Braginskii Equations for Hot Plasmas: Weakly Relativistic Approach

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1 Introduction

- Recently, it was predicted [1,2] that relativistic effects in fluxes of particles and heat in both tokamaks and stellarators are noticeable for T_e of tens keV, i.e. for $T_e \ll m_e c^2$. They appear due to Maxwell–Jüttner distribution function.
- Fusion plasmas means temperatures about 20 – 70 keV. Nevertheless, practically all transport codes in fusion are still non-relativistic.
- The most general and straightforward way to obtain transport and MHD equations with Lorentz invariance is covariant formalism with 4-vectors [3]. However, for plasmas in fusion devices, macroscopic flows are relatively slow and equations can be significantly simplified in non-covariant formulation without reduction of an accuracy.

1. I. Marushchenko *et al.*, PPCF, 55, 085005 (2013); 2. G. Kapper *et al.*, Phys. Plasmas, 25, 122509 (2018); 3. T. Mettens and R. Balescu, Phys. Fluids B, 2, 2076 (1990).

2 Transport Equations in Local Frame

- Relativistic "drifting" with $\mathbf{V}$ Maxwell-Jüttner distribution in weakly relativistic limit:

$$F_{MJ}(u) \simeq \frac{n_e}{\pi^{3/2} p_{te}^3} C_{MJ}(\mu)\, \exp\left[-\mu\left(\gamma - 1 - \frac{V_k u_k}{c^2}\right) - \frac{m_e V^2}{2T_e}\right].$$

Here, $u = v\gamma$, $\gamma_0 \simeq 1 + V^2/2c^2$, $\mu = \dfrac{m_e c^2}{T_e}$, and $C_{MJ}(\mu) = \sqrt{\dfrac{\pi}{2\mu}}\dfrac{\mathrm{e}^{-\mu}}{K_2(\mu)} \simeq 1 - \dfrac{15}{8\mu} + \ldots$

- Below we define relativistic fluxes in the rest frame using definitions given by other authors and adapting it to our notations:

$$\begin{aligned}
&W_e \equiv n_e m_e c^2 \langle \gamma' - 1\rangle, && P^{ei} = m_e c^2 \textstyle\int (\gamma' - 1) C^{ei}(f'_{e0})\mathrm{d}^3u',\\
&q_k = n_e m_e c^2 \langle (\gamma' - 1) v'_k\rangle, && G_k^{ei} = m_e c^2 \textstyle\int v'_k (\gamma' - 1) C^{ei}(f'_e)\mathrm{d}^3u',\\
&n_e m_e \langle u'_k\rangle = \tfrac{q_k}{c^2}, && R_k^{ei} = m_e \textstyle\int u'_k C^{ei}(f'_e)\mathrm{d}^3u',\\
&n_e m_e \langle v'_k u'_j\rangle = p_e\delta_{kj} + \pi_{kj}, && F_{kj}^{ei} = m_e \textstyle\int v'_k u'_j C^{ei}(f'_e)\mathrm{d}^3u'.
\end{aligned}$$

Here, $p_e = n_e T_e$ is a scalar pressure. Internal thermal energy can be also represented as $W_e = (3/2 + \mathcal{R})\, n_e T_e$, which is similar to standard non-relativistic expression, but with relativistic correction $\mathcal{R} = \mu\left(\dfrac{K_3(\mu)}{K_2(\mu)} - 1\right) - \dfrac{5}{2} = \dfrac{15}{8\mu} + \mathcal{O}\left(\dfrac{1}{\mu^2}\right)$.

- Integrating kinetic equation (KE) in local coordinate system, we use a Lorentz invariance of 4-momentum volume: $d^3u/\gamma = d^3u'/\gamma'$. The Lorentz transformation of momentum and energy from local system into the rest frame:

$$u_k = \gamma_0\gamma' V_k + u'_k + (\gamma_0 - 1)\frac{V_k V_j}{V^2} u'_j, \ \text{ with } \gamma = \gamma_0\Big(\gamma' + \frac{V_j u'_j}{c^2}\Big).$$

Formally, these relations are precise, but below we apply it in weakly relativistic approach with respect to V.

- After integrating KE, we obtain the relativistic continuity equation:

$$\frac{\partial}{\partial t}(\gamma_0 n_e) + \frac{\partial}{\partial x_k}(\gamma_0 n_e V_k) = 0.$$

Weakly relativistic expansion $\gamma_0 \simeq 1 - V^2/2c^2$ is supposed here.

- The momentum balance equation in local frame:

$$\frac{\partial}{\partial t}\left[n_e m_e\left(V_k + \frac{1}{\mu}U_k^{(r)}\right)\right] + \frac{\partial}{\partial x_j}\left(\Pi_{kj}^{(0)} + \frac{1}{\mu}\Pi_{kj}^{(r)}\right) = e n_e E_k + \frac{1}{c}[\mathbf{J}\times\mathbf{B}]_k + R_k^{ei} + \frac{1}{\mu}R_k^{(r)}.$$

Here, $\mathbf{J} = e n_e \mathbf{V}$ is electron electric current; $\Pi_{kj}^{(0)} = p_e\delta_{kj} + \pi_{kj} + n_e m_e V_k V_j$ with $\pi_{kj} = n_e m_e\langle v'_k u'_j\rangle - p_e\delta_{kj}$, where $^{(0)}$ is a label for "quasi-classical" term; pure relativistic contributions, labeled by (r), are absent in non-relativistic limit:

$$\begin{aligned}
U_k^{(r)} &= (5/2 + \mathcal{R})\, V_k + \tfrac{1}{p_e}(\pi_{kj}V_j + q_k)\,,\\
\Pi_{kj}^{(r)} &= \tfrac{2}{u_{te}^2}\left[q_k V_j + q_j V_k + (5/2 + \mathcal{R})\, p_e V_k V_j + \tfrac{1}{2}(\pi_{kl}V_j + \pi_{jl}V_k)\, V_l\right],\\
R_k^{(r)} &= \tfrac{3V^2}{u_{te}^2}\left(\tfrac{V_k V_j}{V^2} + \tfrac{1}{3}\delta_{kj}\right) R_j^{ei} + \tfrac{2}{u_{te}^2}\left(V_k P^{ei} + F_{kj}^{ei}\right).
\end{aligned}$$

- The thermal energy balance equation in local frame:

$$\frac{\partial}{\partial t}\left(\mathcal{E}^{(0)} + \frac{1}{\mu}\mathcal{E}^{(r)}\right) + \frac{\partial}{\partial x_k}\left(Q_k^{(0)} + \frac{1}{\mu}Q_k^{(r)}\right) = J_k E_k + P^{ei} + R_k^{ei}V_k + \frac{1}{\mu}P_{ei}^{(r)},$$

with "quasi-classical" terms, which coincide for $1/\mu \to 0$ with non-relativistic ones,

$$\begin{aligned}
\mathcal{E}^{(0)} &= (3/2 + \mathcal{R})\, p_e + n_e\frac{m_e V^2}{2},\\
Q_k^{(0)} &= (5/2 + \mathcal{R})\, p_e V_k + \pi_{kj}V_j + n_e\frac{m_e V^2}{2}V_k,\\
P^{ei} &= P_{(cl)}^{ei} C_{MJ}(\mu)\left(1 + \frac{2}{\mu} + \frac{2}{\mu^2}\right),
\end{aligned}$$

and pure relativistic contributions, absent in non-relativistic limit:

$$\begin{aligned}
\mathcal{E}^{(r)} &= \big(5/2 + \mathcal{R}\big) n_e m_e V^2 + \tfrac{2}{u_{te}^2}\big(\pi_{kj}V_j + 2q_k\big) V_k,\\
Q_k^{(r)} &= \frac{2V^2}{u_{te}^2}\left[(5/2 + \mathcal{R})\, p_e V_k + \frac{1}{2}\left(\frac{V_k V_j}{V^2} + \delta_{kj}\right)\pi_{jl}V_l + \frac{3}{2}\left(\frac{V_k V_j}{V^2} + \tfrac{1}{3}\delta_{kj}\right) q_j\right],\\
P_{ei}^{(r)} &= \tfrac{2V^2}{u_{te}^2}\left[P^{ei} + R_k^{ei}V_k + \tfrac{1}{V^2}\left(G_k^{ei}V_k + V_k V_j F_{kj}^{ei}\right)\right].
\end{aligned}$$

- Formally, the balance equations with a presence of relativistic corrections even in "quasi-classical" terms are identical to standard non-relativistic transport equations. The present form is suitable for implementation in any transport code.

3 Relativistic Momentum Correction

- In this section, following the logic of [4,5], we generalize the momentum correction technique for relativistic approach.
- Solving linearized drift kinetic equation for $\delta f_e = f_e - f_{e0}$,

$$\mathcal{V}(\delta f_e) - C_e^{\mathrm{lin}}(\delta f_e) = -\dot\rho\Big(A_1 + \big(\kappa - 5/2 - \mathcal{R}\big)A_2\Big) f_{e0} - b v_\parallel f_{e0} A_3,$$

where $\kappa = \mu(\gamma - 1)$, $\dot\rho = \mathbf{V}_{dr}\cdot\nabla\rho$, $B = B/B_0$, and thermodynamic forces,

$$A_1 = \frac{p'_e}{p_e} - \frac{e\Phi'}{T_e}, \quad A_2 = \frac{T'_e}{T_e}, \quad A_3 = \frac{e}{T_e}\frac{\langle \mathbf{E}\cdot\mathbf{B}\rangle}{\langle B^2\rangle}B_0,$$

the fluxes of particles, energy and heat, respectively, can be calculated. While

$$\Gamma_k = \int v_k\delta f_e \mathrm{d}^3v \quad \text{and} \quad Q_k = \int m_e c^2(\gamma - 1) v_k \delta f_e \mathrm{d}^3v,$$

the heat flux is $q_k = Q_k - \big(5/2 + \mathcal{R}\big)T_e\Gamma_k = T_e\int\big(\kappa - 5/2 - \mathcal{R}\big)v_k\delta f_e\mathrm{d}^3v$.

- Sonine polynomials $L_n^{3/2}(x^2)$ are perfectly tailored for non-relativistic limit, but in relativistic approach the generalized Laguerre polinomials, $L_n^{3/2+\mathcal{R}}(\kappa)$, are optimal. Considering the parallel fluxes, it is sufficient to account only 1st Legendre harmonic, $\delta f_e = \xi\mathcal{F}_1$ with $\xi = v_\parallel/v$. Then

$$n_e V_\parallel^{(n)} = \int v_\parallel L_n^{3/2+\mathcal{R}}(\kappa)\delta f_e \mathrm{d}^3u \ \text{ with } \ n_e V_\parallel^{(0)} = \Gamma_\parallel \ \text{ and } \ n_e V_\parallel^{(1)} = -q_\parallel/T_e.$$

Representing $\mathcal{F}_1$ and $C_{e,1}(\mathcal{F}_1) = \frac{3}{2}\int_{-1}^{1}\xi C_e(\delta f_e)\mathrm{d}\xi$ as Laguerre series,

$$\mathcal{F}_1 = \frac{mv}{T_e}w(\kappa) f_{e0}\sum_n a_n V_\parallel^{(n)} L_n^{3/2+\mathcal{R}}(\kappa) \ \text{ and } \ C_{e,1} = \frac{mv}{T_e}w(\kappa) f_{e0}\sum_n a_n F_\parallel^{(n)} L_n^{3/2+\mathcal{R}}(\kappa),$$

with weight $w(\kappa) = C_{MJ}^{-1}\kappa^{\mathcal{R}}\gamma\left(\dfrac{2}{\gamma + 1}\right)^{-1}$ and $a_k = \dfrac{\Gamma(5/2 + k)}{\Gamma(5/2 + k + \mathcal{R})}\dfrac{3(2k)!!}{(2k + 3)!!}$,

we get relation between "flow velocities", $V_\parallel^{(n)}$, and collisional "friction forces",

$$F_\parallel^{(n)} = \sum_k a_k c_{nk} V_\parallel^{(k)} \ \text{ with } \ c_{nk} = \tau_{ee}^{-1}\big(M_{nk}^{ee} + N_{nk}^{ee}\big) + \tau_{ei}^{-1}M_{nk}^{ei}.$$

Similar to non-relativistic expressions,

$$\begin{aligned}
\frac{n_a}{\tau_{ab}}M_{nk}^{ab} &= \int \mathrm{d}^3u\ v_\parallel w(\kappa) L_n^{3/2+\mathcal{R}}(\kappa) C^{ab}\Big(\frac{m_a v_\parallel}{T_a}w(\kappa)L_k^{3/2+\mathcal{R}}(\kappa); f_{b0}\Big),\\
\frac{n_a}{\tau_{ab}}N_{nk}^{ab} &= \int \mathrm{d}^3u\ v_\parallel w(\kappa) L_n^{3/2+\mathcal{R}}(\kappa) C^{ab}\Big(f_{b0}; \frac{m_a v_\parallel}{T_a}w(\kappa)L_k^{3/2+\mathcal{R}}(\kappa)\Big),
\end{aligned}$$

where M_{nk}^{ab} and N_{nk}^{ab} correspond to differential and integral parts of Coulomb operator, respectively. Collisions with ions are accounted only by differential part.

- Let us introduce the adjoint monoenergetic kinetic equation,

$$\mathcal{V}(g) + \nu_e(u)\mathcal{L}(g) = R_0^{-1} b v_\parallel f_{e0} \ \text{ with } \ \nu_e(u) = \nu_{ee}(u) + \nu_{ei}(u).$$

Integrating this equation with weight $\widehat{\delta f}_e L_k^{3/2+\mathcal{R}}(\kappa)$, where $\widehat{\delta f_e} = \delta f_e/f_{e0}$, i.e performing $\Big\langle \int \mathrm{d}^3u\widehat{\delta f}_e L_k^{3/2+\mathcal{R}}(\kappa)\ldots\Big\rangle$, where $\langle\ldots\rangle$ is averaging over magnetic surface, and using adjoint properties of $\mathcal{V}$ and C_e, we get system of linear equations with respect to averaged fluxes $\langle bV_\parallel^{(j)}\rangle$ with $J_{e,\parallel} = e n_e\langle bV_\parallel^{(0)}\rangle$ and $q_{e,\parallel}/T_e = -n_e\langle bV_\parallel^{(1)}\rangle$,

$$\begin{aligned}
\sum_{k=0}^{k_{\max}}\left\{\langle bV_\parallel^{(k)}\rangle\left[\delta_{kj} - \frac{2a_k}{\langle b^2\rangle u_{te}^2}\left(\Big[\!\Big[\nu_e w L_k^{3/2+\mathcal{R}} D_{33}^e\Big]\!\Big]_j + \sum_{l=0}^{l_{\max}} a_l c_{lk}\Big[\!\Big[w L_l^{3/2+\mathcal{R}} D_{33}^e\Big]\!\Big]_i\right)\right]\right\}\\
= -\Big[\!\Big[D_{31}^e\Big]\!\Big]_j A_1 + \Big[\!\Big[L_1^{3/2+\mathcal{R}}(\kappa) D_{31}^e\Big]\!\Big]_j A_2 - \Big[\!\Big[D_{33}^e\Big]\!\Big]_j A_3.
\end{aligned}$$

The order of this system, given by $k_{\max} = l_{\max}$, is exactly the same as for Sonine polynomials in non-relativistic limit and is fairly low. Only the convolution of the monoenergetic transport coefficients is required,

$$\Big[\!\Big[\varphi(\kappa)\Big]\!\Big]_i = \frac{2}{\sqrt{\pi}}C_{MJ}\int_0^\infty d\kappa\, \varphi(\kappa)\kappa^{1/2}\gamma\sqrt{\frac{\gamma + 1}{2}}\mathrm{e}^{-\kappa}L_i^{3/2+\mathcal{R}}(\kappa).$$

Since mono-energetic D_{ij} (given, for example, by DKES) are parametrized by only ν/v and E_r/v, convolution $[\![\ldots D_{ij}]\!]$ is trivial with $v = u/\gamma$ and $\nu \equiv \nu_e(u)$.

4. H. Sugama and S. Nishimura, PoP 15, 042502 (2008). 5. H. Maassberg et al., PoP 16, 072504 (2009).

Summary 1: Braginskii equations are derived in mixed approach, with fully relativistic plasma electrons and weakly relativistic mean electron flow.

Summary 2: Using generalized Laguerre polynomials $L_n^{3/2+\mathcal{R}}(\kappa)$, with $\kappa = \mu(\gamma - 1)$ and $\mu = m_e c^2/T_e$, method of momentum correction is generalized for relativistic approach.