# On Ohm's law in reduced plasma fluid models


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## Overview

- Drift-reduced fluid models are routinely used for **edge** simulations
- We review the **impact** of the model **Ohm's law** on the system dispersion relation [1]
- Linear analysis of **supported waves** highlights common **numerical** issues
  - demonstrated with STORM filament simulations
- Wave frequency sets **CFL limit** (explicit), **cost** of iterative inversion (implicit)


## Summary

- Difficulties in electrostatic edge plasma simulations can be traced through the system dispersion relation
- Correctly limiting cold ion system wave speeds requires
  - electromagnetic Ohm's law with finite electron mass
  - space charge contribution in low density regions
- Model selection can mitigate computational cost due to small $\beta$ or $m_e$


## Reduced fluid model

- Consider a minimal reduced fluid model, isothermal, low beta $\mathbf{E}_\perp \approx -\nabla_\perp \phi$

$$\frac{\partial n}{\partial t} = -\nabla \cdot (\mathbf{b} n_0 v_{||e}) \quad \textit{continuity} \qquad \frac{m_i n_0}{B^2}\frac{\partial}{\partial t}\nabla_\perp^2 \phi = \nabla \cdot (\mathbf{b} J_{||}) \quad \textit{vorticity}$$

$$\mu_0 J_{||} = -\nabla^2 A_{||} \quad \textit{Ampère} \qquad \frac{m_i}{e}\frac{\partial v_{||i}}{\partial t} = -\partial_{||}\phi - \frac{\partial A_{||}}{\partial t} - \eta J_{||} \quad \textit{ion parallel momentum}$$

  - isothermal → parallel friction $\mathbf{b} \cdot \mathbf{F} = e n_0 \eta J_{||}$, Spitzer resistivity $\eta = 0.51 m_e / n_0 e^2 \tau_{ei}$
  - we linearise $\partial_{||} \rightarrow i k_{||}$, $\partial_t \rightarrow -i\omega$, $\nabla_\perp^2 \rightarrow -k_\perp^2$ to form dispersion relation
- The system is closed with a model Ohm's law

## Electrostatic model

- Isothermal electrostatic resistive Ohm's law $\eta J_{||} = -\partial_{||}\phi + (T_e/n_0)\partial_{||} n$
  - neglecting electron mass and ion parallel flow, so $J_{||} = -e n v_{||e}$
  - dispersion relation:

$$-i\omega = -k_{||}^2 \frac{T_e}{\eta e n_0}\left(\frac{1}{k_\perp^2 \rho_s^2} + 1\right) = -k_{||}^2 D$$

  - represents parallel diffusion equation, diffusion coefficient D
  - $k_\perp = 0$ modes communicate instantly along field lines: fast diffusion limits timestep

- Retain finite electron mass $\eta J_{||} = -\partial_{||}\phi + (T_e/n_0)\partial_{||} n + (m_e/e)\partial_t v_{||e}$
  - dispersion relation describes waves:

$$\omega^2 + i\omega\eta \frac{v_{te}^2}{\mu_0 V_A^2 \rho_s^2} = k_{||}^2 v_{te}^2 \left(\frac{1}{k_\perp^2 \rho_s^2} + 1\right)$$

  - wave speed diverges as $k_\perp \rightarrow 0$
  - cold plasma, zero resistivity limit recognise electrostatic wave $\omega^2 = \Omega_i^2 \frac{k_{||}^2}{k_\perp^2}\frac{m_i}{m_e}$
    - known to limit timestep in gyrokinetic simulations

- Evolving ion parallel momentum introduces ion acoustic wave
  - neglecting electron mass and resistivity: $\omega^2 = k_{||}^2 c_s^2 \left(\frac{1}{1 + k_\perp^2 \rho_s^2}\right)$
    - recognise finite sound radius corrections
  - ion sound radius couples diffusive mode and acoustic wave at finite $k_\perp$
  - can neglect parallel ion momentum equation when $\omega^2 \gg k_{||}^2 c_s^2$
    - i.e. $m_i \rightarrow \infty$ or $k_\perp \rightarrow 0$ at finite $k_{||}$, recover diffusive mode
  - electrostatic wave dispersion relation only multiplied by $(1 + m_e/m_i)$
  - wave speed still diverging as $k_\perp \rightarrow 0$

## Electromagnetic model

- Electromagnetic Ohm's law neglecting electron mass $\eta J_{||} = -\partial_{||}\phi + (T_e/n_0)\partial_{||} n - \partial_t A_{||}$
  - dispersion relation no longer diverging as $k_\perp \rightarrow 0$
  - parallel wave speed now diverges as $k_\perp \rightarrow \infty$
  - neglecting resistivity and ion acoustic wave at low $\beta = c_s^2/V_A^2$ , $V_A^2 = B^2/\mu_0 m_i n$:
    - recognize cause as kinetic Alfvén wave $\omega^2 = k_{||}^2 V_A^2 (1 + k_\perp^2 \rho_s^2)$

- With finite electron mass $\eta J_{||} = -\partial_{||}\phi + (T_e/n_0)\partial_{||} n - \partial_t A_{||} + (m_e/e)\partial_t v_{||e}$
  - neglecting resistivity:
    - recognize combination of inertial and kinetic Alfvén wave

$$\omega^2 = k_{||}^2 V_A^2 \left(\frac{k^2}{k_\perp^2} + k_\perp^2 \rho_s^2\right) \Bigg/ \left(1 + \frac{k^2 c^2}{\omega_{pe}^2}\frac{1}{\left(1 + \frac{m_e}{m_i}\right)}\right)$$

  - no longer diverging:
    - as $k_\perp \rightarrow \infty$ full dispersion relation → $\omega^2 + i\omega\eta \frac{v_{te}^2}{\mu_0 V_A^2 \rho_s^2} = k_{||}^2 v_{te}^2$

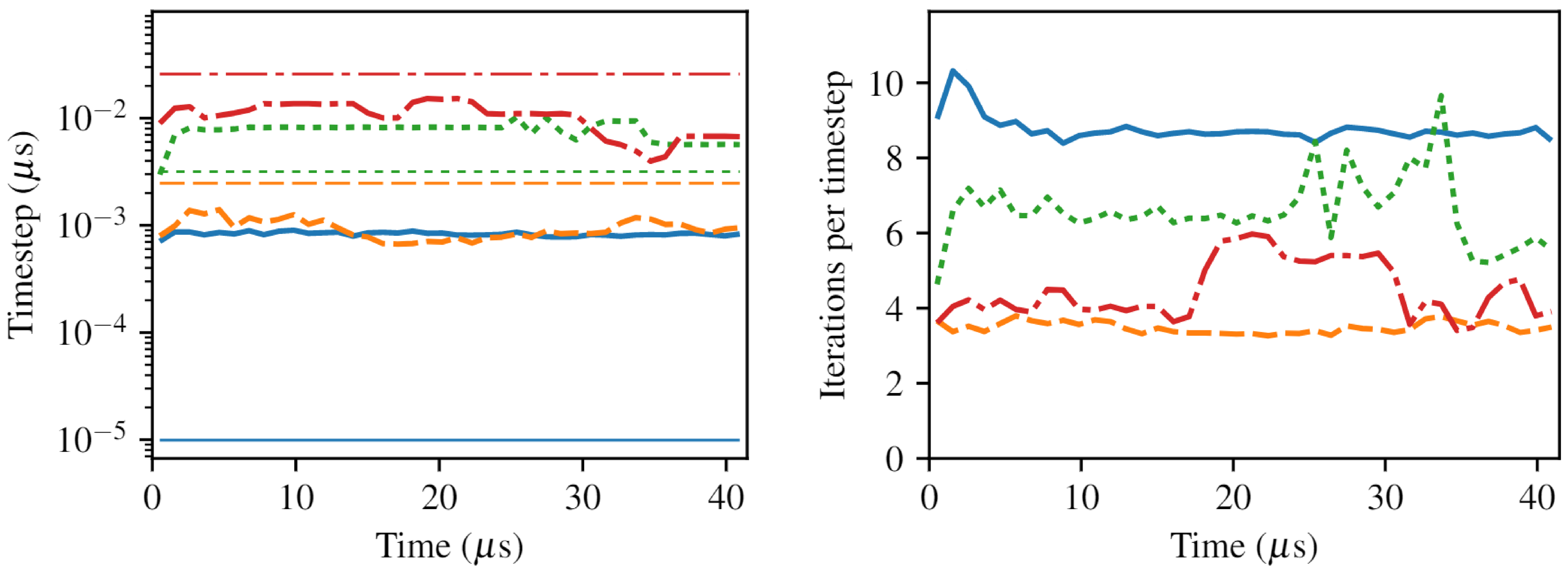
## Nonlinear timestep benchmark

- Test with STORM code, typical of BOUT++ drift-reduced fluid models
  - extended with zero-$m_e$ options for both electrostatic and electromagnetic modes
  - implicit time stepping: CVODE (adaptive step-size, order) from SUNDIALS suite [2]

- Simulation of isolated SOL filament, in slab geometry
  - moderate computational cost
  - includes features of SOL turbulence: highly nonlinear, sheath boundary conditions
  - simulation results insensitive to choice of Ohm's law due to low $\beta$

- Time stepping in simulations follows expectation from dispersion relations
  - higher frequency or damping rate makes implicit solve more expensive
  - implicit solver can step over strongly damped modes
  - time step consistent during simulation, despite evolution and break-up of filament

| Model | $1/\lvert\omega_{\mathrm{analytic}}\rvert$ (ns) | time step (ns) | iterations/step | wall-clock time (hrs) |
|---|---|---|---|---|
| Electrostatic zero-$m_e$ | 0.0091 | 0.828 | 8.76 | 30.4 |
| Electrostatic finite-$m_e$ | 2.47 | 0.899 | 3.64 | 11.5 |
| Electromagnetic zero-$m_e$ | 3.18 | 7.31 | 6.41 | 3.21 |
| Electromagnetic finite-$m_e$ | 25.8 | 9.25 | 4.35 | 2.36 |




***Left:*** time step (thick) compared to inverse analytic mode frequency (thin) and ***right:*** iteration count for the different models (see table for colours) as a function of simulation time

## Low density : space charge and displacement current

- Scrape-off layer (SOL) plasma can reach low density
  - electromagnetic Ohm's law without electron mass
    - resolving $\rho_s$ gives $k_\perp \sim 2\pi/\rho_s \rightarrow$ wave speeds $\sim 10 V_A$
    - exceeds speed of light at $B = 1$T for density below $n_0 \sim 2.6 \times 10^{17}\,\mathrm{m}^{-3}$
  - electromagnetic Ohm's law with electron mass
    - fastest wave speeds ~ $v_{te}$ or ~ $V_A$
    - exceeds speed of light at $B = 5$T for density below $n_0 \sim 0.7 \times 10^{17}\,\mathrm{m}^{-3}$

- Reconsider space charge and displacement current
  - contribution from parallel displacement current partially cancels space charge
  - remaining space charge effect modifies vorticity
    - no parallel coupling
    - perpendicular electric field energy bounded

$$\nabla \cdot \left(\frac{m_i n}{B^2}\frac{d\nabla_\perp \phi}{dt} + \varepsilon_0 \frac{\partial}{\partial t}\nabla_\perp \phi\right) = \nabla \cdot (\mathbf{b} J_{||})$$

  - waves limited to less than light speed
  - dispersion relation $\omega^2 \gg k_{||}^2 c_s^2$ : $\omega^2 \left(1 + k_\perp^2 \rho_s^2 \frac{V_A^2}{V_{te}^2}\right) + i\omega \frac{k_\perp^2 \eta}{\mu_0} = k_{||}^2 V_A^2 \left(\frac{c^2}{c^2 + V_A^2} + k_\perp^2 \rho_s^2\right)$

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[1] B.D. Dudson, S.L. Newton, J.T. Omotani and J. Birch, "On Ohm's law in reduced plasma fluid models", *accepted by PPCF* (2021). doi:10.1088/1361-6587/ac2af9
[2] A.C. Hindmarsh et al., *ACM Transactions on Mathematical Software* **31**(3), 363-396 (2005)