Sparse Basis Polynomial Chaos Analysis of Radio Frequency Wave Scattering by Random Interfaces in the Fusion-Plasma Edge

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Introduction

Radio frequency (RF) electromagnetic waves enter the strongly turbulent edge region before passing into the fusion plasma. Whether used for diagnostics or for heating and current drive, it is important to quantify the spectral changes of the RF waves. The magnetized fusion-plasma and edge (via homogenization [1]) regions are defined through the cold plasma, anisotropic permittivity tensor. Experimental evidence [2] suggest that drift waves and rippling modes are present in the edge region. Therefore it is assumed that the edge region is separated from a low density plasma and the fusion plasma regions by periodic density interfaces (plasma gratings) formed as a superposition of spatial modes with varying periodicity and random amplitudes [3]. The ScaRF [3] full-wave, 3D electromagnetic code has been developed for analyzing scattering scenarios of this form. ScaRF can be used for scattering analysis of any cold plasma RF wave and consequently for the scattering of electron cyclotron waves in ITER-type and medium-sized tokamaks. Since the density interfaces are random (Fig3: 10 random amplitudes), the power reflection coefficient (R), obtained by ScaRF, is a random variable and is calculated for different realizations of the density interface. In this work, the uncertainty of R is rigorously quantified by use of the Polynomial Chaos Expansion [4] method using Sparse Basis (SB-PCE) and Hyperbolic truncation schemes [5]. The SB-PCE method is accurate and faster that the PCE-SG (Smolyak Grid) method in [6], and much more efficient compared to the Monte Carlo (MC) reference method.

ScaRF-FDFD 3D Code

In Fig. 1 the geometry of a spatial mode of the interface is shown, where the magnetic field and incident wavevector are the lines are of arbitrary orientation, in red and blue respectively.

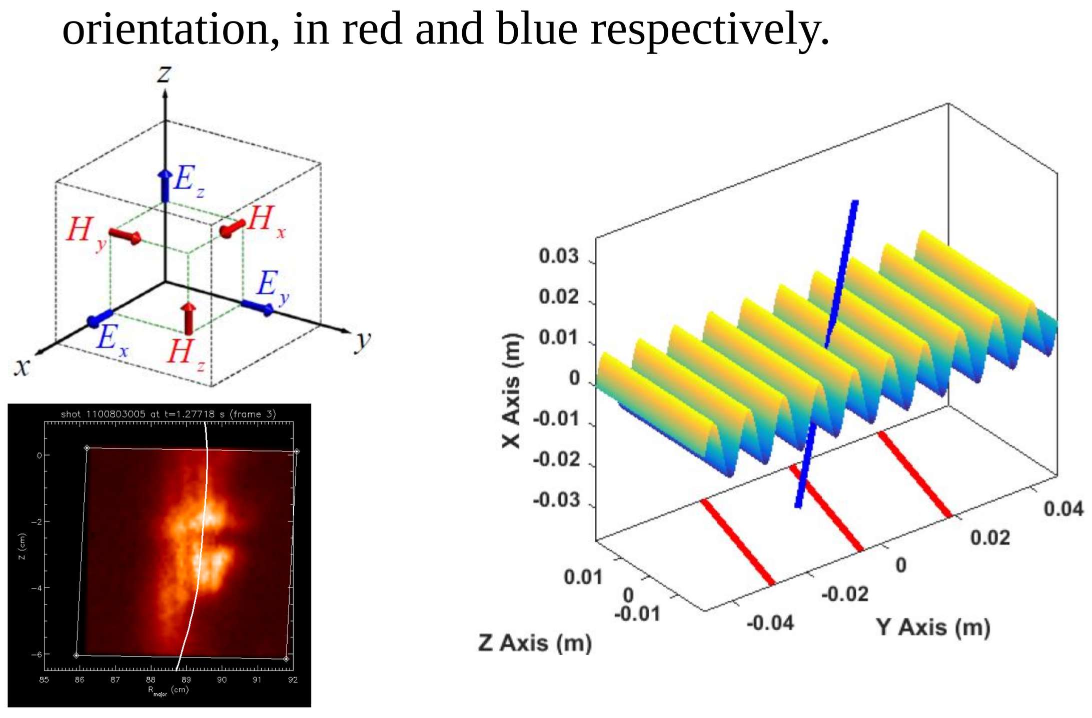

Figure 1: 3D Yee Grid (Left-Up) [6], Imaging the Edge Turbulence (Left-Down: *Courtesy of J. Terry & Wallace (MIT)* [8]), Structure Geometry (Right)

The ScaRF method solves Maxwell's equations in the frequency domain using finite differences. Field nodes are arranged in space on the staggered 3D Yee grid [7]. For density fluctuations near the plasma edge, the permittivity is a tensor [3], and FDFD is formulated for anisotropic media. This complicates Maxwell's equations, resulting in the extension of the 3D-Yee grid with additional nodes, whose fields values are calculated by interpolation operators. Considering the magnetic field normalization: ,

$$\vec{\tilde{H}} = -j\eta_0 \vec{H}$$

Maxwell's equations become:

$$\nabla \times \vec{E} = k_0 [\mu_r][s]\vec{\tilde{H}} \qquad [s] = \begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix}$$

$$\nabla \times \vec{\tilde{H}} = k_0 [\varepsilon_r][s]\vec{E}$$

where s_w, w=x,y,z are the stretching factors of the PML medium, spatially varying in the w direction, and $[\mu_r]$ and $[\varepsilon_r]$ are relative permeability and permittivity tensors.

- *ScaRF excitation*: Plane wave or beam by the computationally efficient *Total-Field Scattered-Field* (TFSF) method [4].
- *ScaRF Boundary Conditions (BC):* Dirichlet, *Bloch-Floquet BC*: $\vec{E}(\vec{r}) = \vec{A}_{\vec{\beta}}(\vec{r})e^{j\vec{\beta}\bullet\vec{r}}$

where A is a y-periodic function (Fig. 1), and β is the incident wavector.

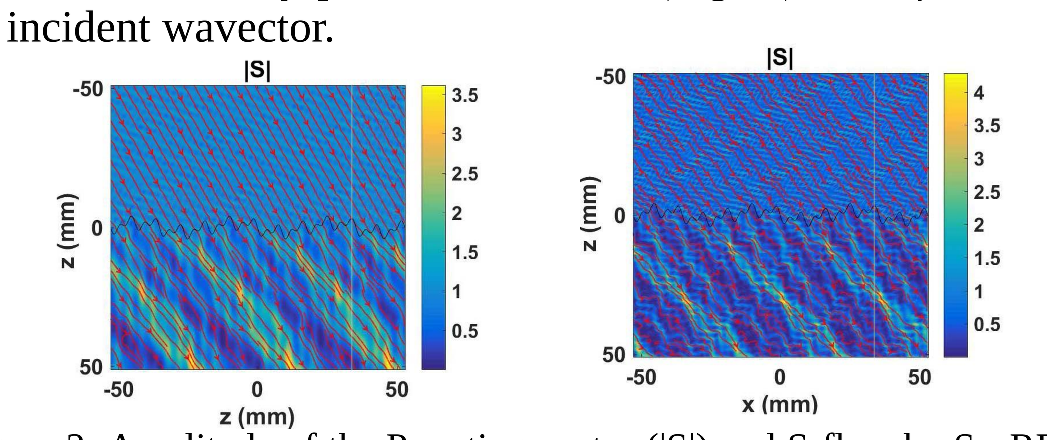

Figure 2: Amplitude of the Poynting vector (|S|) and S-flow by ScaRF, for a plane wave (O-mode, X-mode, Right and Left respectively) incident on a realization of a multimode interface [3].

Polynomial Chaos Expansion-Smolyak Sparse Grid

- Multimode Interface (Figure 3): $x(y) = \sum_{i=1}^{n} \xi_i \sin\left(\frac{2\pi y}{c_i \lambda_0}\right)$
- Incident Wavelength: λ_0
- Model assumption (PCE): $R(\theta;\boldsymbol{\xi}) \simeq \sum_{k=0}^{P} \beta_k(\theta)\Psi_k(\boldsymbol{\xi})$
- Independent random variables (ξ_i), i=1:d, $\boldsymbol{\xi}=(\xi_1,\xi_2,\ldots,\xi_d)$
- Polynomial order: p ($\boldsymbol{p=3}$) — Uniformly Distributed
- Total number of basis functions: P+1, P=(p+d)!/(p!d!)
- Basis functions: $\Psi_k(\boldsymbol{\xi}) = \prod_{m=1}^{d} \varphi_{\alpha_m^k}(\xi_m)$, $\left\|\boldsymbol{\alpha}^k\right\|_1 \le p$, $\left\|\boldsymbol{\alpha}^k\right\|_1 \equiv \sum_{m=1}^{d}\left|\alpha_m^k\right|$, $k=0,\ldots,P$
- Exponential convergence of PCE→1D polynomials $\varphi(\xi_k)$ according to Wiener-Askey scheme, [3]:

Random Variable (ξ_i)	1D Polynomials ($\varphi(\xi_k)$)
Uniform	**Legendre**
Gaussian	Hermite
Gamma	Laguerre
Beta	Jacobi

- Hyperbolic polynomial index set: $A^q_p = \{\boldsymbol{\alpha}: \|\boldsymbol{\alpha}\|_q \le p\}$, $\|\cdot\|_q$ =q-quasi norm: $\|\boldsymbol{\alpha}\|_q = \sqrt[q]{\left(\sum_{i=1}^{d}|a_i|^q\right)}$ => **sparser** index set than $\|\boldsymbol{\alpha}\|_1 \le p$
- Goal: Calculation of PCE expansion coefficients $\beta_k(\theta)$
- Solution: Solve underdetermined linear system: $\boldsymbol{\Psi\beta}=\boldsymbol{R}(\theta)$, $\boldsymbol{\Psi}_{i,j}=\Psi_j(\xi^{(i)})$, $\boldsymbol{\beta}=[\beta(\xi^{(1)}, \ldots, \beta(\xi^{(N)})]$, $M>>N$. $\boldsymbol{\Psi}$ $\mathbb{R}^{NxM}$. (where $\boldsymbol{\beta}$ is a sparse solution, with s nonzero coefficients) by the:
- Orthogonal Matching Pursuit Alg. (OMP): OMP seeks a sparse solution $\boldsymbol{\beta}^*$:

$$\boldsymbol{\beta}^* = \min\{\|\boldsymbol{\beta}\|_0\} \text{ such that } \boldsymbol{\Psi\beta}=\boldsymbol{R}(\theta)$$

- OMP is a greedy algorithm that iteratively finds the columns of $\boldsymbol{\Psi}$ that participate in the creation of $\boldsymbol{R(\theta)}$. These are the columns most correlated with the residual residual : $\|\boldsymbol{\Psi\beta}-\boldsymbol{R}(\theta)\|_2$. For a $\boldsymbol{\beta}^*$ recovery of s non-zero coefficients there will be s- iterations in the OMP. Alternatively the iterations continue until the, residual at the i-th OMP iteration, is smaller than a given error level .

Statistics of Reflection :

$$E[R] = \beta_0$$

$$\sigma^2[R] = \sum_{k=1}^{P} \beta_k^2 \|\Psi_k\|^2$$

5%, 95% percentiles : Calculated by the Nearest-Rank method.

Probability Density Function, $f_{\boldsymbol{R}}(R)$, Calculation :

$$f_{\mathbf{R}}(R) = \frac{1}{N_k h_K}\sum_{i=1}^{N_K} K\left(\frac{R-R_i}{h_K}\right)$$

K=Gaussian Kernel, h_K=Bandwidth and R_i samples from PCE model

Sobol Total Sensitivity Indices (STI) :

$$ST_\xi = \sum_{u \ni \xi} S_u, \qquad S_u = \frac{\sum_{k\in K_u}\beta_k^2\|\Psi_k\|^2}{\sum_{k=1}^{P}\beta_k^2\|\Psi_k\|^2}$$

$$K_u = \left\{k \in \{1,\ldots,P\} | \Psi_k(\boldsymbol{\xi}) = \prod_{i=1}^{|u|}\phi_{\alpha_i^k}(\xi_{u_k}), \alpha_i^k > 0\right\} \qquad u \subseteq \{1,2,\ldots,d\}$$

Results

The scattering of an incident O mode at 170GHz (electron-cyclotron (EC) freq. range) by a random periodic interface separating Blob-Background anisotropic media is analyzed using ScaRF in conjunction with the SB-PCE method. The interface is a superposition of 10 spatial sinusoidal modes of wavelengths **c**=[0.125 0.25, 0.375, 0.5, 0.75, 1, 1.75, 2.5, 3.75, 5]λ_0 and random amplitudes ξ_i, i=1:10, with a 5% variation around ξ_i=1. The toroidal, B_t, and poloidal B_p magnetic fields are 4.5T and 0.473T respectively. The incident wave has polar angle φ=174 deg. and experiments are performed for samples of azimuthal angles ±10% around θ=30 deg. The low density plasma, edge density and fusion plasma density are 1.0e17m^{-3}, 1.5e19m^{-3}, 1.0e19m^{-3} respectively.

In Fig. 3 the amplitude of the Poynting vector **S**, and the **S**-flow is shown for an incident O-mode at θ=30deg, scattering of the random density interface. In Fig.4 the MV & Std of R is shown as a function of θ. The minimum MV of R ≈0.03 occurs at θ≈29.1deg. The Pdf of R is shown in Fig. 5. It is observed the maximum of the Pdf, for θ=30 is roughly at R ≈ 0.035 which on the average occurs (Fig. 4) for θ≈30deg as expected.

In Fig. 6 (left) the Sobol Total Indices (STi) vs. θ are presented for the ten ξ_i random variables. R is more sensitive to the ξ_i with the highest STi. In Fig 6 (right) the 5%, 95% percentiles are calculated and observe that the R values lower than the minimum MV of R ≈0.03 at θ≈29.1 deg could occur with probability less than 5%.

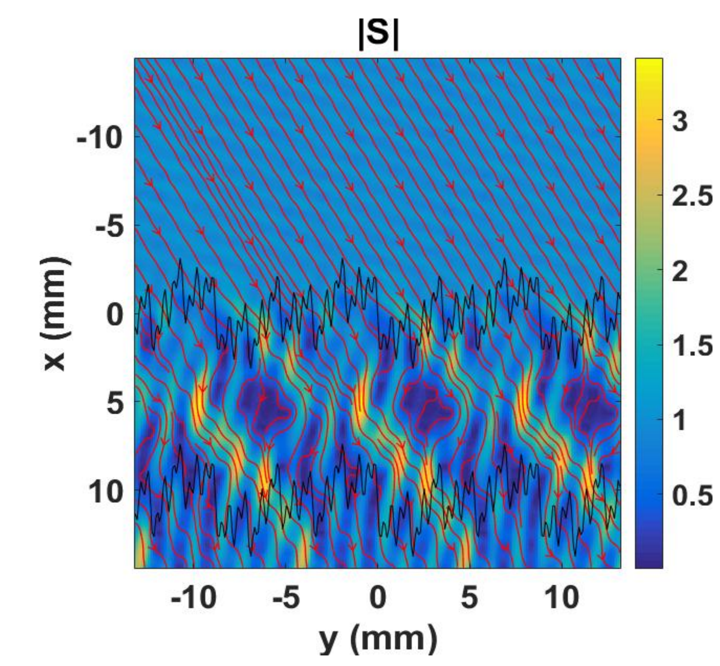

Figure 3: Amplitude of the Poynting vector (|**S**|) and S-flow, for a plane wave (O-mode), incident on a realization of a 10-mode interfaces with random amplitudes

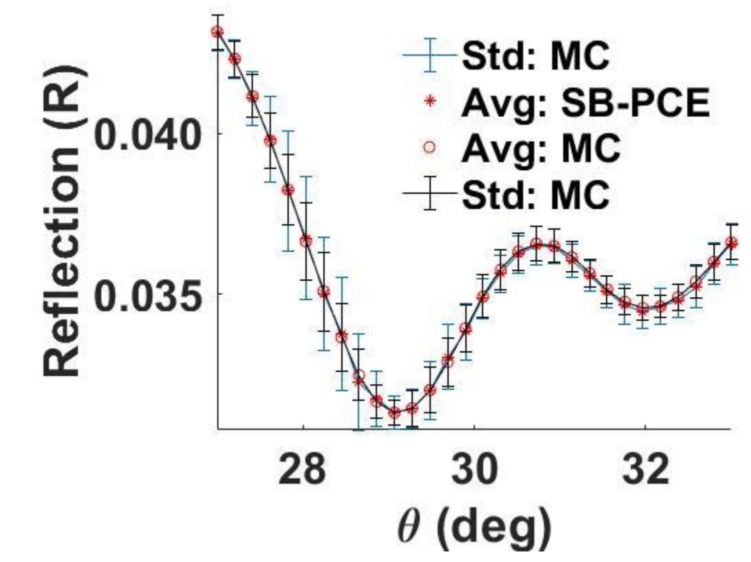

Figure 4:Mean Value (MV) & Standard Deviation (Std) of Reflection as a function of the angle of incidence (θ) of the incident wave by the MC and SB-PCE methods.

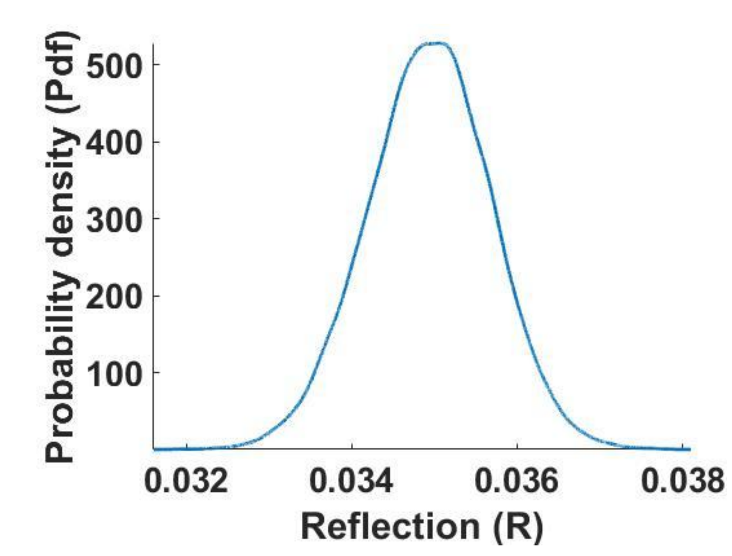

Figure 5:Probability Density Function of Reflection for θ =30deg.

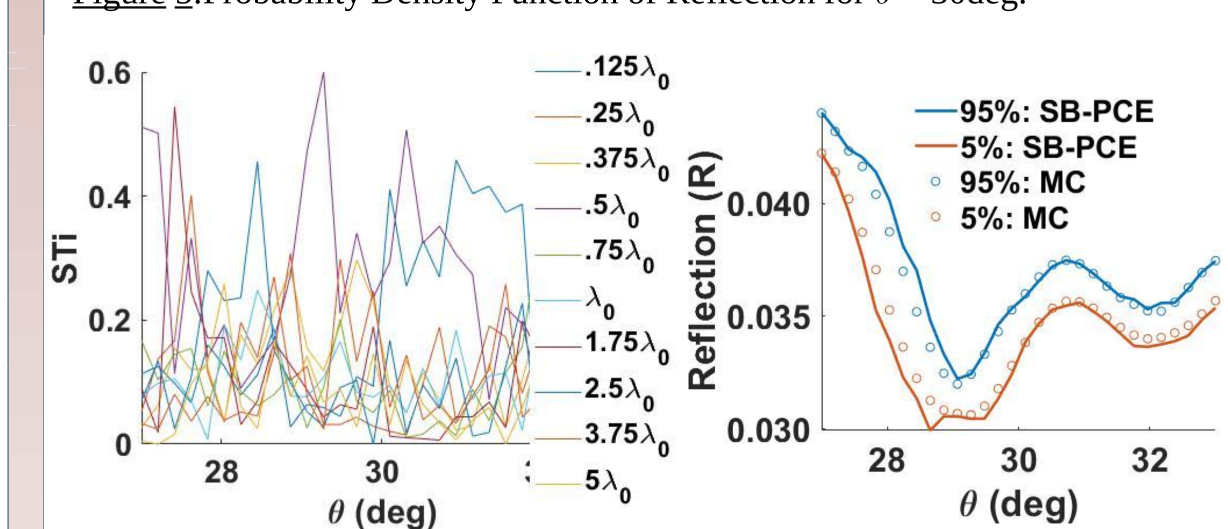

Figure 6: (left) Sobol Total Indices for the 10 random amplitudes of the spatial modes that define the density interface, (right) 5% and 95% percentile calculations by the MC and SB-PCE methods

Discussion

We have carried out a statistical analysis of the reflection (R) for a plane wave O-mode (in the electron cyclotron range of frequencies) incident on a dual periodic plasma-density interface. The Maxwell's equations are solved for different realizations of the turbulence using ScaRF. The mean value, the standard deviation, the 5% and 95% percentiles and the probability distribution function of R are calculated using the SB-PCE method. Also the contribution to the variance of R of each random design variable has been presented by calculation of the STi indices. The SB-PCE method is much more computationally efficient for the statistical analysis of R, than a MC method that would require orders of magnitude more R samples (**MC: 1000, SB-PCE: 38**) from the ScaRF code.

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