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Ghosts without Runaway

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Ghosts without Runaway Instabilities

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We present a simple class of mechanical models where a canonical degree of freedom interacts with another one with a negative kinetic term, i.e., with a ghost. We prove analytically that the classical motion of the system is completely stable for all initial conditions, notwithstanding that the conserved Hamiltonian is unbounded from below and above. This is fully supported by numerical computations. Systems with negative kinetic terms often appear in modern cosmology, quantum gravity, and high energy physics and are usually deemed as unstable. Our result demonstrates that for mechanical systems this common lore can be too naive and that living with ghosts can be stable.

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Main Message

- Some systems with ghosts are manifestly stable.

Plan

- What is a ghost?
- Why are we interested in ghosts?
- Our model
- Proof of stability and finiteness of motion
- Our model as system with higher derivatives
- Conclusions

Ghosts are systems with negative mass or kinetic energy unbounded from below

Ghosts and gradient instabilities

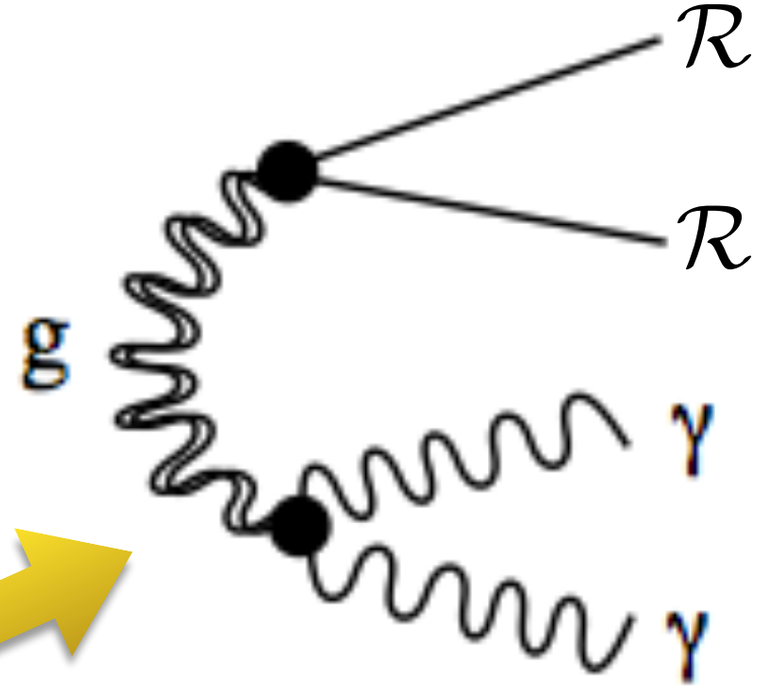
$$S[\mathcal{R}] = \frac{1}{2}\int d\tau d^3\mathbf{x}\, Z\left(\left(\mathcal{R}'\right)^2 - c_{\rm S}^2\left(\partial_i\mathcal{R}\right)^2\right) \qquad \mathcal{R} = \Phi + H\frac{\delta\varphi}{\dot\varphi}\,,$$

$$H_{\mathbf{k}} = \frac{\left|P_{\mathbf{k}}\right|^2}{2Z} + \frac{Zc_S^2k^2\left|\mathcal{R}_{\mathbf{k}}\right|^2}{2} \qquad R_{\mathbf{k}} \sim \exp\left(\left|c_s\right|k\tau\right)$$

Gradient instability $c_s^2 < 0$

ghost $Z(t) < 0$

*ghosts - modes (oscillators) with the **negative mass***



$$\Gamma_{0\to 2\gamma 2\phi} \sim \frac{\Lambda^8}{M_{\rm Pl}^4}$$

Cline, Jeon, Moore, (2003)

Why are we interested in ghosts?

- May be interesting for cosmology:
  Phantom DE, Bouncing universe etc.

- Higher derivative systems have ghosts
  due to Ostrogradski theorem

- Renormalisability of Gravity

Our Model

Hamiltonian

$$H = \frac{1}{2}(p_x^2 + x^2) - \frac{1}{2}(p_y^2 + y^2) + V_I(x, y)$$

Normal Oscillator | Ghosty Oscillator | Interaction Potential

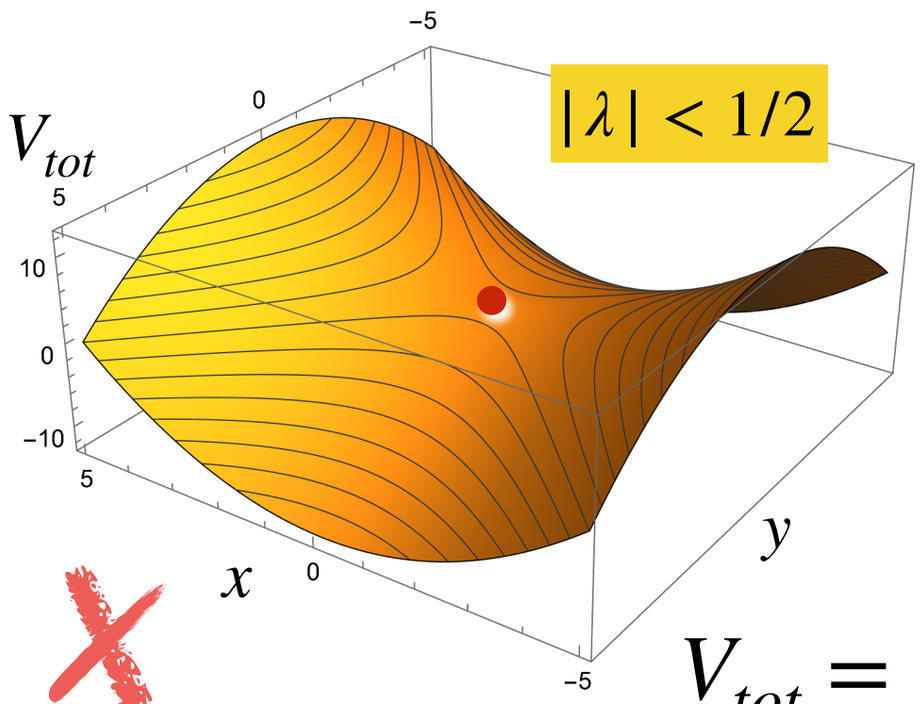
$$V_I(x, y) = \lambda \left[1 + 2\left(y^2 + x^2\right) + \left(y^2 - x^2\right)^2\right]^{-1/2}$$

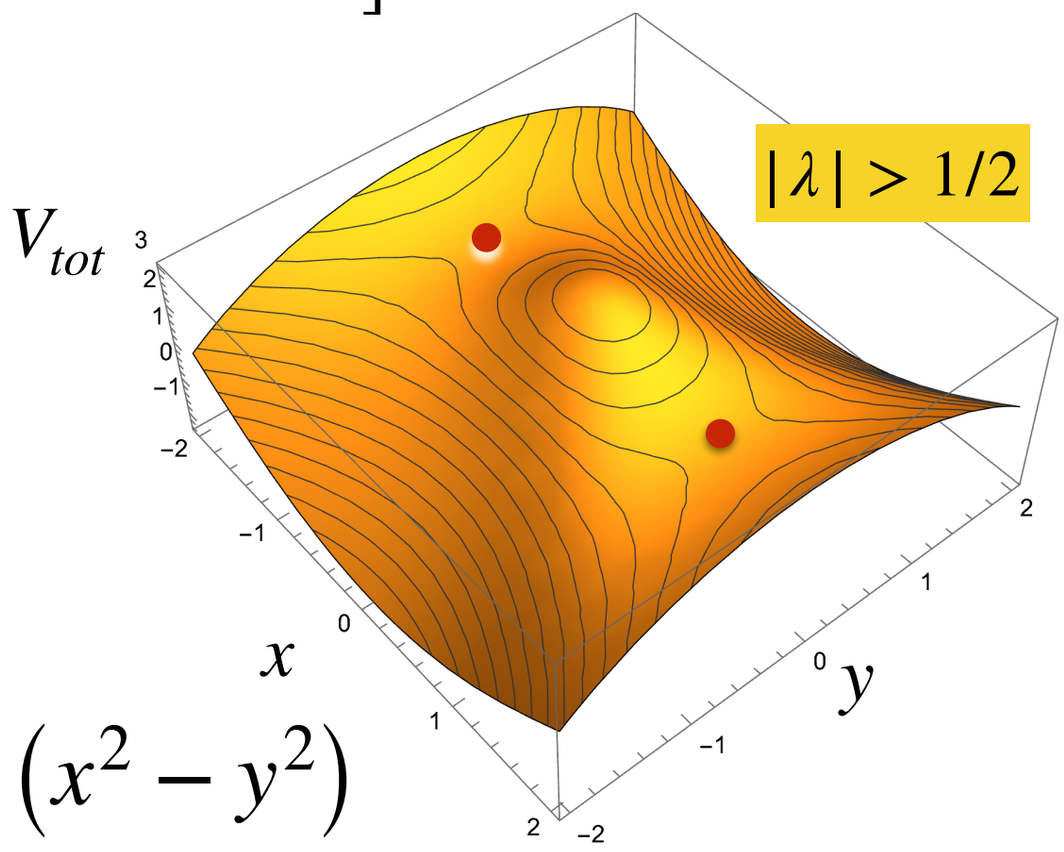
Coupling Constant

Interaction is bounded $0 < V_I(x, y)\, \lambda^{-1} \leq 1$

Potential

$$V_I(x,y) = \lambda\left[\left(x^2 - y^2 - 1\right)^2 + 4x^2\right]^{-1/2}$$





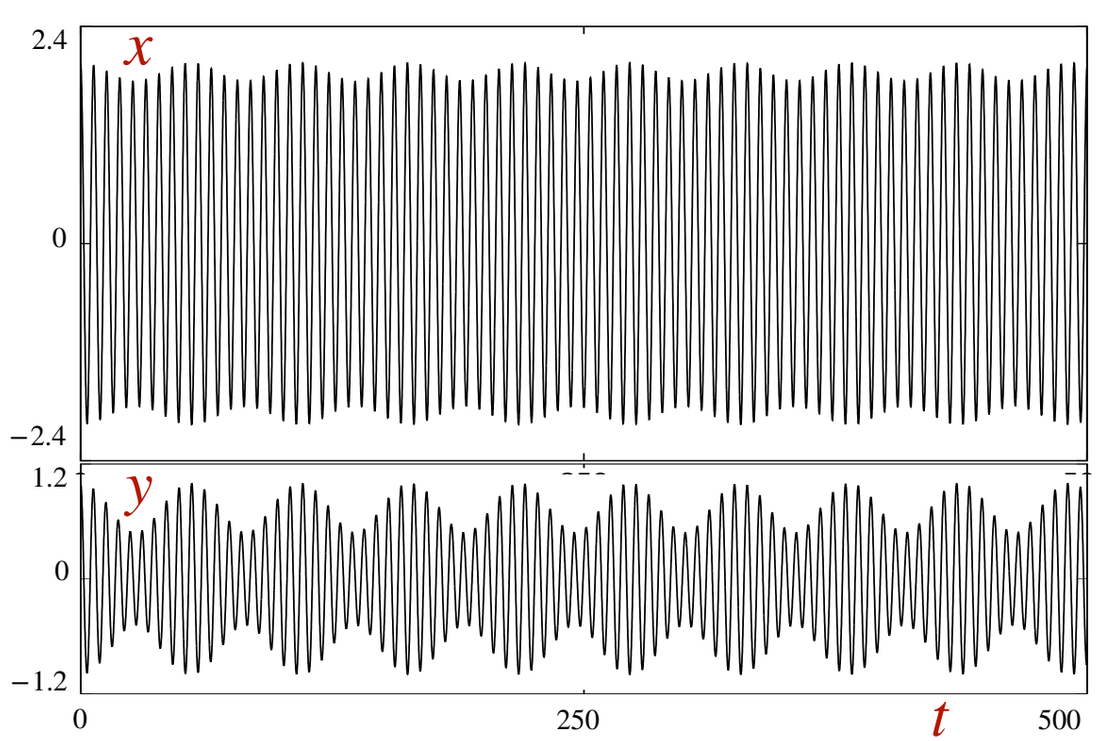
$$V_{tot} = V_I + \frac{1}{2}\left(x^2 - y^2\right)$$

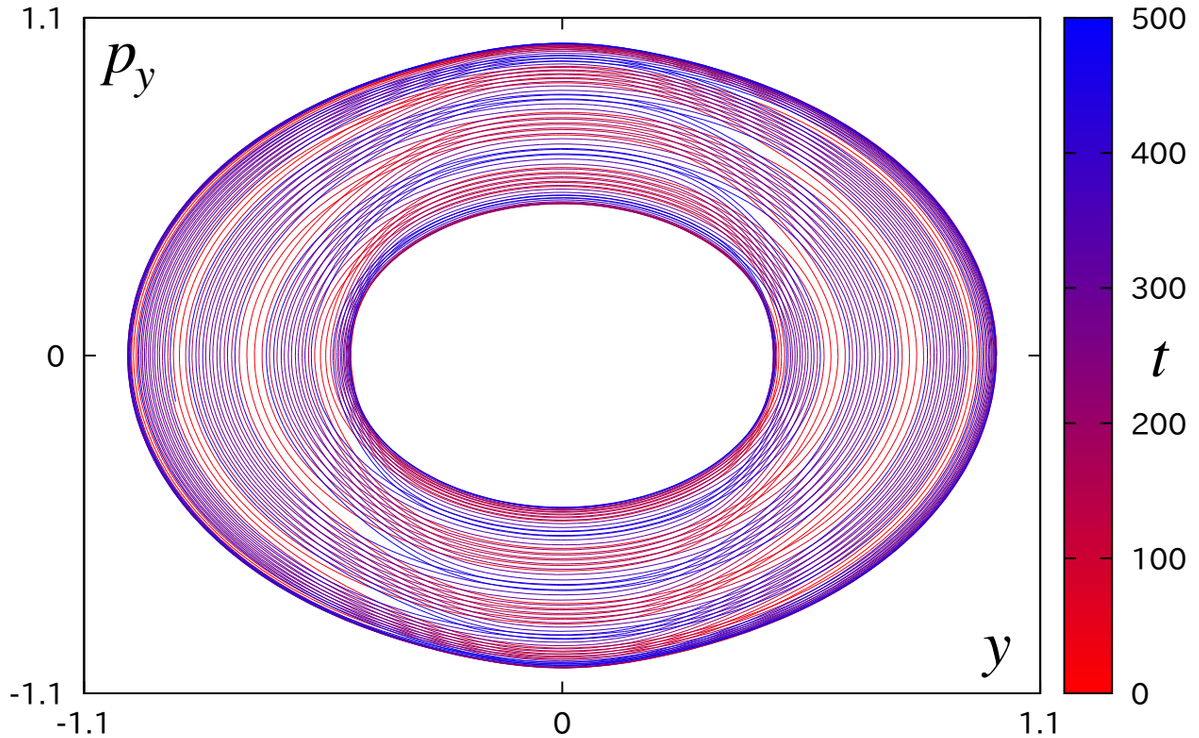
$$V_{tot} = \frac{\omega_x^2}{2}x^2 - \frac{\omega_y^2}{2}y^2 + \lambda\left(x^4 + 4y^2x^2 + y^4\right) + \ldots$$

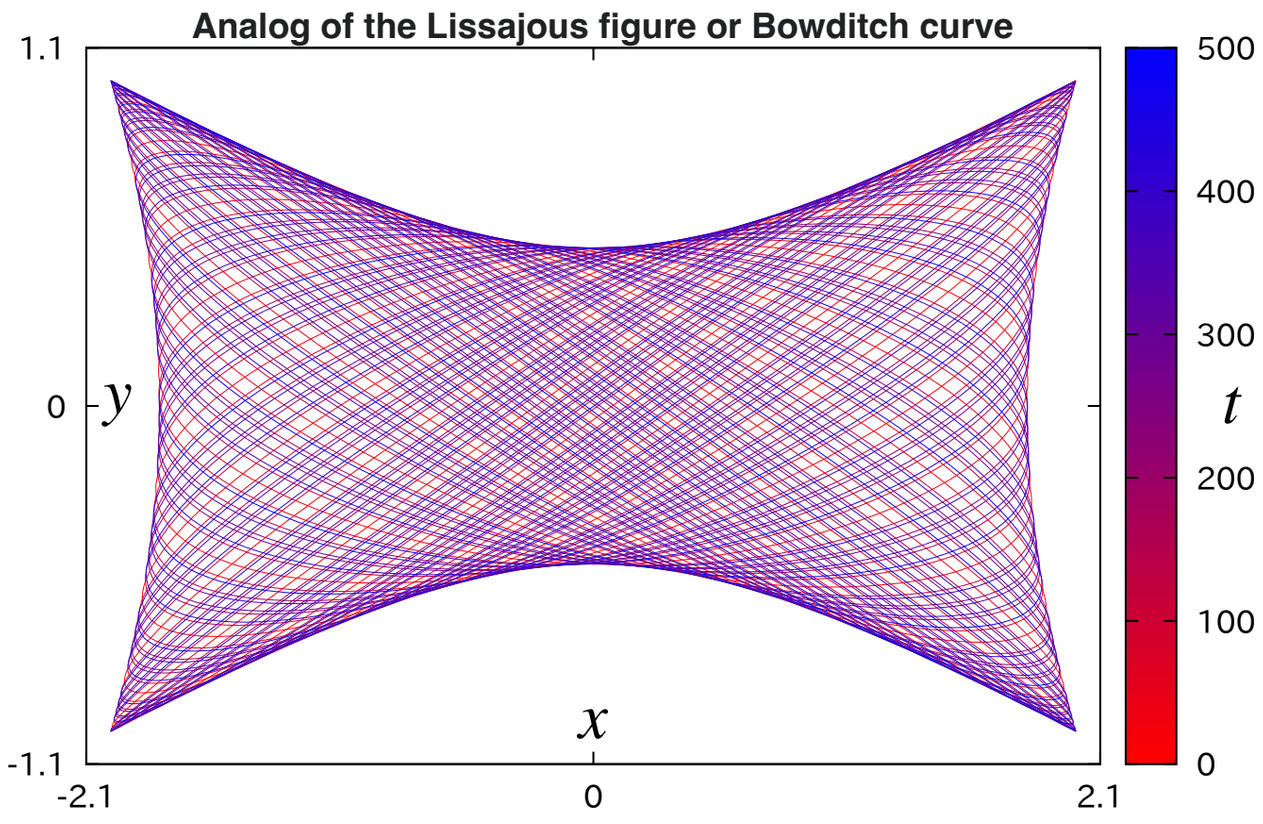
$$\omega_x^2 = 1 - 2\lambda\,, \quad \text{and} \quad \omega_y^2 = 1 + 2\lambda$$

Stable motion not at a minimum, but at a saddle point of the potential!

Numerical Solutions







Numerical solutions look pretty stable
for hundreds of oscillations,
and some initial data.

Maybe it is just a stable system?

Can one analytically prove stability
and the absence of the runaway solutions?

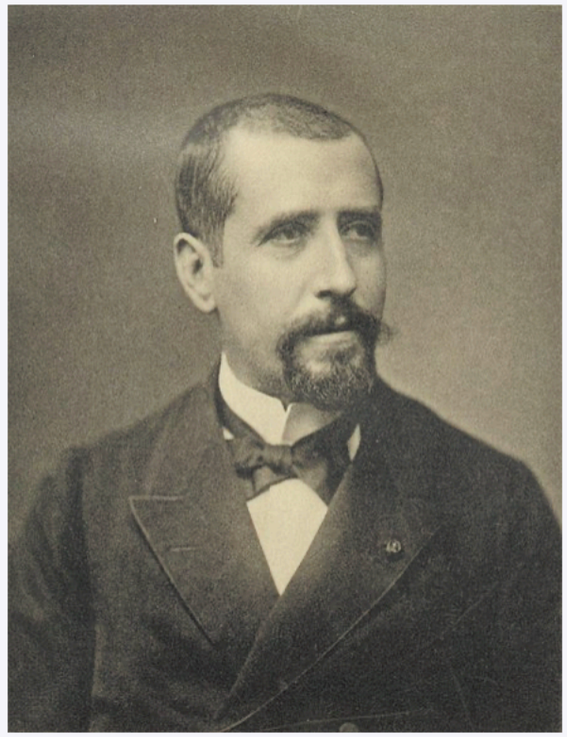
First Integral and the Power of Imagination

$$C = K^2 + \left(p_x^2 + x^2\right) - \left(x^2 - y^2 - 1\right) V_I\left(x, y\right)$$

generator for hyperbolic rotations $K = p_y x + p_x y$

$$\frac{dC}{dt} = 0$$

Jean-Gaston Darboux



One can obtain our system via complex canonical transformation

$$y = i\tilde{y}\,, \quad \text{and} \quad p_y = -i\tilde{p}_y$$

(so that $[y, p_y] = [\tilde{y}, \tilde{p}_y] = 1$ etc.)

from a ghost-free integrable system introduced by Darboux in 1901

Is there any symmetry behind this conserved quantity C?

Another First Integral: $\mathscr{E}$

$$\mathscr{E} = C - H = \Sigma + \left(y^2 - x^2\right) V_I\left(x, y\right)$$

where

$$\Sigma = \left(p_y x + p_x y\right)^2 + \frac{1}{2}\left(p_x^2 + x^2\right) + \frac{1}{2}(p_y^2 + y^2)$$

the interaction part is always bounded

$$-|\lambda| \leq \left(y^2 - x^2\right) V_I\left(x, y\right) \leq |\lambda|$$

as

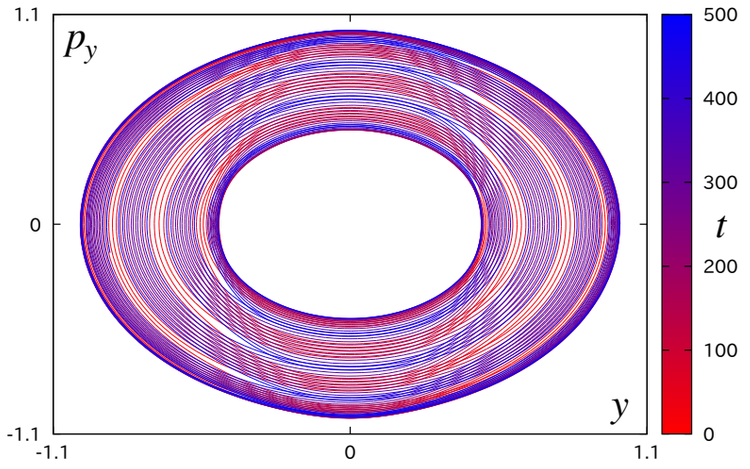
$$V_I\left(x, y\right) = \lambda\left[1 + 2\left(y^2 + x^2\right) + \left(y^2 - x^2\right)^2\right]^{-1/2}$$

for all times $\Sigma - |\lambda| \leq \mathscr{E} \leq \Sigma + |\lambda|$

Finiteness of motion

at initial point of time t_a $\quad \Sigma_a - |\lambda| \le \mathscr{E} \le \Sigma_a + |\lambda|$

at any later point in time t_b $\quad \Sigma_b - |\lambda| \le \mathscr{E} \le \Sigma_b + |\lambda|$



Σ is positive definite and
is confined in a stripe

$$\Sigma_a - 2|\lambda| \le \Sigma_b \le \Sigma_a + 2|\lambda|$$

Thus the trajectory is confined in a stripe, as for $\xi = \left(x, y, p_x, p_y\right)$ we have $\left|\xi\right|^2 \le 2\Sigma$

System always evolves in a finite region of phase space

One can show that $\mathscr{E}$ is a Lyapunov function so that the system is stable at the origin.

Realisation through Higher Derivatives

$$L(q,\ddot{q}) = \left(\ddot{q} + q\right)\left(2p_2 + (2p_2)^{-1}\right)$$

where $p_2 \equiv p_2\left(q,\ddot{q}\right)$ is the solution of

$$(\ddot{q} + q)\sqrt{2q^2 + 1} = -2\lambda p_2(2p_2^2 + 1)^{-3/2}$$

In this way $p_2 = \partial L/\partial\ddot{q}$

Why have not we seen such systems so far in nature?

Thanks a lot for attention!