# On the cosmological constant appearing as an initial condition for inflationary models

Eric Ling
University of Copenhagen (GeoTop and QMath)

(joint with Ghazal Geshnizjani and Jerome Quintin)

Copernicus Webinar

July 11th, 2023

Most of the material presented in these slides is based on the following two works:

- E. Ling, *Remarks on the cosmological constant appearing as an initial condition for Milne-like spacetimes*, Gen. Rel. Grav. **54** (2022).
- G. Geshnizjani, E. Ling, J. Quintin, *On the initial singularity and extendibility of flat quasi-de Sitter spacetimes*, arXiv:2305.01676, (2023).

# Outline

# Outline

(1) Coordinate singularities

(2) Milne-like spacetimes

(3) $\Lambda$ appears as an initial condition

(4) Applications to inflationary theory

(5) Past-asymptotically flat de Sitter spacetimes

Coordinate singularities

# Coordinate singularities

- Roughly, a spacetime contains a coordinate singularity when a defining set of coordinates fails to capture all the geometry of the spacetime, and another set of coordinates *exposes* this geometry.

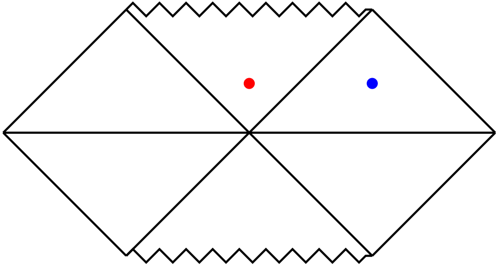
- Example: In the usual coordinates of the Schwarzschild spacetime, $r = 2m$ is a coordinate singularity.

$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

  The coordinates $(t, r, \theta, \varphi)$ fail to capture all the geometry of the spacetime, namely the event horizon. Other coordinates (e.g. Eddington–Finkelstein or Kruskal coordinates) exposes the geometry of the event horizon.

# Coordinate singularities

Penrose diagram for (maximal) Schwarzschild:



- The blue dot represents a sphere in the $r > 2m$ region.
- The red dot represents a sphere in the $r < 2m$ region. These are trapped surfaces.
- Trapped surfaces play a fundamental role in Penrose's singularity theorem (2020 Nobel Prize). They capture the focusing effects of black holes in the *non-spherically symmetric* setting.

# Coordinate singularities

- Story:
  1. Recognize that $r = 2m$ is a coordinate singularity.
  2. Explore the geometry of the other coordinates.
  3. Use the geometry to develop theories in non-symmetrical settings.
- Questions:
  - Is there a similar story in cosmology?
  - Can the big bang be a coordinate singularity?
  - Is there geometry to explore at the big bang?
  - If so, does the geometry suggest special initial conditions for the universe?
- The goal of this presentation is to provide a convincing argument that the answer these questions is *yes* and that exploring the geometry of these big bang coordinate singularities could help us in understanding early universe cosmology.

Milne-like spacetimes

# Milne-like spacetimes

**Definitions**

- A $C^k$ spacetime $(M, g)$ is a four-dimensional time oriented Lorentzian manifold with $g \in C^k$.

- A $C^0$ spacetime $(M_{\text{ext}}, g_{\text{ext}})$ is a $C^0$ extension of a $C^\infty$ spacetime $(M, g)$ if there is an isometric embedding

$$(M, g) \quad \hookrightarrow \quad (M_{\text{ext}}, g_{\text{ext}})$$

  such that $M \subset M_{\text{ext}}$ is a proper subset.

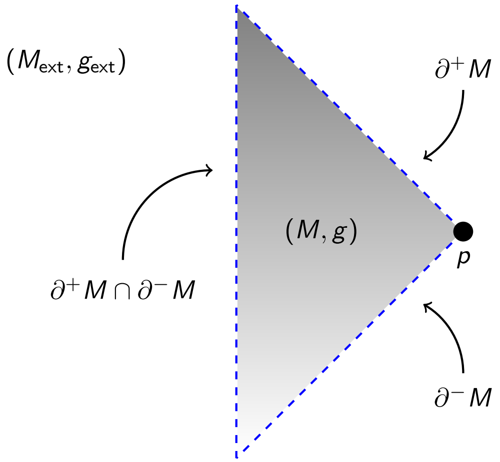
- A f.d. timelike curve $\gamma\colon [a, b] \to M_{\text{ext}}$ is future terminating for $p \in \partial M$ if $\gamma(b) = p$ and $\gamma\big([a, b)\big) \subset M$.

- The future and past boundaries of $M$ within $M_{\text{ext}}$ are defined as

$$\partial^+ M \,=\, \{p \in \partial M \mid \text{there is a future terminating timelike curve for } p\}$$

$$\partial^- M \,=\, \{p \in \partial M \mid \text{there is a past terminating timelike curve for } p\}.$$

# Milne-like spacetimes



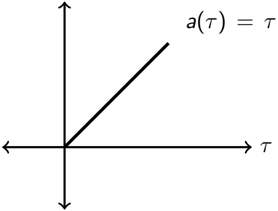
# Milne-like spacetimes

The Milne universe is the $k = -1$ FLRW spacetime with scale factor

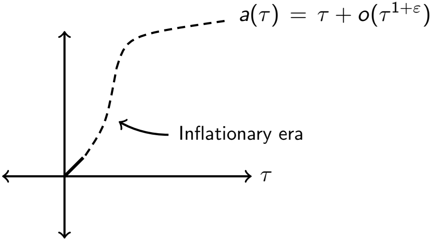
$$a(\tau) = \tau.$$

Milne-like spacetimes are $k = -1$ FLRW spacetimes with scale factor satisfying the following limiting condition near the big bang:

$$a(\tau) = \tau + o(\tau^{1+\varepsilon}).$$




The Milne universe




A Milne-like spacetime

**Important:** For this talk, $\tau$ will *always* denote the time of the comoving observers, *not* conformal time.

# Milne-like spacetimes

**Theorem (Galloway and L. '17, L. '20)**

*Milne-like spacetimes admit continuous extensions through the big bang.*

*Proof.* The metric is

$$\begin{aligned} g &= -d\tau^2 + a^2(\tau) h_{\mathrm{hyper}} \\ &= -d\tau^2 + a^2(\tau)\big[dR^2 + \sinh^2(R)(d\theta^2 + \sin^2\theta d\phi^2)\big]. \end{aligned}$$

Introduce new coordinates

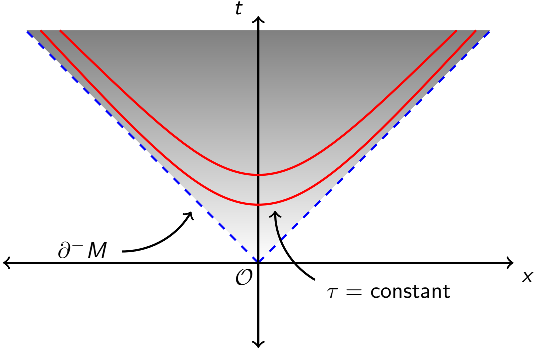
$$t = b(\tau)\cosh R \quad \text{and} \quad r = b(\tau)\sinh R$$

where $b(\tau) = \exp(\int_{\tau_0}^{\tau} \frac{1}{a})$ for some $\tau_0 > 0$. Then the metric becomes

$$\begin{aligned} g &= \left(\frac{a}{b}\right)^2 \big[ -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\big] \\ &= \left(\frac{a}{b}\right)^2 g_{\mathrm{Mink}}. \end{aligned}$$

$a(\tau) = \tau + o(\tau^{1+\varepsilon})$ implies $a/b \to \tau_0$ as $\tau \to 0^+$. Therefore $g$ extends continuously through the lightcone at the origin $\mathcal{O} = (0,0,0,0)$. $\square$

# Milne-like spacetimes




$$g \,=\, \Omega^2(\tau)\big[-dt^2 + dx^2 + dy^2 + dz^2\big]$$

# Milne-like spacetimes

**Theorem (L. '20)**

*Suppose the scale factor of a Milne-like spacetime is analytic*

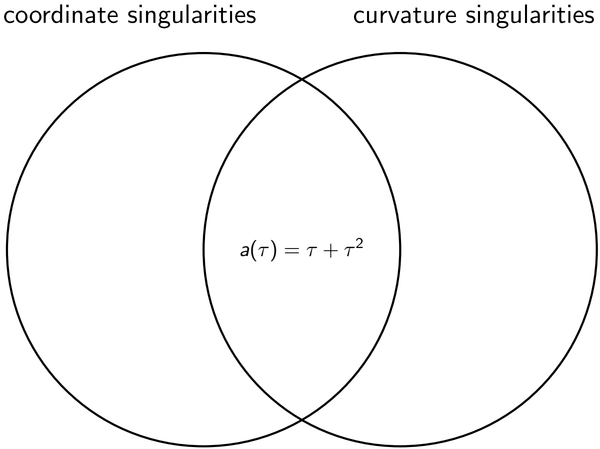
$$a(\tau) = \tau + \sum_{n=2}^{\infty} c_n \tau^n .$$

*If $c_2 = c_4 = 0$, then the Milne-like spacetime admits a $C^2$ extension through the big bang.*

*Consequently, there are no curvature singularities towards the big bang.*

Example: Consider $a(\tau) = \tau + \tau^2$. This produces a Milne-like spacetime and hence a continuous extension. However, the scalar curvature diverges as $\tau \to 0^+$. Therefore coordinate singularities and curvature singularities are *not* mutually exclusive.

# Milne-like spacetimes



# Milne-like spacetimes

Note that: $a(\tau) = \tau + o(\tau^{1+\varepsilon}) \quad \Longrightarrow \quad a(\tau) = \tau + o(\tau)$.

Questions: (1) Is $a(\tau) = \tau + o(\tau)$ sufficient?

(2) Is $a(\tau) = \tau + o(\tau)$ necessary?

(3) Is the $\varepsilon$ in $a(\tau) = \tau + o(\tau^{1+\varepsilon})$ necessary?

**Theorem (L. and Piubello '23)**

*Let $(M, g)$ be a $k = -1$ FLRW spacetime with $a(\tau) \to 0$ as $\tau \to 0$.*

1. *There are examples of scale factors satisfying $a(\tau) = \tau + o(\tau)$ such that the metric does not extend continuously to the lightcone.*
2. *If the metric extends continuously to the lightcone, then $a(\tau) = \tau + o(\tau)$ provided $\lim_{\tau \to 0} a'(\tau)$ exists.*
3. *There are examples of scale factors satisfying $a(\tau) = \tau + o(\tau)$ but $a(\tau) \neq \tau + o(\tau^{1+\varepsilon})$ for any $\varepsilon > 0$ such that the metric does extend continuously to the lightcone.*

$\Lambda$ appears as an initial condition for Milne-like spacetimes

# $\Lambda$ appears as an initial condition

FLRW spacetimes satisfy the Einstein equations with a perfect fluid:

$$\mathrm{Ric} - \frac{1}{2}Rg \;=\; 8\pi T \;=\; 8\pi\big[(\rho + p)u_* \otimes u_* + pg\big]$$

where

$$\begin{aligned} u_* \;&=\; g(u,\cdot) \quad (\text{where } u = \partial_\tau) \\ \rho(\tau) \;&=\; \frac{3}{8\pi}\left[\left(\frac{\dot{a}(\tau)}{a(\tau)}\right)^2 + \frac{k}{a^2(\tau)}\right] \\ p(\tau) \;&=\; -\frac{1}{8\pi}\left[2\frac{\ddot{a}(\tau)}{a(\tau)} + \left(\frac{\dot{a}(\tau)}{a(\tau)}\right)^2 + \frac{k}{a^2(\tau)}\right]. \end{aligned}$$

Here $k = +1, 0, -1$ depending on the space form.

Regardless of the value of $k$, the equations for $\rho$ and $p$ imply Friedmann's second equation:

$$\frac{\ddot{a}}{a} \;=\; -\frac{4\pi}{3}(\rho + 3p).$$

# $\Lambda$ appears as an initial condition

**Equations of state (physics)**

To close the Einstein equations, one imposes an equation of state. This is a relationship between $\rho$ and $p$, and it's usually determined via thermodynamic principles. Physically important equations of state are

$$p_{\mathrm{dust}} = 0 \quad \text{and} \quad p_{\mathrm{rad}} = \frac{1}{3}\rho_{\mathrm{rad}}.$$

- Both of these equations of state imply

  $$\ddot{a}(\tau) < 0$$

  which follows from Friedmann's 2nd eq. $\ddot{a}/a = -\frac{4\pi}{3}(\rho + 3p)$.

- However, as we know, observations show that

  $$\ddot{a}(\tau_0) > 0$$

  where $\tau_0$ is our current cosmic time.

- To explain this effect, cosmologists suggested that there is an invisible dark energy throughout the cosmos with equation of state:

  $$p_\Lambda = -\rho_\Lambda = -\frac{\Lambda}{8\pi} \quad \text{where } \Lambda > 0 \text{ is called the cosmological constant.}$$

# $\Lambda$ appears as an initial condition

**The cosmological constant appears as an initial condition for Milne-like spacetimes.**

Suppose $(M, g)$ is a Milne-like spacetime. For simplicity assume $a(\tau)$ is analytic near $\tau = 0$:

$$a(\tau) = \tau + \sum_{n=2}^{\infty} c_n \tau^n.$$

## Proposition

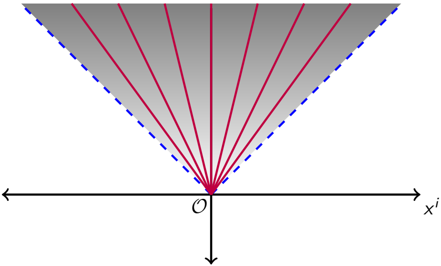
$$c_2 = 0 \implies \rho(0) = -p(0) = \frac{3}{8\pi}(6c_3).$$

# $\Lambda$ appears as an initial condition

**Motivation:** Generalize the previous proposition to spacetimes which share similar geometrical properties with Milne-like spacetimes but without any isotropic or homogeneous assumptions.

Two geometrical properties we will want to keep are the following.

- FLRW spacetimes solve the Einstein equations with a perfect fluid $(u, \rho, p)$.
- The comoving observers are the integral curves of $u = \partial_\tau$. For Milne-like spacetimes, they all emanate from $\mathcal{O}$:



# $\Lambda$ appears as an initial condition

**Setting:** Let $(M_{\text{ext}}, g_{\text{ext}})$ be a $C^0$ extension of a $C^\infty$ spacetime $(M, g)$ such that

$$M = I^+(\mathcal{O}, M_{\text{ext}})$$

for some point $\mathcal{O} \in \partial^- M$. We will call $\mathcal{O}$ the origin.

**Definitions**

- Let $f\colon M \to \mathbb{R}$ be a $C^\infty$ function. We say $f$ extends continuously to $M \cup \{\mathcal{O}\}$ provided there is a $C^0$ function

$$\widetilde{f}\colon M \cup \{\mathcal{O}\} \to \mathbb{R} \quad \text{such that} \quad \widetilde{f}|_M = f.$$

- More generally, let $T$ be a $C^\infty$ tensor field on $M$. We say $T$ extends continuously to $M \cup \{\mathcal{O}\}$ provided there is a coordinate neighborhood $U \subset M_{\text{ext}}$ of $\mathcal{O}$ such that all the components of $T$ within $U \cap M$ extend continuously to $(U \cap M) \cup \{\mathcal{O}\}$.

# $\Lambda$ appears as an initial condition

**Theorem (L. '22)**

*Suppose* $(M_{ext}, g_{ext})$ *is a* $C^0$ *extension of a* $C^\infty$ *spacetime* $(M, g)$ *such that* $M = I^+(\mathcal{O}, M_{ext})$ *for some* $\mathcal{O} \in \partial^- M$. *Assume*

(a) $(M, g)$ *solves the Einstein equations with a perfect fluid* $(u, \rho, p)$.

(b) *The integral curves of* $u$ *have past endpoint* $\mathcal{O}$.

(c) $\mathrm{Ric}$, $\rho$, *and* $p$ *extend continuously to* $M \cup \{\mathcal{O}\}$.

(d) $(M_{ext}, g_{ext})$ *is strongly causal at* $\mathcal{O}$.

*Then the continuous extensions of* $\rho$ *and* $p$ *satisfy*

$$\widetilde{\rho}(\mathcal{O}) = -\widetilde{p}(\mathcal{O}).$$

*Moreover, the continuous extension of* $\mathrm{Ric}$ *at* $\mathcal{O}$ *is*

$$\widetilde{\mathrm{Ric}}|_{\mathcal{O}} = 8\pi \widetilde{\rho}\, g_{ext}|_{\mathcal{O}}.$$

*Proof.* Suppose $\widetilde{\rho}(\mathcal{O}) \neq -\widetilde{p}(\mathcal{O})$. Then

$$\mathrm{Ric} - \frac{1}{2}Rg \;=\; 8\pi T \;=\; 8\pi\big[(\rho + p)u_* \otimes u_* + pg\big]$$

implies

$$u_* \otimes u_* \;=\; \frac{1}{\rho + p}(T - pg)$$

in $U \cap M$ where $U$ is a coordinate neighborhood of $\mathcal{O}$ where $\widetilde{\rho} \neq -\widetilde{p}$. The RHS extends continuously to $(U \cap M) \cup \{\mathcal{O}\}$. Therefore so does the LHS.

$$\Longrightarrow \quad u_* \text{ extends continuously to } M \cup \{\mathcal{O}\}.$$
$$\Longrightarrow \quad u \text{ extends continuously to } M \cup \{\mathcal{O}\}.$$

However, the assumptions

- integral curves of $u$ have past endpoint $\mathcal{O}$
- $M = I^+(\mathcal{O}, M_{\mathrm{ext}})$
- strong causality at $\mathcal{O}$

$$\Longrightarrow \quad u \text{ does not extend continuously to } M \cup \{\mathcal{O}\}.$$

□

Applications to inflationary theory

# Applications to inflationary theory

- Many cosmologists argue that the scale factor underwent an inflationary era:
$$\ddot{a}(\tau) \,>\, 0,$$
after the big bang but before the radiation dominated era.

- Recall Friemdann's second equation: $\frac{\ddot{a}}{a} = -\left(\frac{4\pi}{3}\right)(\rho + 3p)$.

- Therefore
$$\rho(0) = -p(0) \quad \text{and} \quad \rho(0) > 0 \quad \Longrightarrow \quad \ddot{a}(\tau) > 0$$
for $\tau$ near $\tau = 0$. This can be thought of as a "quasi de Sitter" expansion for the early universe.

- The main theorem can be used to generalize this "quasi de Sitter" expansion to spacetimes without any isotropic or homogeneous assumptions.

# Applications to inflationary theory

Generalizing Friedmann's 2nd equation:

- For FLRW spacetimes, Friedmann's 2nd eq. can be expressed as

$$3\frac{\ddot{a}}{a} = -\mathrm{Ric}(u, u).$$

- For arbitrary spacetimes with a unit timelike vector field $u$, the generalization of Friedmann's 2nd eq. is the Raychaudhuri eq.:

$$3\frac{\ddot{\mathfrak{a}}}{\mathfrak{a}} = -\mathrm{Ric}(u, u) + \omega^2 - \sigma^2 + \mathrm{div}(\nabla_u u),$$

where $\mathfrak{a}(\tau)$ is defined via $\frac{\dot{\mathfrak{a}}}{\mathfrak{a}} = H$ where $H = \frac{1}{3}\mathrm{div}(u)$.

  - $\mathfrak{a}(\tau)$ generalizes the scale factor.
  - $H(\tau)$ generalizes the Hubble parameter.
  - $\tau$ is the flower parameter of $u$, i.e. its proper time.

Also

  - $\omega^2 = \omega_{ij}\omega^{ij} \geq 0$ is the vorticity scalar.
  - $\sigma^2 = \sigma_{ij}\sigma^{ij} \geq 0$ is the shear scalar.

# Applications to inflationary theory

- From the theorem, we have

$$\widetilde{\mathrm{Ric}}|_{\mathcal{O}} = 8\pi\widetilde{\rho}\, g_{\mathrm{ext}}|_{\mathcal{O}}.$$

- Therefore

$$\mathrm{Ric}(u,u) \approx -8\pi\widetilde{\rho}(\mathcal{O})$$

  for points in $M$ sufficiently close to $\mathcal{O}$.

- Inserting this into the Raychaudhuri equation gives

$$3\frac{\ddot{\mathfrak{a}}}{\mathfrak{a}} \approx 8\pi\widetilde{\rho}(\mathcal{O}) + \omega^2 - \sigma^2 + \mathrm{div}(\nabla_u u).$$

- $\sigma^2$ and $\mathrm{div}(\nabla_u u)$ represent anisotropic terms. For a universe that begins nearly isotropic (but not perfectly isotropic), then we expect these terms to be small. Then $\widetilde{\rho}(\mathcal{O}) > 0$ implies

$$\ddot{\mathfrak{a}} > 0$$

  for points in $M$ sufficiently close to $\mathcal{O}$.

- This yields a generalization of an inflationary era in the nonhomogeneous setting.

Past-asymptotically flat de Sitter spacetimes

(joint with Ghazal Geshnizjani and Jerome Quintin)

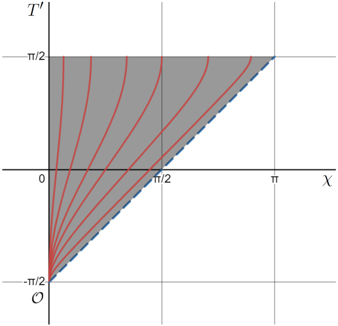
# Past-asymptotically flat de Sitter spacetimes

- Consider flat ($k = 0$) FLRW spacetimes:

$$M = I \times \mathbb{R}^3 \qquad \text{and} \qquad g = -d\tau^2 + a^2(\tau)[dx^2 + dy^2 + dz^2].$$

- If $I = \mathbb{R}$ and $a(\tau) = e^{h\tau}$ for some $h > 0$, then we will call $(M, g)$ a flat de Sitter spacetime.

- Flat de Sitter spacetimes conformally embed into the Einstein static universe:



# Past-asymptotically flat de Sitter spacetimes

**Theorem (Yoshida, Quintin '18 and Geshnizjani, L., Quintin '23)**

*Consider a $k = 0$ FLRW spacetime with scale factor $a(\tau)$ defined on $(-\infty, \tau_{\max})$. Let $h > 0$.*

(1) *If $a(\tau) = e^{h\tau} + o(e^{h\tau})$ or if $H(\tau) \to h$ as $\tau \to -\infty$, then the spacetime admits a continuous extension through the big bang.*

(2) *If $H(\tau) \to h$ and $\dot{H}(\tau) \to 0$, then*

$$R, \quad R_{\mu\nu}R^{\mu\nu}, \quad R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$$

*all converge as $\tau$ approaches the big bang, $\tau = -\infty$. We called these* past asymptotically flat de Sitter *spacetimes.*

(3) *If $a(\tau) = e^{h\tau} + o(e^{(2+\varepsilon)h\tau})$ and $\dot{a}(\tau) = he^{h\tau} + o(e^{(2+\varepsilon)h\tau})$ as $\tau \to -\infty$, then the spacetime admits a $C^1$ extension through the big bang. If also $\dot{H}/a^2$ converges as $\tau \to -\infty$, then the spacetime admits a $C^2$ extension through the big bang. In this case, there are no curvature singularities at the big bang.*

(4) *Sufficient conditions exist for gaining $C^k$ regularity for any $k$.*

# Past-asymptotically flat de Sitter spacetimes

**The cosmological constant appears as an initial condition.**

Proposition (Geshnizjani, L., Quintin '23)

*Under appropriate assumptions of the scale factor, a $k = 0$ FLRW spacetime is past-asymptotically de Sitter if and only if*

$$\rho(-\infty) = -p(-\infty).$$

*Proof of the forward direction:*

The Friedmann equations imply

$$\dot{H} = -4\pi(\rho + p).$$

If the spacetime is past-asymptotically de Sitter, then $\dot{H} \to 0$ as $\tau \to -\infty$. Moreover, $\rho$ and $p$ must converge as $\tau \to -\infty$. $\square$

# Past-asymptotically flat de Sitter spacetimes

**Motivation:** Analogous to Milne-like spacetimes, we wish to generalize the previous proposition to spacetimes which share similar geometrical properties with past-asymptotically flat de Sitter spacetimes but without any isotropic or homogeneous assumptions.

**Definition:** Suppose $(M, g)$ is a $C^\infty$ spacetime and $(\tilde{M}, \tilde{g})$ is a $C^0$ spacetime. We call $(\tilde{M}, \tilde{g})$ a $C^0$ conformal extension of $(M, g)$ if there is an embedding $M \hookrightarrow \tilde{M}$ and a $C^0$ function $\Omega \colon \overline{M} \to [0, \infty)$ such that

(i) $M \subset \tilde{M}$ is a proper subset,

(ii) $\Omega|_M > 0$ and $\partial_0 M \neq \emptyset$ where $\partial_0 M := \{p \in \partial M \mid \Omega(p) = 0\}$,

(iii) the pull back of $\tilde{g}$ under the embedding equals $\Omega^2 g$.

We define the future and past conformal boundaries:

$$\partial_0^+ M = \{p \in \partial_0 M \mid \text{there is a future-terminating timelike curve for } p\}\,,$$
$$\partial_0^- M = \{p \in \partial_0 M \mid \text{there is a past-terminating timelike curve for } p\}\,.$$

# Past-asymptotically flat de Sitter spacetimes

**Theorem (Geshnizjani, L., Quintin '23)**

*Let $(\tilde{M}, \tilde{g})$ be a $C^0$ conformal extension of $(M, g)$ with conformal factor $\Omega$ such that $M = I^+_{\tilde{g}}(\mathcal{O}, M \cup \{\mathcal{O}\})$ for some point $\mathcal{O} \in \partial_0^- M$. Assume:*

(a) *$(M, g)$ solves the Einstein equations with a perfect fluid $(u, \rho, p)$.*

(b) *Each of the integral curves of $u$ have past endpoint $\mathcal{O}$ within $\tilde{M}$.*

(c) *$\rho$, $p$, and $\Omega^2 \mathrm{Ric}_g$ extend continuously to $M \cup \{\mathcal{O}\}$.*

(d) *$(\tilde{M}, \tilde{g})$ is strongly causal at $\mathcal{O}$.*

*Then the continuous extensions of $\rho$ and $p$ satisfy*

$$\widetilde{\rho}(\mathcal{O}) = -\widetilde{p}(\mathcal{O}).$$

*Moreover, the continuous extension of $\Omega^2 \mathrm{Ric}$ at $\mathcal{O}$ is*

$$\widetilde{\Omega^2 \mathrm{Ric}}|_{\mathcal{O}} = 8\pi \tilde{\rho}(\mathcal{O}) \tilde{g}|_{\mathcal{O}}.$$

As in the Milne-like case, this theorem can be used to generalize inflationary eras provided the anisotropic terms are small.

## Questions/Future directions

(1) Are there nonhomogeneous examples of Milne-like or past-asymptotically flat de Sitter spacetimes which satisfy the hypotheses of theorems on the cosmological constant?

(2) Can one formulate an initial value problem for Milne-like or past-asymptotically flat de Sitter spacetimes analogous to the Choquet-Bruhat-Geroch theorem for globally hyperbolic developments of initial data sets?

(3) Can one glue a past-asymptotically flat de Sitter spacetime together with a time-reversed copy to produce a *bouncing* cosmology as in the full de Sitter model?

Thank you!

References


- Gregory J. Galloway and Eric Ling, *Some remarks on the $C^0$-inextendibility of spacetimes*, Ann. H. Poincaré **18** (2017).
- D. Yoshida and J. Quintin, *Maximal extensions and singularities in inflationary spacetimes*, Class. Quant. Grav. **35** (2018).
- E. Ling, *The Big Bang is a Coordinate Singularity for $k = -1$ Inflationary FLRW Spacetimes*, Found. of Phys. **50** (2020).
- E. Ling, *Remarks on the cosmological constant appearing as an initial condition for Milne-like spacetimes*, Gen. Rel. Grav. **54** (2022).
- E. Ling and A. Piubello, *On the asymptotic assumptions for Milne-like spacetimes*, Gen. Rel. Grav. **55** (2023).
- G. Geshnizjani, E. Ling, J. Quintin, *On the initial singularity and extendibility of flat quasi-de Sitter spacetimes*, arXiv:2305.01676, (2023).