{array}{l} \nabla\cdot\overline{\mathbb{D}} = 4\pi\,\overline{\varrho} \\ c\,\nabla\times\overline{\mathbb{H}} - \partial\overline{\mathbb{D}}/\partial t = 4\pi\,\overline{\mathbf{J}} \end{array}\right.$$

- Dynamical reduction guarantees reduced charge conservation law

$$\begin{aligned} \frac{\partial\varrho}{\partial t} + \nabla\cdot\mathbf{J} &= \frac{\partial}{\partial t}\left(\overline{\varrho} \;-\; \nabla\cdot\overline{\mathbb{P}}\right) + \nabla\cdot\left(\overline{\mathbf{J}} + \frac{\partial\overline{\mathbb{P}}}{\partial t} + c\,\nabla\times\overline{\mathbb{M}}\right) \\ &= \frac{\partial\overline{\varrho}}{\partial t} \;+\; \nabla\cdot\overline{\mathbf{J}} \;=\; 0 \end{aligned}$$

Reduced energy-momentum conservation laws

- Reduced Noether equation

$$\frac{\partial}{\partial t}\left(\overline{\mathcal{P}}\cdot\delta\mathbf{x} \;-\; \overline{\mathcal{E}}\,\delta t\right) \;+\; \nabla\cdot\left(\overline{\mathsf{T}}\cdot\delta\mathbf{x} \;-\; \overline{\mathbf{S}}\,\delta t\right) \;=\; 0$$

- Reduced energy-momentum densities

$$\overline{\mathcal{E}} \;=\; \int_{\overline{\mathbf{p}}} \overline{f}\,\overline{K} \;+\; \frac{1}{4\pi}\,\mathbf{E}\cdot\overline{\mathbb{D}} \;-\; \frac{1}{8\pi}\left(|\mathbf{E}|^2 \;-\; |\mathbf{B}|^2\right)$$

$$\overline{\mathcal{P}} \;=\; \int_{\overline{\mathbf{p}}} \overline{f}\,\frac{d\overline{\mathbf{x}}}{dt}\,m\,\overline{\mathbf{v}} \;+\; \frac{\overline{\mathbb{D}}\times\mathbf{B}}{4\pi\,c}$$

- Reduced energy-density flux

$$\overline{\mathbf{S}} \;=\; \int_{\overline{\mathbf{p}}} \overline{f}\,\frac{d\overline{\mathbf{x}}}{dt}\,\overline{K} \;+\; \frac{c}{4\pi}\,\mathbf{E}\times\overline{\mathbb{H}}$$

◦ Reduced stress tensor → **Not Manifestly Symmetric**

$$\overline{\mathsf{T}} \;=\; \int_{\overline{\mathbf{p}}} \overline{f}\,\frac{d\overline{\mathbf{x}}}{dt}\; m\,\overline{\mathbf{v}} + \frac{\mathbf{I}}{4\pi}\left[\frac{1}{2}\left(|\mathbf{E}|^2 \;-\; |\mathbf{B}|^2|\right) + \;\mathbf{B}\cdot\overline{\mathbb{H}}\right] - \frac{1}{4\pi}\left(\mathbf{B}\,\overline{\mathbb{H}} \;+\; \overline{\mathbb{D}}\,\mathbf{E}\right)$$

◦ Apparent asymmetry of the reduced stress tensor implies that the azimuthal angular momentum may not be conserved:

$$\frac{\partial\overline{\mathcal{P}}_{\varphi}}{\partial t} \;+\; \nabla\cdot\overline{\mathbf{T}}_{\varphi} \;=\; \overline{\mathsf{T}}^{\top} : \nabla(\partial\mathbf{x}/\partial\varphi) \;\equiv\; \widehat{\mathsf{z}}\cdot\left(\int_{\mathbf{p}} \overline{f}\;\overline{\mathbf{N}}\right)$$

unless the reduced torque $\overline{\mathbf{N}}$ vanishes identically:

$$\overline{\mathbf{N}} \;\equiv\; \frac{d\overline{\mathbf{x}}}{dt}\times m\,\overline{\mathbf{v}} \;-\; \left(\overline{\boldsymbol{\pi}}\times\mathbf{E} \;+\; \overline{\boldsymbol{\mu}}\times\mathbf{B}\right)$$

Ponderomotive & polarization-magnetization conspire to produce a symmetric reduced stress tensor

Guiding-center dynamics with higher-order corrections

- Guiding-center Lagrangian (Tronko & Brizard, 2015)

$$L_{\rm gc} = \left[\frac{q}{c\epsilon} \mathbf{A} + p_{\|} \,\widehat{\rm b} - \epsilon\, J \left(\mathbf{R} + \frac{1}{2} \nabla \times \widehat{\rm b} \right) \right] \cdot \dot{\mathbf{X}} + \epsilon\, J\, \dot{\zeta} - \left(q\, \Phi + \frac{p_{\|}^2}{2m} + \mu\, B \right)$$

- Gyrogauge invariance $\mathbf{R} = \nabla\widehat{1}\cdot\widehat{2}$ and polarization $\frac{1}{2}\nabla \times \widehat{\rm b}$

- Guiding-center magnetic-moment adiabatic invariant (for magnetic dipole field)

$$\mu = J\,\frac{\Omega}{B} = \mu_0 + \epsilon \left[\left(\mu_0 + \frac{p_{\|}^2}{2\,mB} \right) \widehat{\rm b}\cdot\nabla\widehat{\rm b} + \frac{p_{\|}}{2B}\,\frac{d\widehat{\rm b}}{dt} \right] \cdot \boldsymbol{\rho}_0 + \cdots$$

100 MeV proton orbiting in geomagnetic dipole field at $2\,r_{\mathrm{E}}$
(Brizard & Markowski, 2021)

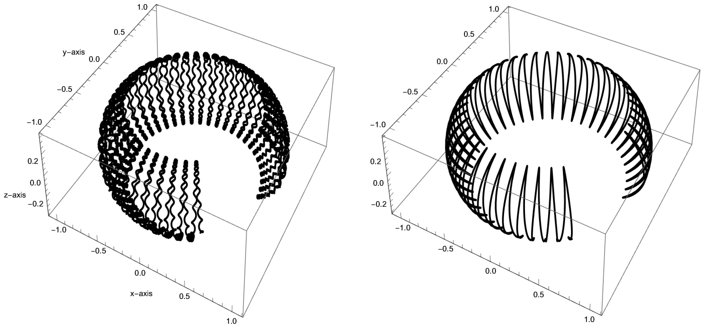

Figure: Normalized 3D Orbits ($\Omega_{\mathrm{e}}^{-1} d/dt = \epsilon\, d/d\tau$ with $\epsilon = 1/50$): exact particle orbit (left) and guiding-center orbit (right).

Magnetic-moment adiabatic invariance & Azimuthal angular canonical momentum

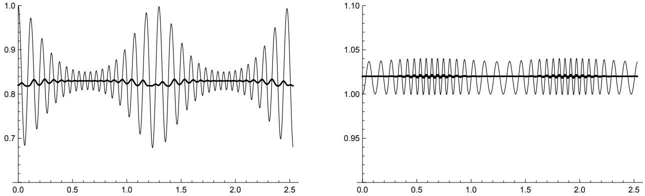

Figure: (Left) Normalized magnetic moment: μ_0 (light) $\mu_0 + \epsilon\,\mu_1$ (dark). (Right) Normalized azimuthal canonical momentum: $\psi/B_{\rm e} r_{\rm e}^2$ (light) guiding-center pullback $\mathsf{T}_{\rm gc} P_{{\rm gc}\varphi}$ (dark) with P_φ (dark horizontal line).

- Axisymmetric dipole field $\rightarrow$ P_φ is an exact invariant

$$\mathsf{T}_{\rm gc} P_{{\rm gc}\varphi} \;=\; \frac{q}{c}\,\mathsf{T}_{\rm gc}\Psi \;-\; \epsilon^2\,J\left(\mathbf{R} \;+\; \frac{1}{2}\,\nabla\times\widehat{\mathsf{b}}\right)\cdot\frac{\partial\mathbf{X}}{\partial\varphi} + \cdots = P_\varphi$$

Lowest-order Guiding-center Lagrangian

(Brizard & Tronci, 2016)

$$L_{\rm gc} = \left(\frac{q}{c}\,\mathbf{A} + p_{\|}\,\widehat{\rm b}\right)\cdot\dot{\mathbf{X}} + J\,\dot{\zeta} - \left(q\,\Phi + \frac{p_{\|}^2}{2m} + \mu\,B\right)$$

- Guiding-center Euler-Lagrange equations

$$\dot{p}_{\|}\,\widehat{\rm b} - \dot{\mathbf{X}}\times q\mathbf{B}^*/c = q\,\mathbf{E}^* \;\;\text{and}\;\; \widehat{\rm b}\cdot\dot{\mathbf{X}} = p_{\|}/m$$

yield the guiding-center equations of motion

$$\dot{\mathbf{X}} \;=\; \frac{p_{\|}}{m}\,\frac{\mathbf{B}^*}{B_{\|}^*} \;+\; \mathbf{E}^*\times\frac{c\widehat{\rm b}}{B_{\|}^*} \;\;\text{and}\;\; \dot{p}_{\|} \;=\; q\,\mathbf{E}^*\cdot\frac{\mathbf{B}^*}{B_{\|}^*}$$

where (Jacobian $\mathcal{J}_{\rm gc}$) $B_{\|}^* \equiv \widehat{\rm b}\cdot\mathbf{B}^*$ and

$$\begin{aligned} q\mathbf{E}^* \;&\equiv\; q\mathbf{E} \;-\; \mu\,\nabla B \;-\; p_{\|}\,\partial\widehat{\rm b}/\partial t \\ q\mathbf{B}^*/c \;&\equiv\; q\mathbf{B}/c \;+\; p_{\|}\,\nabla\times\widehat{\rm b} \end{aligned}$$

Symmetric guiding-center stress tensor

- Guiding-center canonical momentum $\overline{\mathbf{p}} = (e/c)\,\mathbf{A} + \overline{p}_{\|}\,\widehat{\mathsf{b}}$
- Guiding-center electric and magnetic dipole moments

$$\begin{aligned} \overline{\boldsymbol{\pi}} &= (e\widehat{\mathsf{b}}/\Omega) \times d\overline{\mathbf{x}}/dt \;=\; (e\widehat{\mathsf{b}}/\Omega) \times \partial\overline{\Psi}/\partial\overline{\mathbf{p}} \\ \overline{\boldsymbol{\mu}} &= -\,\overline{\mu}\,\widehat{\mathsf{b}} \;+\; \overline{\boldsymbol{\pi}} \times (\overline{p}_{\|}\widehat{\mathsf{b}}/mc) \end{aligned}$$

- Guiding-center torque

$$\begin{aligned} \overline{\mathbf{N}} &= \frac{d\overline{\mathbf{x}}}{dt} \times m\,\overline{\mathbf{v}} \;-\; \left(-\,\overline{\mu}\,\widehat{\mathsf{b}} + \overline{\boldsymbol{\pi}} \times \frac{\overline{p}_{\|}\,\widehat{\mathsf{b}}}{mc} \right) \times \mathbf{B} \\ &= \left(-\,m\,\overline{\mathbf{v}} \;+\; \overline{p}_{\|}\,\widehat{\mathsf{b}} \right) \times \frac{d\overline{\mathbf{x}}}{dt} \;\equiv\; 0 \end{aligned}$$

- Guiding-center stress tensor

$$\mathsf{T}_{\rm gc} \;=\; \frac{1}{4\pi}\left(\frac{\mathbf{I}}{2}\,|\mathbf{B}|^2 - \mathbf{B}\,\mathbf{B} \right) + \mathsf{P}_{\rm CGL} + \int_{\overline{\mathbf{p}}} \overline{f}\;\left[\overline{p}_{\|} \left(\frac{\partial\overline{\Psi}}{\partial\overline{\mathbf{p}}}\,\widehat{\mathsf{b}} + \widehat{\mathsf{b}}\,\frac{\partial\overline{\Psi}}{\partial\overline{\mathbf{p}}} \right) \right],$$

where $\mathsf{P}_{\rm CGL} = \int_{\overline{\mathbf{p}}} \overline{f}[(\overline{p}_{\|}^2/m)\widehat{\mathsf{b}}\widehat{\mathsf{b}} + \overline{\mu}B\,(\mathbf{I} - \widehat{\mathsf{b}}\widehat{\mathsf{b}})]$

Gauge-free Gyrokinetic Vlasov-Maxwell Theory

◦ Dynamical Reduction: Particle → Guiding-center → Gyrocenter

Gyrocenter Lagrangian (derived from gyrokinetic orderings)

$$L_{\rm gy} \equiv \left(\frac{q}{c}\,\mathbf{A}_0^* + p_\|\,\widehat{\sf b}_0 + \mathbf{\Pi}_{\rm gy} \right)\cdot \dot{\mathbf{X}} + J\,\dot{\zeta} - \left(\frac{p_\|^2}{2m} + \mu\,B_0 + q\,\Psi_{\rm gy} \right)$$

◦ Unperturbed background magnetic vector potential $\mathbf{A}_0^*$ contains higher-order guiding-center corrections

◦ Gyrocenter symplectic momentum $\mathbf{\Pi}_{\rm gy} = \epsilon\,\mathbf{\Pi}_{1\rm gy} + \cdots$ and the gyrocenter potential $\Psi_{\rm gy} = \epsilon\,\Psi_{1\rm gy} + \cdots$ may include first-order electromagnetic potential perturbations $(\Phi_1, \mathbf{A}_1)$ and field perturbations $(\mathbf{E}_1, \mathbf{B}_1)$, which are selected on the basis of specific theoretical or numerical considerations.

Symplectic gyrocenter Hamilton equations

$$\dot{\mathbf{X}} = \frac{c\mathsf{b}^*_{\rm gy}}{q\mathcal{J}_{\rm gy}} \times \left(\nabla H_{\rm gy} + \frac{\partial \boldsymbol{\Pi}_{\rm gy}}{\partial t}\right) + \frac{\mathbf{B}^*_{\rm gy}}{\mathcal{J}_{\rm gy}}\,\frac{\partial H_{\rm gy}}{\partial p_\parallel}$$

$$\dot{p}_\parallel = -\,\frac{\mathbf{B}^*_{\rm gy}}{\mathcal{J}_{\rm gy}} \cdot \left(\nabla H_{\rm gy} + \frac{\partial \boldsymbol{\Pi}_{\rm gy}}{\partial t}\right)$$

where $\mathcal{J}_{\rm gy} \equiv \mathsf{b}^*_{\rm gy} \cdot \mathbf{B}^*_{\rm gy}$ and

$$\mathsf{b}^*_{\rm gy} \equiv \widehat{\mathsf{b}}_0 + \partial \boldsymbol{\Pi}_{\rm gy}/\partial p_\parallel$$

$$\mathbf{B}^*_{\rm gy} \equiv \mathbf{B}^*_0 + (c/q)\,\nabla \times (p_\parallel\,\widehat{\mathsf{b}}_0 + \boldsymbol{\Pi}_{\rm gy})$$

- We construct a gyrocenter transformation that produces gauge-free equations of motion (see Burby & Brizard, 2019)

- Guiding-center push-forward of perturbed electromagnetic fields

$$(\mathbf{E}_{1\mathrm{gc}},\ \mathbf{B}_{1\mathrm{gc}}) \ \equiv\ \left(\mathsf{T}_{\mathrm{gc}}^{-1}\mathbf{E}_1,\ \mathsf{T}_{\mathrm{gc}}^{-1}\mathbf{B}_1\right) = \left(\mathbf{E}_1(\mathbf{X}+\boldsymbol{\rho}_0),\ \mathbf{B}_1(\mathbf{X}+\boldsymbol{\rho}_0)\right)$$

- First-order gyrocenter symplectic momentum & Hamiltonian (Brizard, *Symplectic gyrokinetic Vlasov-Maxwell theory*, 2020)

$$\begin{aligned} \mathbf{\Pi}_{1\mathrm{gy}} &= \frac{q}{c}\langle\mathbf{A}_{1\mathrm{gc}}\rangle \ +\ \left(\langle\mathbf{E}_{1\mathrm{gc}}\rangle + \frac{p_\parallel\widehat{\mathsf{b}}_0}{mc}\times\langle\mathbf{B}_{1\mathrm{gc}}\rangle\right)\times\frac{q\widehat{\mathsf{b}}_0}{\Omega_0} \\ q\,\Psi_{1\mathrm{gy}} &= q\,\langle\Phi_{1\mathrm{gc}}\rangle \ +\ \mu\,\langle\langle B_{1\|\mathrm{gc}}\rangle\rangle \end{aligned}$$

- Second-order gyrocenter Hamiltonian (zero-Larmor-radius limit)

$$H_{2\mathrm{gy}}^{\mathrm{ZLR}} \ =\ \frac{mc^2}{2\,B_0^2}\left|\mathbf{E}_1 \ +\ \frac{p_\parallel\widehat{\mathsf{b}}_0}{mc}\times\mathbf{B}_1\right|^2 \ +\ \frac{\mu\,|\mathbf{B}_1|^2}{2\,B_0} \ \equiv\ K_{2\mathrm{gy}}$$

- First-order gyrocenter polarization and magnetization

$$\frac{\partial K_{2\mathrm{gy}}}{\partial \mathbf{E}_1} = \frac{mc^2}{B_0^2}\left(\mathbf{E}_1 + \frac{p_\| \widehat{\mathsf{b}}_0}{mc} \times \mathbf{B}_1\right)$$

$$\frac{\partial K_{2\mathrm{gy}}}{\partial \mathbf{B}_1} = \mu\,\frac{\mathbf{B}_1}{B_0} + \frac{mc^2}{B_0^2}\left(\mathbf{E}_1 + \frac{p_\| \widehat{\mathsf{b}}_0}{mc} \times \mathbf{B}_1\right) \times \frac{p_\| \widehat{\mathsf{b}}_0}{mc}$$

$$= \mu\,\frac{\mathbf{B}_1}{B_0} + \frac{\partial K_{2\mathrm{gy}}}{\partial \mathbf{E}_1} \times \frac{p_\| \widehat{\mathsf{b}}_0}{mc}$$

- Gyrocenter magnetization includes intrinsic magnetic-dipole and moving electric-dipole contributions.

Gauge-free gyrocenter equations of motion $(\dot{\mathbf{X}}, \dot{p}_\parallel)$

$$
\begin{aligned}
\mathbf{P}_{1\mathrm{gy}} &= \left(\langle \mathbf{E}_{1\mathrm{gc}} \rangle + \frac{p_\parallel \widehat{\mathsf{b}}_0}{mc} \times \langle \mathbf{B}_{1\mathrm{gc}} \rangle \right) \times \frac{q \widehat{\mathsf{b}}_0}{\Omega_0} \\
\mathsf{b}^*_{\mathrm{gy}} &\equiv \widehat{\mathsf{b}}_0 \;+\; \epsilon\, \partial \mathbf{P}_{1\mathrm{gy}} / \partial p_\parallel \\
&= \widehat{\mathsf{b}}_0 \;+\; \epsilon \left(\widehat{\mathsf{b}}_0 \times \langle \mathbf{B}_{1\mathrm{gc}} \rangle \right) \times \widehat{\mathsf{b}}_0 / B_0 \\
\mathbf{B}^*_{\mathrm{gy}} &\equiv \mathbf{B}^*_0 \;+\; \epsilon \,\langle \mathbf{B}_{1\mathrm{gc}} \rangle \;+\; (c/q)\, \nabla \times \left(p_\parallel \widehat{\mathsf{b}}_0 + \epsilon\, \mathbf{P}_{1\mathrm{gy}} \right) \\
\nabla H_{\mathrm{gy}} + \frac{\partial \mathbf{\Pi}_{\mathrm{gy}}}{\partial t} &= \nabla K_{\mathrm{gy}} \;-\; \epsilon\, \langle \mathbf{E}_{1\mathrm{gc}} \rangle \;+\; \epsilon\, \partial \mathbf{P}_{1\mathrm{gy}} / \partial t
\end{aligned}
$$

Gyrokinetic Vlasov equation

$$
\frac{\partial F}{\partial t} \;+\; \dot{\mathbf{X}} \cdot \nabla F \;+\; \dot{p}_\parallel \,\frac{\partial F}{\partial p_\parallel} \;=\; 0
$$

Gyrokinetic Maxwell equations

$$\nabla \cdot \epsilon\, \mathbf{E}_1 = 4\pi \left(\varrho_{\rm gy} - \nabla \cdot \mathbb{P}_{\rm gy} \right)$$

$$\nabla \times (\mathbf{B}_0 + \epsilon\, \mathbf{B}_1) - \frac{\epsilon}{c} \frac{\partial \mathbf{E}_1}{\partial t} = \frac{4\pi}{c} \left(\mathbf{J}_{\rm gy} + \frac{\partial \mathbb{P}_{\rm gy}}{\partial t} + c\, \nabla \times \mathbb{M}_{\rm gy} \right)$$

- Gyrocenter polarization and magnetization

$$\mathbb{P}_{\rm gy} \equiv \int_{\mathcal{Z}} \mathcal{J}_{\rm gy}\, F \left(\langle \delta^3(\mathbf{X} + \boldsymbol{\rho}_0 - \mathbf{x}) \rangle\, \boldsymbol{\pi}_{\rm gy} - \epsilon\, \delta^3\, \frac{\partial K_{2\rm gy}}{\partial \mathbf{E}_1} \right)$$

$$\mathbb{M}_{\rm gy} \equiv \int_{\mathcal{Z}} \mathcal{J}_{\rm gy}\, F \left[-\mu \left(\widehat{\mathsf{b}}_0 \, \langle\langle \delta^3(\mathbf{X} + \boldsymbol{\rho}_0 - \mathbf{x}) \rangle\rangle + \epsilon\, \delta^3\, \frac{\mathbf{B}_1}{B_0} \right) + \left(\langle \delta^3(\mathbf{X} + \boldsymbol{\rho}_0 - \mathbf{x}) \rangle\, \boldsymbol{\pi}_{\rm gy} - \epsilon\, \delta^3\, \frac{\partial K_{2\rm gy}}{\partial \mathbf{E}_1} \right) \times \frac{p_{\|} \widehat{\mathsf{b}}_0}{mc} \right]$$

where $\boldsymbol{\pi}_{\rm gy} = (q\widehat{\mathsf{b}}_0/\Omega_0) \times \dot{\mathbf{X}}$ and

$$\left(\frac{\delta \mathbf{P}_{1\rm gy}}{\delta \mathbf{E}_1(\mathbf{x})}, \frac{\delta \mathbf{P}_{1\rm gy}}{\delta \mathbf{B}_1(\mathbf{x})} \right) \equiv \langle \delta^3(\mathbf{X} + \boldsymbol{\rho}_0 - \mathbf{x}) \rangle \left(\boldsymbol{\pi}_{\rm gy},\; \boldsymbol{\pi}_{\rm gy} \times \frac{p_{\|} \widehat{\mathsf{b}}_0}{mc} \right)$$

Gyrokinetic energy-momentum conservation laws

(Brizard, 2021)

- Gyrokinetic energy conservation law $\partial\mathcal{E}_{\rm gy}/\partial t + \nabla\cdot\mathbf{S}_{\rm gy} = 0$

  ◦ Gyrokinetic energy density

$$\begin{aligned}\mathcal{E}_{\rm gy} &= \int_{\mathbf{P}} \mathcal{J}_{\rm gy}\, F\, K_{\rm gy} + \frac{\epsilon}{4\pi}\,\mathbf{E}_1\cdot\mathbb{D}_{\rm gy} - \frac{1}{8\pi}\left(\epsilon^2|\mathbf{E}_1|^2 - |\mathbf{B}|^2\right) \\ &= \int_{\mathbf{P}} \mathcal{J}_{\rm gy}\, F\left[\frac{p_\parallel^2}{2m} + \mu\left(B_0 + \epsilon\,\langle\langle B_{1\parallel{\rm gc}}\rangle\rangle + \frac{\epsilon^2}{2}\,\frac{|\mathbf{B}_1|^2}{B_0}\right)\right. \\ &\quad \left. + \mathbf{E}_1(\mathbf{x})\cdot\left(\frac{\delta\mathbf{\Pi}_{\rm gy}}{\delta\mathbf{E}_1(\mathbf{x})} - \frac{\delta K_{\rm gy}}{\delta\mathbf{E}_1(\mathbf{x})}\right)\right] + \frac{1}{8\pi}\left(\epsilon^2|\mathbf{E}_1|^2 + |\mathbf{B}|^2\right)\end{aligned}$$

  ◦ Gyrokinetic energy-density flux

$$\mathbf{S}_{\rm gy} = \int_{\mathbf{P}} \mathcal{J}_{\rm gy}\, F\, \dot{\mathbf{X}}\, K_{\rm gy} + \frac{c}{4\pi}\,\epsilon\,\mathbf{E}_1\times\mathbb{H}_{\rm gy}$$

- Gyrokinetic Noether momentum equation

$$\frac{\partial \boldsymbol{\mathcal{P}}^{*}_{\rm gy}}{\partial t} + \nabla\cdot\mathsf{T}^{*}_{\rm gy} \;=\; \int_{\mathbf{P}} \mathcal{J}_{\rm gy} F \left[\left(\frac{q}{c}\nabla\mathbf{A}_0^{*} \;+\; \nabla\boldsymbol{\Pi}_{\rm gy}\right)\cdot\dot{\mathbf{X}} \;-\; \nabla K_{\rm gy}\right] - \epsilon\left(\nabla\mathbf{E}_1\cdot\mathbb{P}_{\rm gy} \;+\; \nabla\mathbf{B}_1\cdot\mathbb{M}_{\rm gy}\right) \;-\; \nabla\mathbf{B}_0\cdot\frac{\mathbf{B}}{4\pi}$$

  - Gyrokinetic canonical momentum density

$$\boldsymbol{\mathcal{P}}^{*}_{\rm gy} \;=\; \int_{\mathbf{P}} \mathcal{J}_{\rm gy}\, F\,\left(\frac{q}{c}\,\mathbf{A}_0^{*} \;+\; \boldsymbol{\Pi}_{\rm gy}\right) \;+\; \frac{\mathbb{D}_{\rm gy}}{4\pi c}\times\epsilon\,\mathbf{B}_1$$

  - Gyrokinetic canonical stress tensor (not manifestly symmetric)

$$\mathsf{T}^{*}_{\rm gy} \;=\; \int_{\mathbf{P}} \mathcal{J}_{\rm gy}\, F\,\dot{\mathbf{X}}\,\left(\frac{e}{c}\,\mathbf{A}_0^{*} \;+\; \boldsymbol{\Pi}_{\rm gy}\right) \;-\; \frac{\epsilon}{4\pi}\left(\mathbb{D}_{\rm gy}\,\mathbf{E}_1 \;+\; \mathbf{B}_1\,\mathbb{H}_{\rm gy}\right) + \mathbf{I}\left[\frac{1}{8\pi}\left(\epsilon^2\,|\mathbf{E}_1|^2 \;-\; |\mathbf{B}|^2\right) + \frac{\epsilon}{4\pi}\,\mathbf{B}_1\cdot\mathbb{H}_{\rm gy}\right]$$

Gyrokinetic angular-momentum conservation law

- Axisymmetric background magnetic field $\mathbf{B}_0 \;\rightarrow$ Gyrokinetic canonical angular-momentum conservation law

$$
\begin{aligned}
\frac{\partial \mathcal{P}^{*}_{\mathrm{gy}\varphi}}{\partial t} + \nabla\cdot\left(\mathsf{T}^{*}_{\mathrm{gy}}\cdot\frac{\partial \mathbf{x}}{\partial \varphi}\right) \\
= \; \mathsf{T}^{*\top}_{\mathrm{gy}} : \nabla\left(\frac{\partial \mathbf{x}}{\partial \varphi}\right) - \frac{\partial \mathbf{B}_0}{\partial \varphi}\cdot\frac{\mathbf{B}}{4\pi} \\
+ \int_{\mathbf{P}} \mathcal{J}_{\mathrm{gy}}\, F\left(\frac{e}{c}\,\frac{\partial \mathbf{A}^{*}_{0}}{\partial \varphi}\cdot\dot{\mathbf{X}} + \frac{\partial' \mathbf{\Pi}_{\mathrm{gy}}}{\partial \varphi}\cdot\dot{\mathbf{X}} - \frac{\partial' K_{\mathrm{gy}}}{\partial \varphi}\right)
\end{aligned}
$$

where $\partial'/\partial\varphi$ denotes toroidal derivatives of background vector fields (i.e., $\partial \mathbf{B}_0/\partial\varphi = \widehat{\mathsf{z}}\times\mathbf{B}_0$) appearing in $\mathbf{\Pi}_{\mathrm{gy}}$ and K_{gy}.

Upon using the gyrokinetic Vlasov-Maxwell equations, we obtain

$$\frac{\partial \mathcal{P}^{*}_{\mathrm{gy}\varphi}}{\partial t} + \nabla\cdot\left(\mathsf{T}^{*}_{\mathrm{gy}}\cdot\frac{\partial \mathbf{x}}{\partial \varphi}\right) \;=\; \int_{\mathbf{P}} \mathcal{J}_{\mathrm{gy}}\, F\, \widehat{\mathsf{z}}\cdot\mathbf{N}$$

where the gyrocenter torque vanishes as a result of the Jacobi vector identity for vector fields

$$\mathbf{N} \;\equiv\; \sum_{i=1}^{3}\Big[\mathbf{U}_i\times(\mathbf{V}_i\times\mathbf{W}_i) \;+\; \mathbf{V}_i\times(\mathbf{W}_i\times\mathbf{U}_i) \;+\; \mathbf{W}_i\times(\mathbf{U}_i\times\mathbf{V}_i)\Big] \;\equiv\; 0$$

with

$$\begin{aligned}
(\mathbf{U}_1,\mathbf{V}_1,\mathbf{W}_1) &= \left(\epsilon\langle\mathbf{B}_{1\mathrm{gc}}\rangle,\, \boldsymbol{\pi}_{\mathrm{gy}},\, p_{\|}\widehat{\mathsf{b}}_0/mc\right)\\
(\mathbf{U}_2,\mathbf{V}_2,\mathbf{W}_2) &= \left(\epsilon\mathbf{B}_1,\, -\epsilon\,\boldsymbol{\pi}_2,\, p_{\|}\widehat{\mathsf{b}}_0/mc\right)\\
(\mathbf{U}_3,\mathbf{V}_3,\mathbf{W}_3) &= \left(\dot{\mathbf{X}},\, \epsilon\langle\mathbf{E}_{1\mathrm{gc}}\rangle + p_{\|}\widehat{\mathsf{b}}_0/mc\times\epsilon\langle\mathbf{B}_{1\mathrm{gc}}\rangle,\, e\widehat{\mathsf{b}}_0/\Omega_0\right)
\end{aligned}$$

Summary and Ongoing Work

Nonlinear gyrokinetic Vlasov-Maxwell theory can be constructed in terms of perturbation electric and magnetic fields only.

◦ From a variational (Lagrangian) formulation, exact conservation laws can be derived by Noether method.

◦ Hamiltonian formulations have recently been derived for the guiding-center Vlasov-Maxwell equations and the gyrokinetic Vlasov-Maxwell equations (to be published soon).

A.J. Brizard, *Hamiltonian structure of the guiding-center Vlasov-Maxwell equations*, Phys. Plasmas **28**, 102303 (2021)

Hybrid kinetic-gyrokinetic codes can now be built, in which each particle dynamics (exact or reduced) is expressed in terms of electromagnetic fields only.