

The separate-universe approach and sudden transitions during inflation

Joe Jackson

arXiv:2311.03281

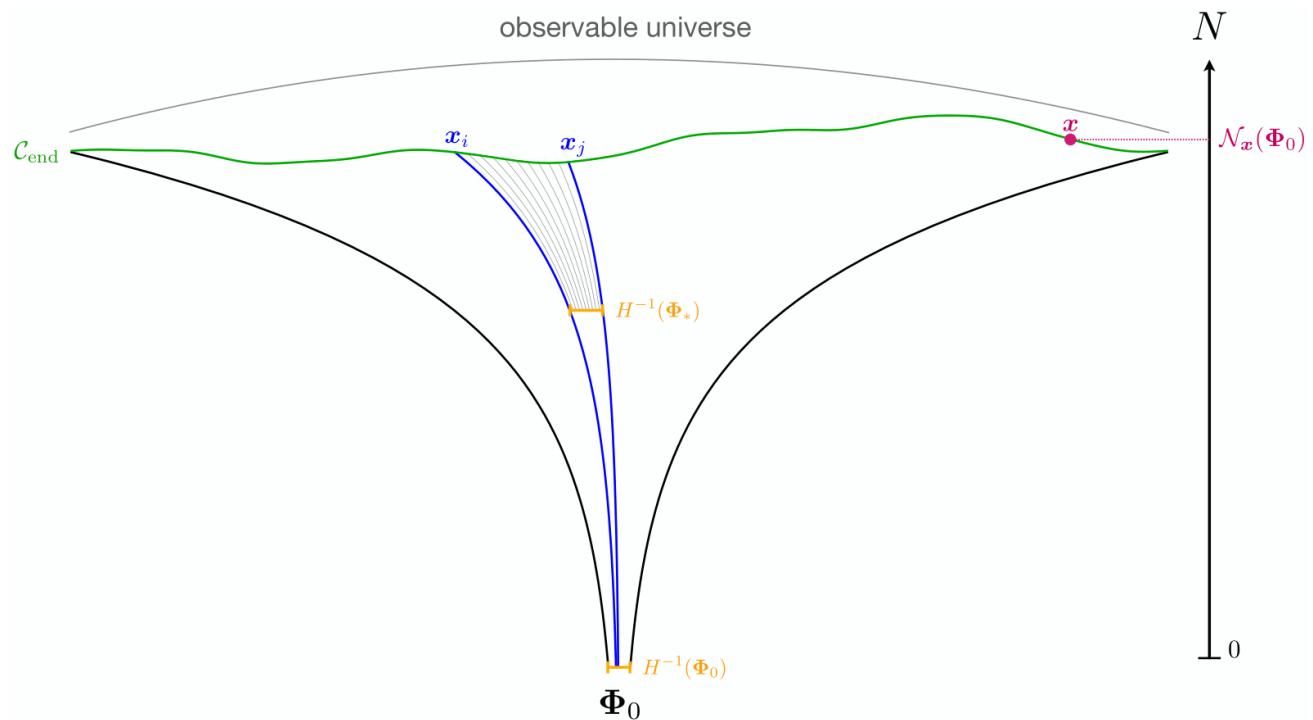
With Hooshyar Assadullahi, Andrew Gow, Kazuya Koyama, Vincent
Vennin and David Wands



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Inflation



$$N = \ln a$$

Inflation Dynamics

The equation of motion is (in $M_{\text{pl}} = 1$ units)

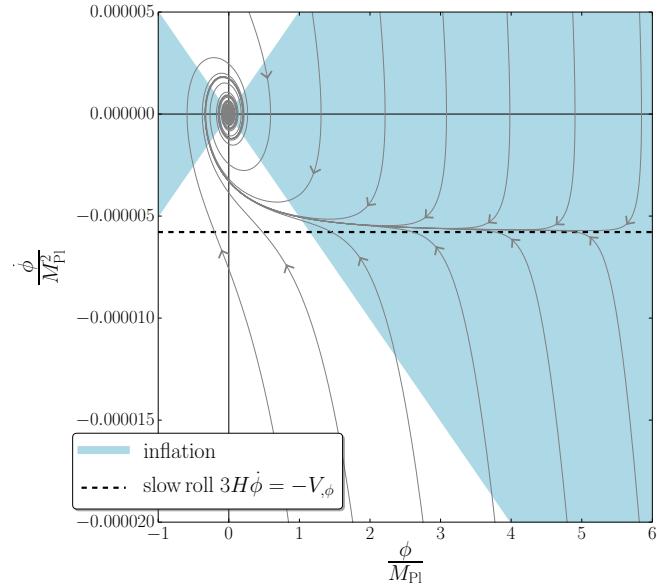
$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0, \quad \& \quad H^2 = \frac{\dot{\phi}^2}{6} + \frac{V}{3}. \quad (1)$$

The Hubble-flow parameters characterise the dynamics

$$\epsilon_{i+1} = \frac{d \ln \epsilon_i}{dN}, \quad \epsilon_0 = \frac{H_{\text{init}}}{H}, \quad (2)$$

$\epsilon_1 < 1$ corresponds to inflation.

Slow-roll

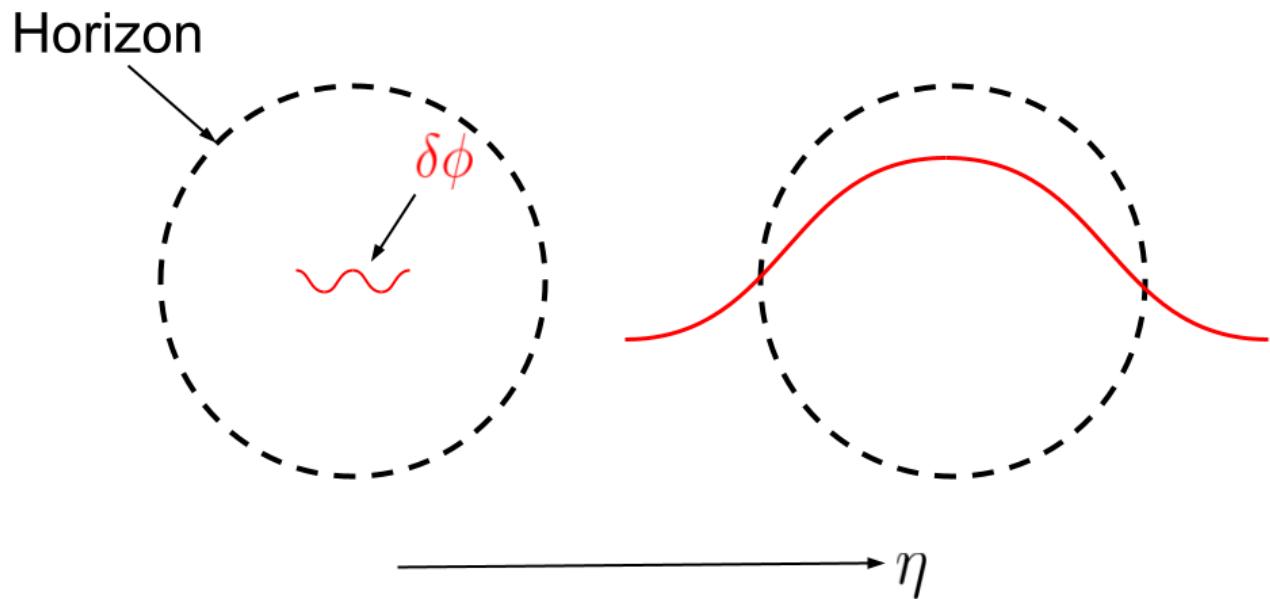


If Hubble-friction dominates

$$\dot{\phi} \approx -\frac{1}{3H} \frac{dV(\phi)}{d\phi} \quad \& \quad H^2 \approx \frac{V}{3}. \quad (3)$$

or more succinctly $|\epsilon_i| \ll 1 \ \forall i \geq 1$.

Quantum field fluctuations



$$\eta = \int \frac{dt}{a(t)} . \quad (4)$$

Linear perturbations

The E.O.M. for the curvature perturbation $\mathcal{R} = H\delta\phi/\dot{\phi}$ is

$$\mathcal{R}_k'' + 2\frac{z'}{z}\mathcal{R}_k' + k^2\mathcal{R}_k = 0 \quad (5)$$

Prime denotes η derivative

where

$$z = \frac{a\dot{\phi}}{H} \quad \& \quad \frac{z'}{z} = aH \left(1 + \frac{\epsilon_2}{2}\right). \quad (6)$$

For a massless field in de Sitter

$$\mathcal{R}_k = \frac{\text{sign}(\dot{\phi})}{2a\sqrt{k\epsilon_1}} \left[\alpha_k \left(1 - \frac{i}{k\eta}\right) e^{-ik\eta} + \beta_k \left(1 + \frac{i}{k\eta}\right) e^{ik\eta} \right]. \quad (7)$$

Bunch–Davies (BD) vacuum $\implies \alpha_k = 1$ and $\beta_k = 0$.

Cosmic Microwave Background

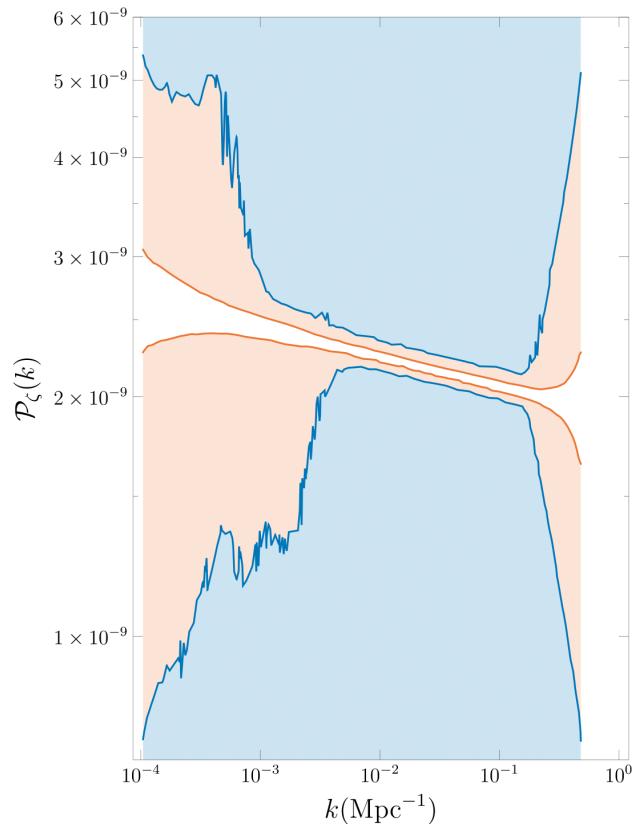
On super-horizon scales ($-k\eta \ll 1$)

$$\frac{k^{3/2}}{\pi\sqrt{2}} |\mathcal{R}_k| \approx \frac{H}{2\pi\sqrt{2\epsilon_1}} \Big|_{k=\sigma aH}. \quad (8)$$

This gives

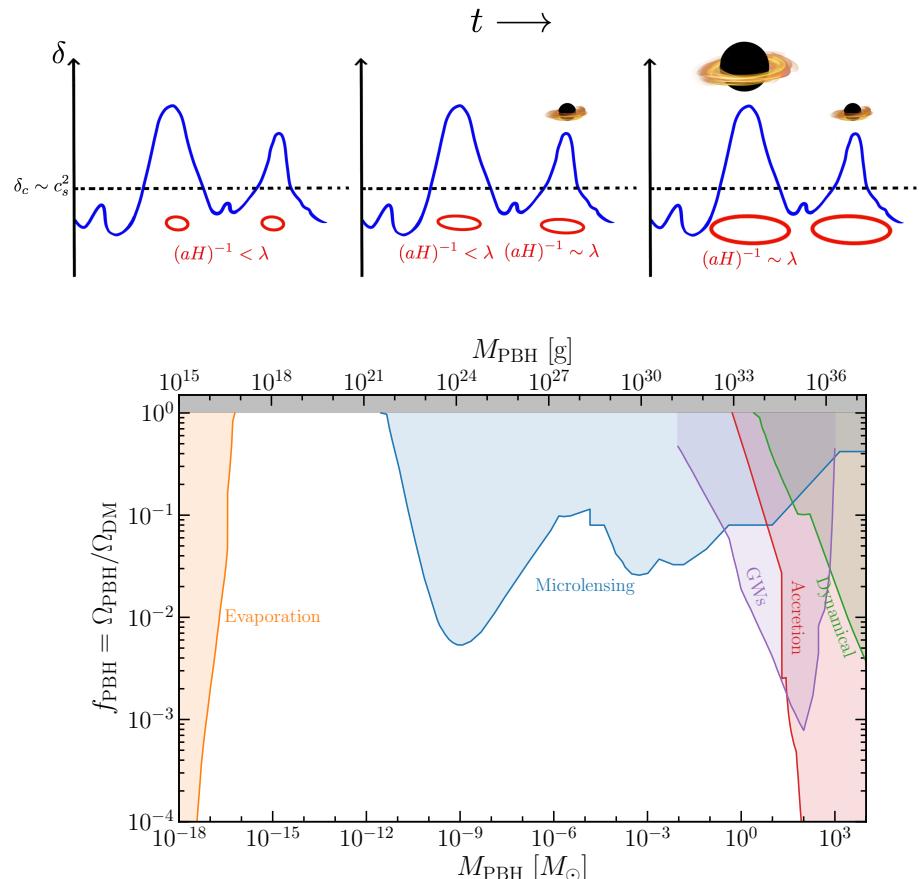
$$\mathcal{P}_{\mathcal{R}}(k) \approx \frac{H^2}{8\pi^2\epsilon_1} \Big|_{k=\sigma aH}. \quad (9)$$

Normally $\sigma = 1$ is used for the matching scale σaH .

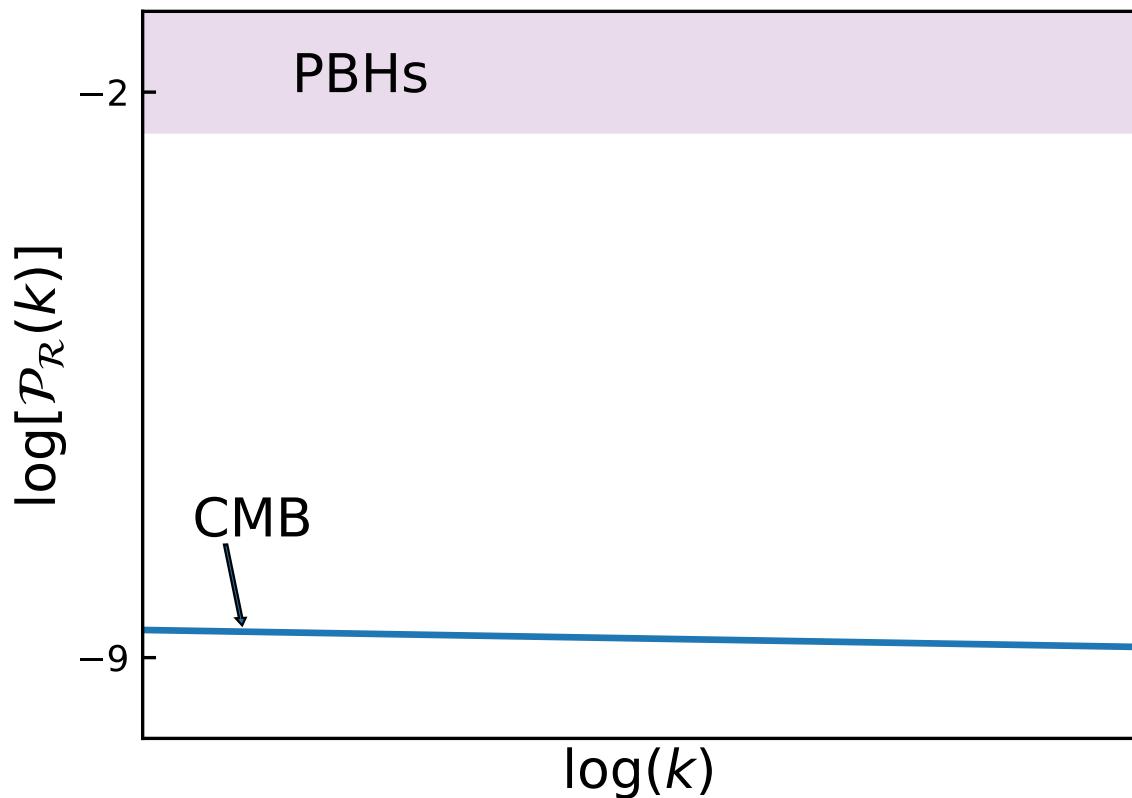


Planck 2018

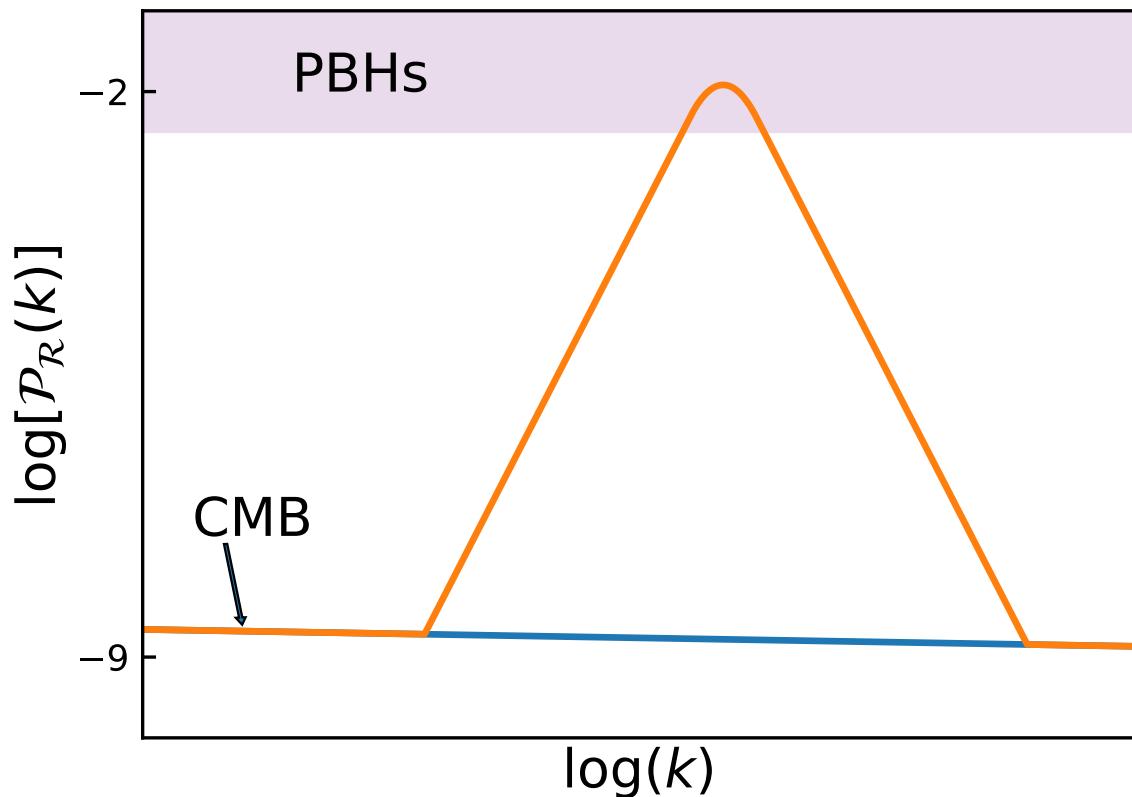
Primordial black holes



Primordial black holes - formation I

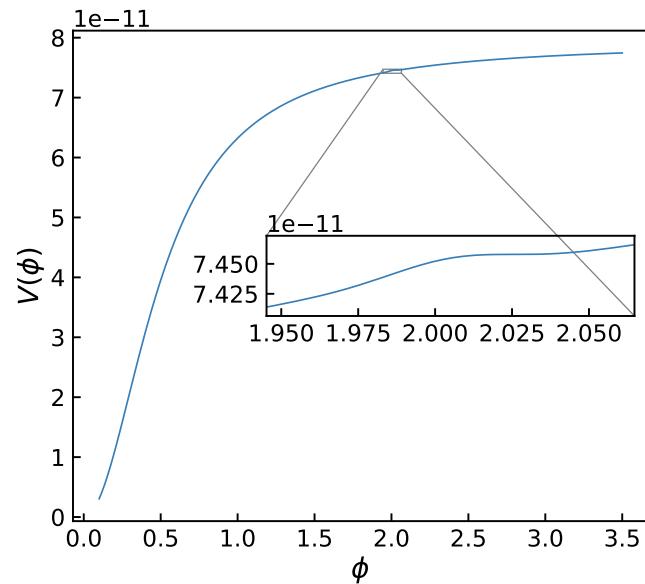
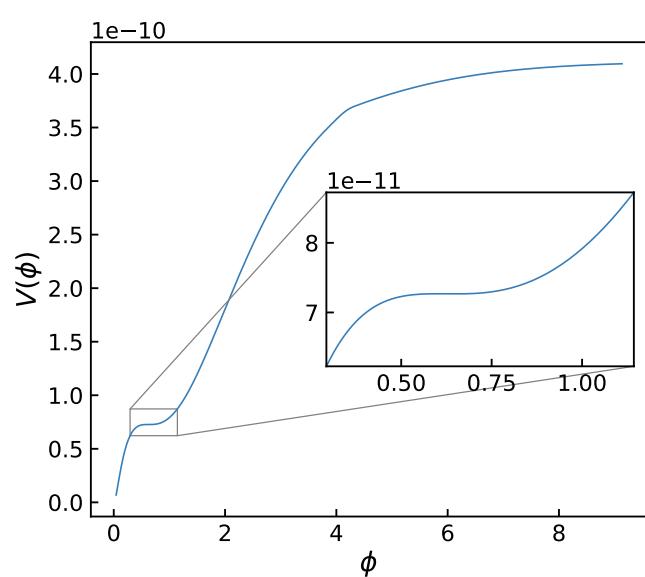


Primordial black holes - formation II



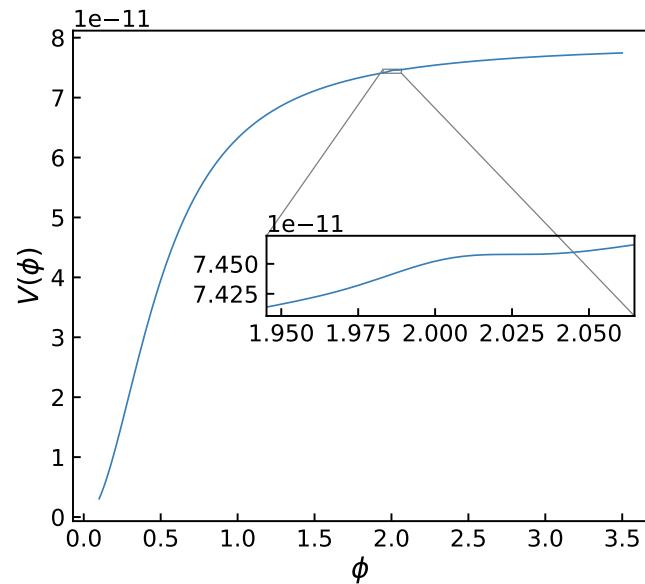
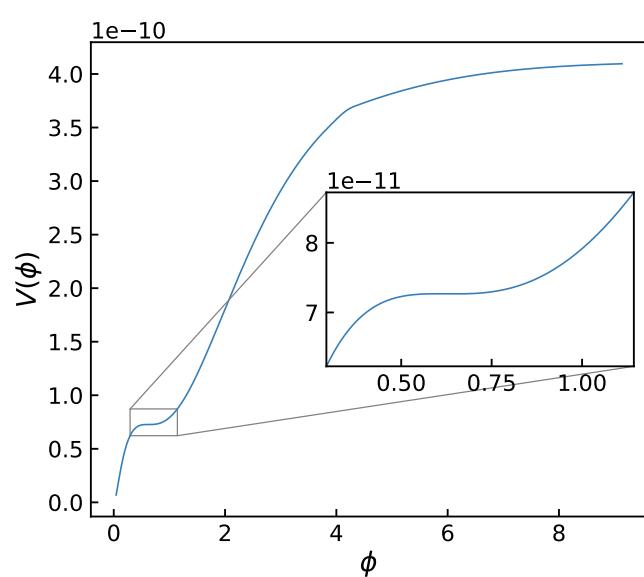
A feature in the potential is needed.

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Ultra-slow-roll ($\epsilon_2 = -6$) is induced with $\epsilon_1 \sim e^{-6N_{\text{USR}}} \implies \mathcal{P}_{\mathcal{R}} \sim e^{6N_{\text{USR}}}$

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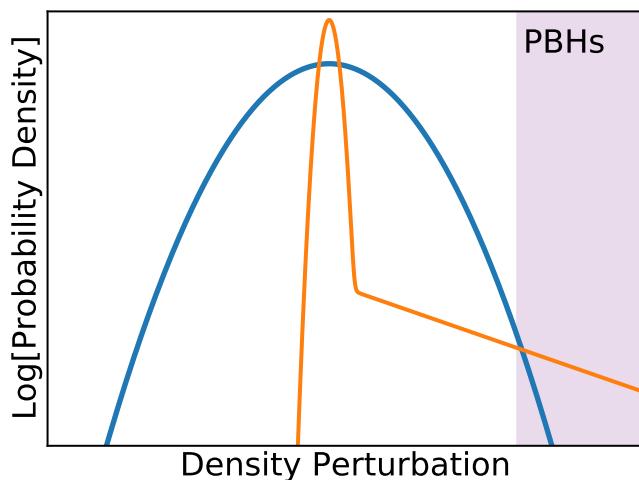
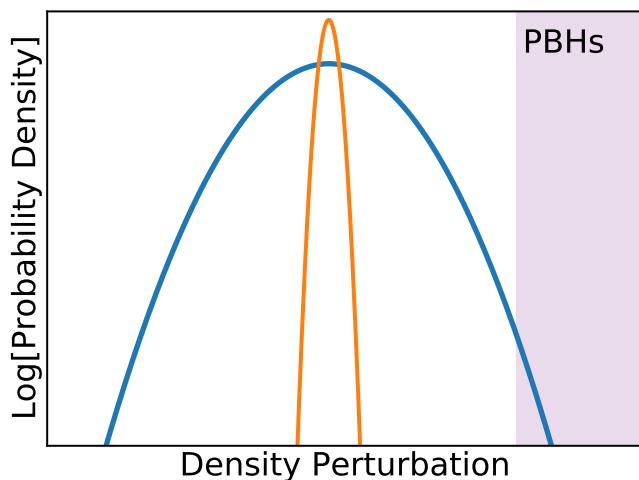
Ultra-slow-roll ($\epsilon_2 = -6$) is induced with $\epsilon_1 \sim e^{-6N_{\text{USR}}} \implies \mathcal{P}_{\mathcal{R}} \sim e^{6N_{\text{USR}}}$

Suffers from a fine-tuning problem (see Cole et al. 2304.01997)

Enckell et al (2012.03660) and Mishra et al (1911.00057)

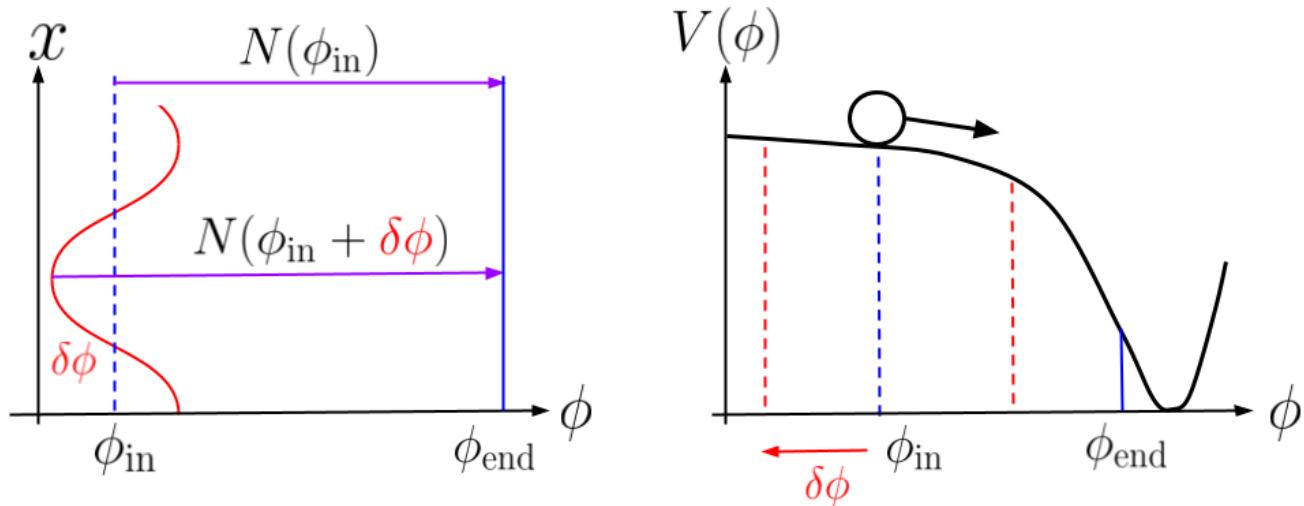
Strong non-Gaussianity

Non-Gaussianity might be a solution.



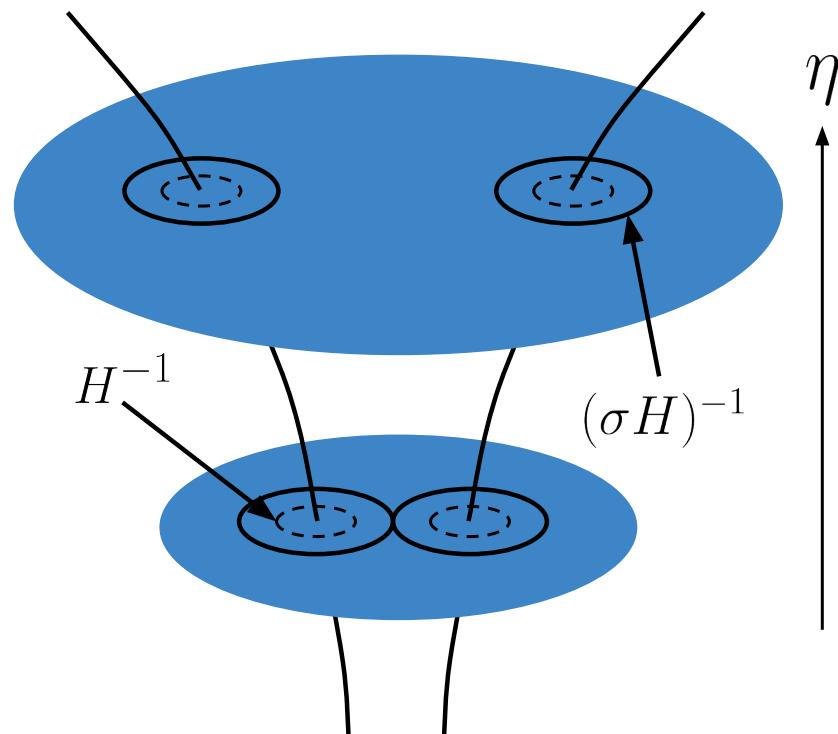
The δN formalism

$$-\mathcal{R} = \zeta = N(\phi_{\text{in}} + \delta\phi) - N(\phi_{\text{in}}) \quad (10)$$

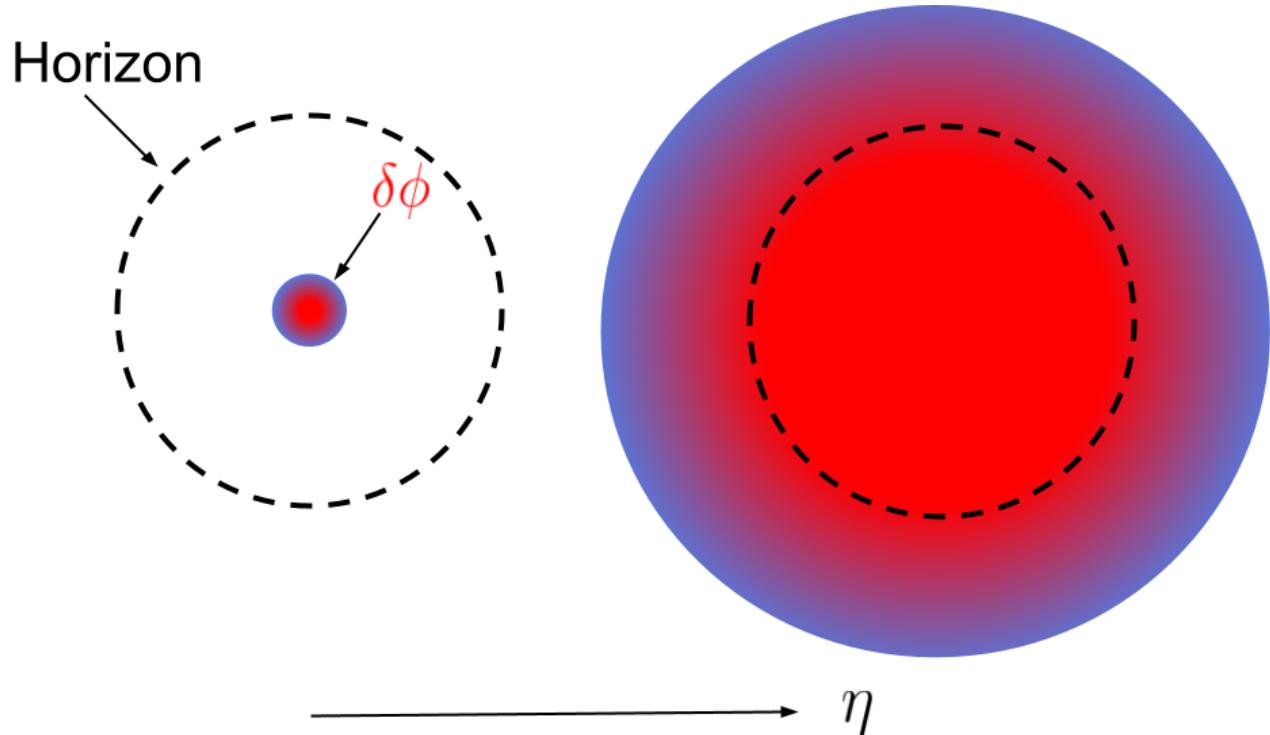


Allows one to go beyond linear theory \implies primordial black holes.

Separate universe approach I



Separate universe approach II



See Wands *et al* (astro-ph/0003278) for details

Separate universe approach II

Perturbing a small *homogeneous* patch by $\phi \rightarrow \phi + \delta\phi_h$, gives

$$\boxed{\mathcal{R}_h'' + 2\frac{z'}{z}\mathcal{R}_h' = 0} \quad (11)$$

This is the same as the $k \rightarrow 0$ limit of $\mathcal{R}_k'' + 2\frac{z'}{z}\mathcal{R}_k' + k^2\mathcal{R}_k = 0$.

For de Sitter + BD ($\alpha_k = 1$, $\beta_k = 0$ and $z'/z = -\eta^{-1}$)

$$\mathcal{R}_k \approx \frac{iH\text{sign}(\dot{\phi})}{2\sqrt{k^3\epsilon_1}} \left[\underbrace{\left(1 - i\frac{(k\eta)^3}{3}\right)}_{\text{homogeneous}} + \underbrace{\left(\frac{(k\eta)^2}{2} - \frac{(k\eta)^4}{8} + \dots\right)}_{\text{gradients}} \right]. \quad (12)$$

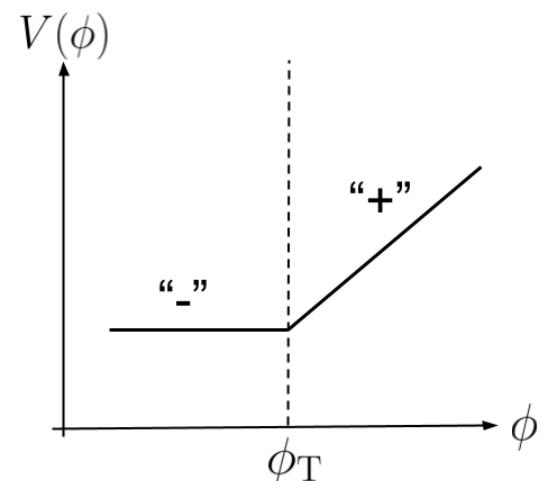
Sudden transition

Starobinsky's piece-wise potential is defined by

$$V(\phi) = \begin{cases} V_0 + A_+(\phi - \phi_T) & \text{for } \phi \geq \phi_T, \\ V_0 + A_-(\phi - \phi_T) & \text{for } \phi < \phi_T, \end{cases} \quad (13)$$

and produces a sharp transition to ultra-slow-roll inflation.

$$\epsilon_2(\eta) = \begin{cases} 0 & \text{for } \eta \leq \eta_T, \\ -\frac{6(A_- - A_+)k_T^3\eta^3}{(A_- - A_+)k_T^3\eta^3 + A_-} & \text{for } \eta > \eta_T. \end{cases} \quad (14)$$



Starobinsky JETP Lett. 55 (1992) 489–494

Sudden transition: implications

This gives a non-adiabatic transition, changing the behaviour of \mathcal{R}_k

$$\mathcal{R}_k = -\frac{iH}{2\sqrt{k^3\epsilon_1(\eta)}} \left[\alpha_{k-} (1 + ik\eta) e^{-ik\eta} + \beta_{k-} (1 - ik\eta) e^{ik\eta} \right]. \quad (15)$$

We are no longer in the BD vacuum

$$\alpha_{k-} \neq 1 \quad \& \quad \beta_{k-} \neq 0.$$

The new homogeneous solution is

$$\mathcal{R}_h = -\frac{iH}{2\sqrt{k^3\epsilon_1(\eta)}} \left[(\alpha_{k-} - \beta_{k-}) - i(\alpha_{k-} + \beta_{k-}) \frac{(k\eta)^3}{3} \right]. \quad (16)$$

Discontinuous homogeneous solution

For $k < k_T$

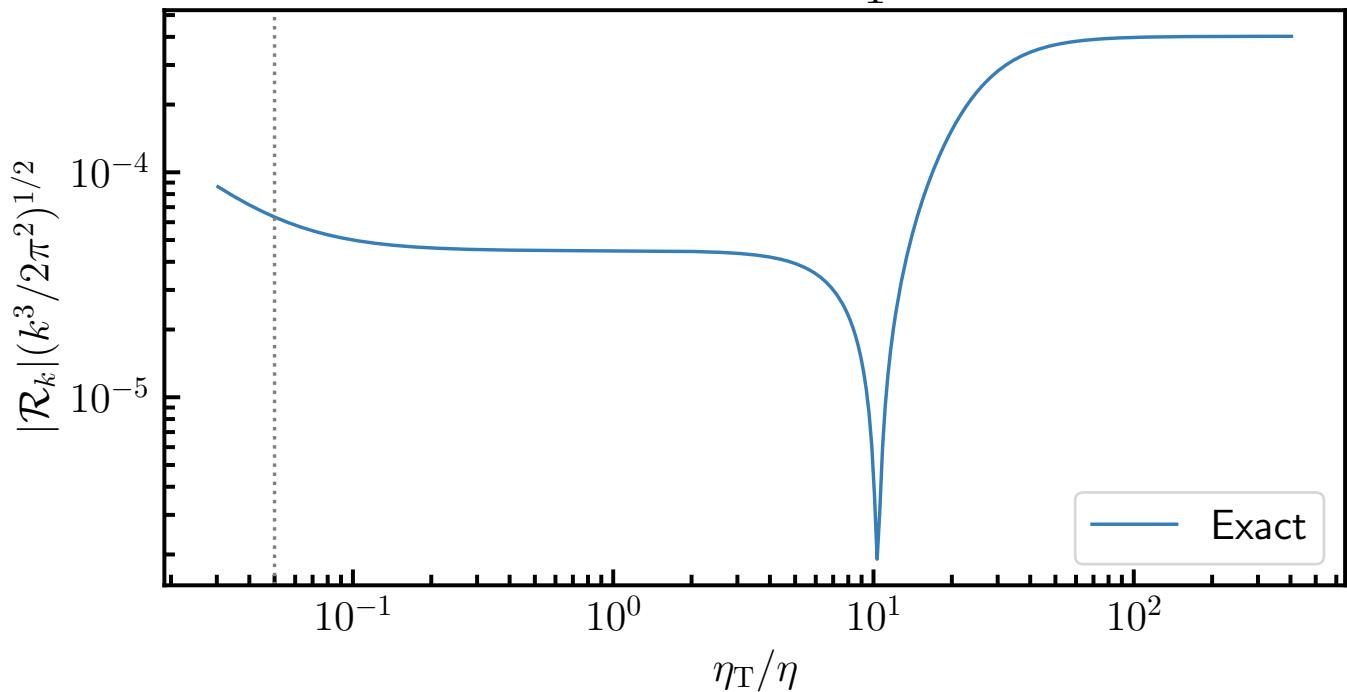
$$\frac{\mathcal{R}_h(\eta_T) - \tilde{\mathcal{R}}_h(\eta_T)}{\tilde{\mathcal{R}}_h(\eta_T)} \simeq -\frac{3}{5} \frac{A_- - A_+}{A_+} \left(\frac{k}{k_T} \right)^2, \quad (17)$$

$$\frac{\mathcal{R}'_h(\eta_T) - \tilde{\mathcal{R}}'_h(\eta_T)}{\tilde{\mathcal{R}}'_h(\eta_T)} \simeq 3i \frac{A_- - A_+}{A_+} \left(1 + \frac{3}{5} \frac{A_- - A_+}{A_+} \right) \frac{k_T}{k}. \quad (18)$$

⇒ The separate universe approach is broken *at the transition*.

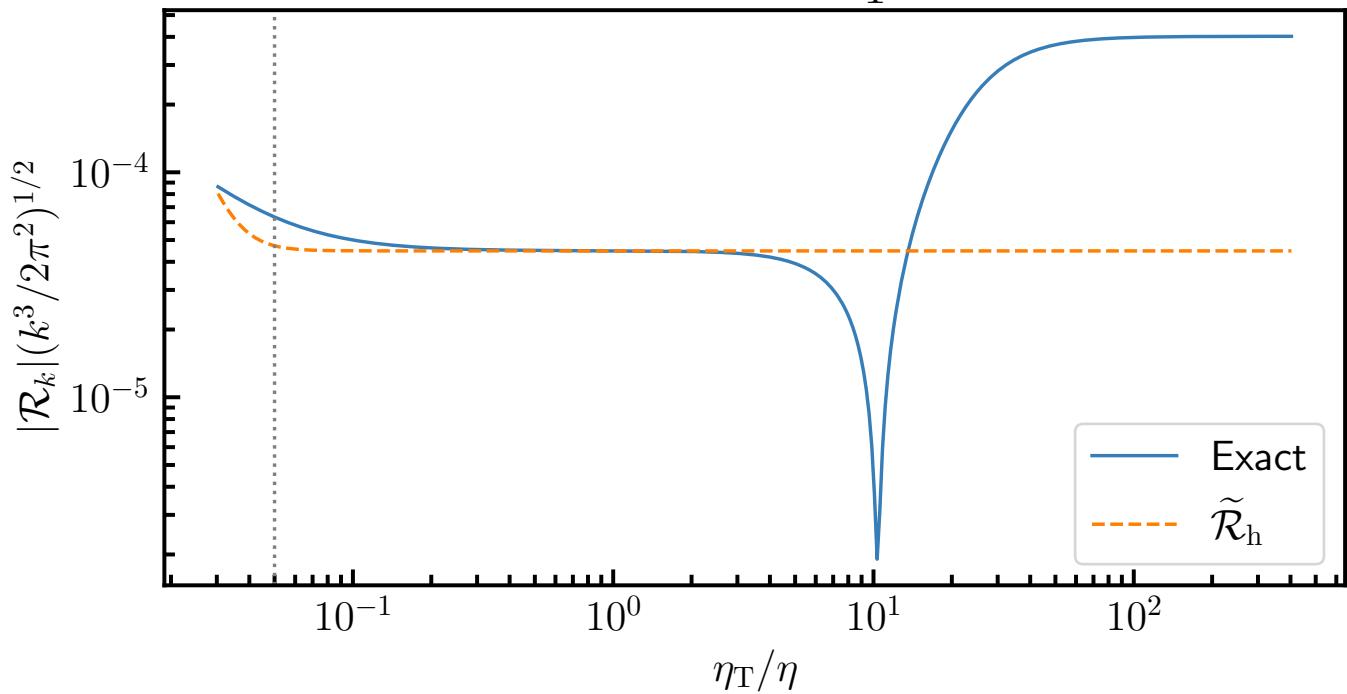
New homogeneous solution II

$$k = 0.05k_T$$



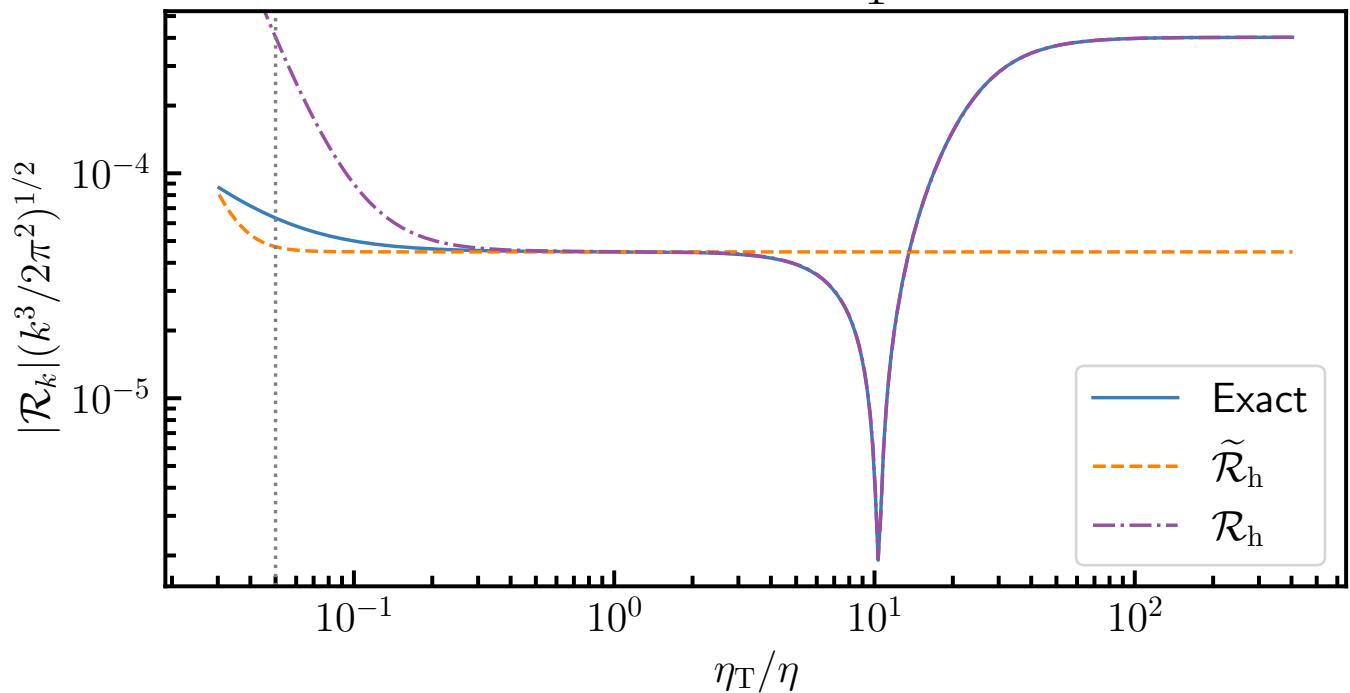
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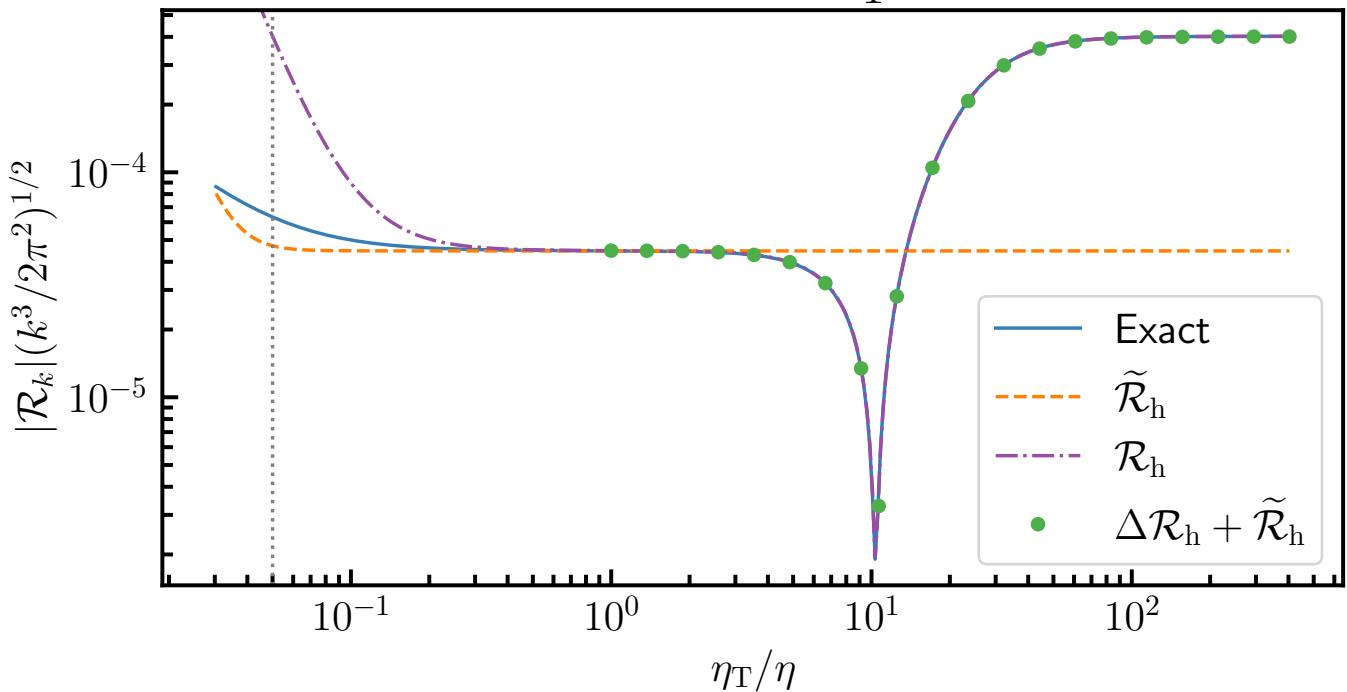
New homogeneous solution II

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New homogeneous solution II

$$k = 0.05k_T$$



Using $\Delta\mathcal{R}_h$ and $\Delta\mathcal{R}'_h$ as the initial condition of

$$\mathcal{R}''_h + 2(z'/z)\mathcal{R}'_h = 0 .$$

Homogeneous matching I

Below

$$\mathcal{R}_h = C + D \int_{\eta}^{\eta_1} \frac{d\tilde{\eta}}{z^2(\tilde{\eta})}, \quad (19)$$

is the general solution of (remember $z^2 = 2a^2\epsilon_1$)

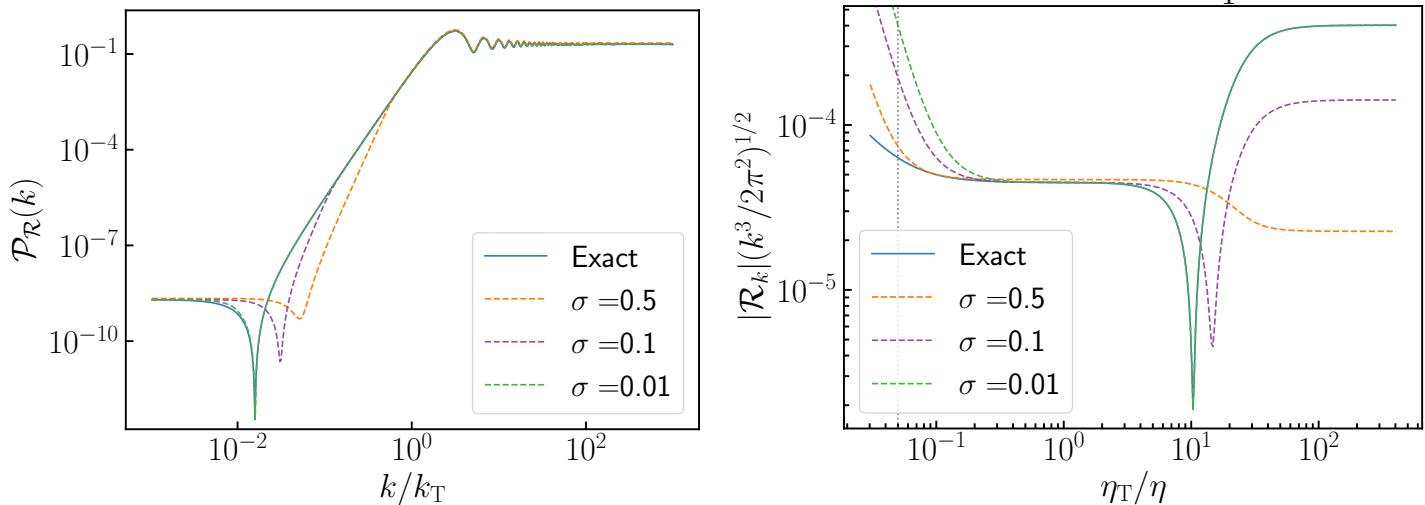
$$\mathcal{R}_h'' + 2(z'/z)\mathcal{R}_h' = 0.$$

Matching is done by

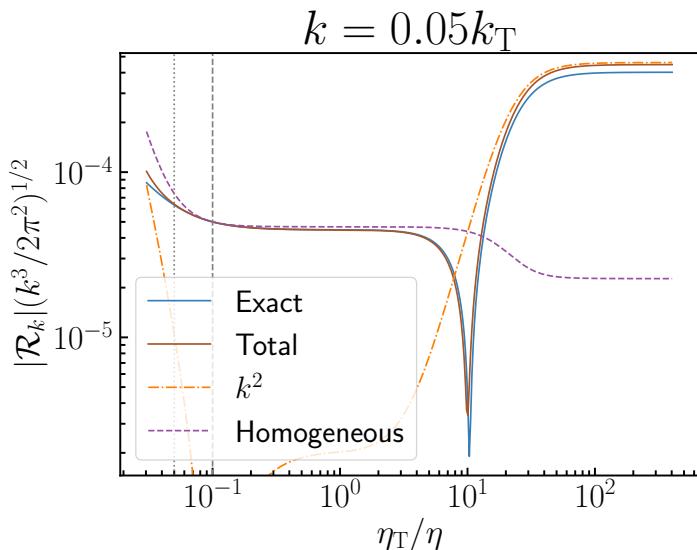
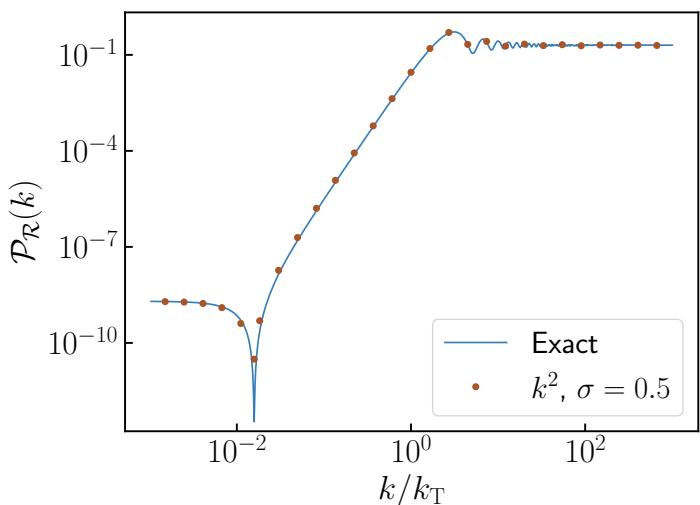
$$\hat{C}_k = \mathcal{R}_{k*} + z_*^2 \int_{\eta_*}^0 \frac{d\tilde{\eta}}{z^2(\tilde{\eta})} \mathcal{R}'_{k*} \quad \text{and} \quad \hat{D}_k = -z_*^2 \mathcal{R}'_{k*}. \quad (20)$$

at the time η_* corresponding to $k = \sigma aH$.

Homogeneous matching II

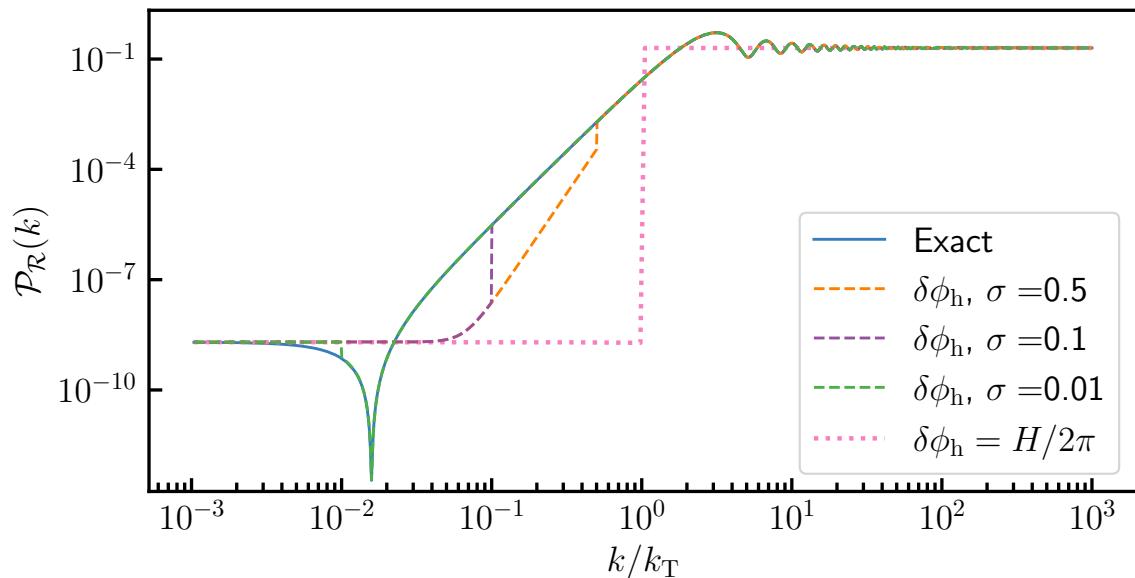


k^2 correction



$$\hat{\mathcal{R}}_k(\eta) = \mathcal{R}_{k*} \left[1 - k^2 \int_{\eta_*}^{\eta} \frac{d\tilde{\eta}}{z^2(\tilde{\eta})} \int_{\eta_*}^{\tilde{\eta}} z^2(\tilde{\eta}) d\tilde{\eta} \right] + z_*^2 \mathcal{R}'_{k*} \int_{\eta_*}^{\eta} \frac{d\tilde{\eta}}{z^2(\tilde{\eta})} \quad (21)$$

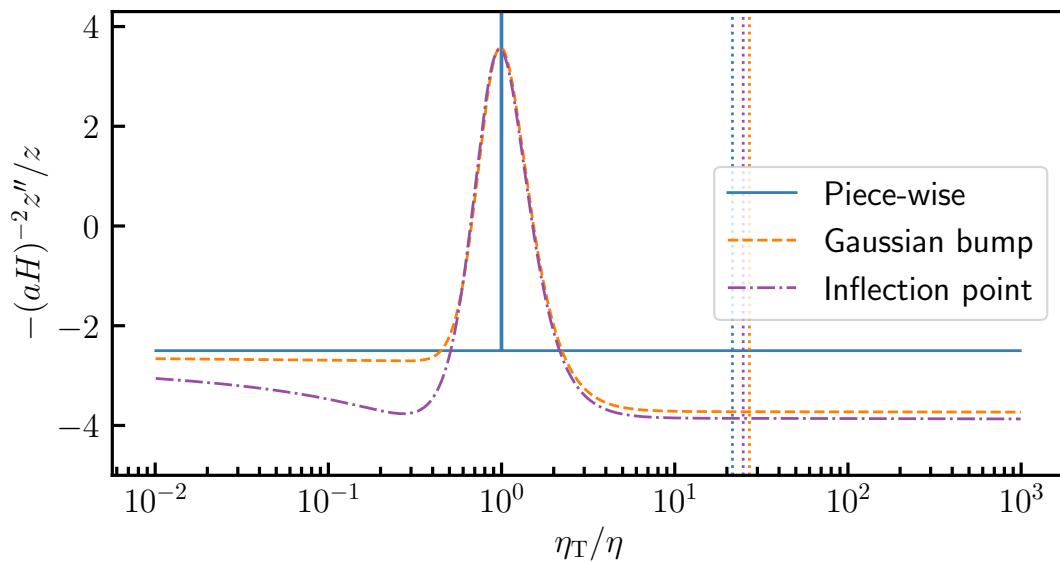
Implications - δN formalism



$$\delta N_k \simeq \frac{\partial N}{\partial \phi_{\text{in}}} (\phi_*, \dot{\phi}_*) \delta \phi_{k*} + \frac{\partial N}{\partial \dot{\phi}_{\text{in}}} (\phi_*, \dot{\phi}_*) \delta \dot{\phi}_{k*} \quad (22)$$

$\implies H/(2\pi)$ cannot describe the peak!

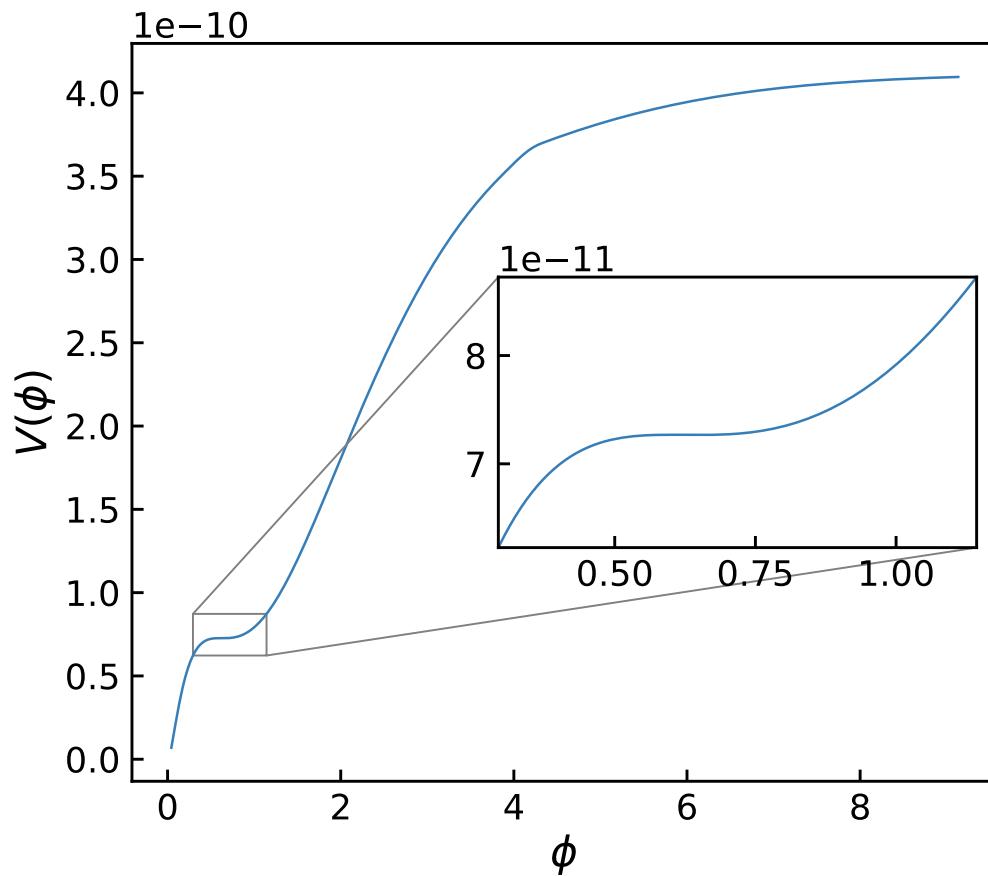
Realistic models: background evolution



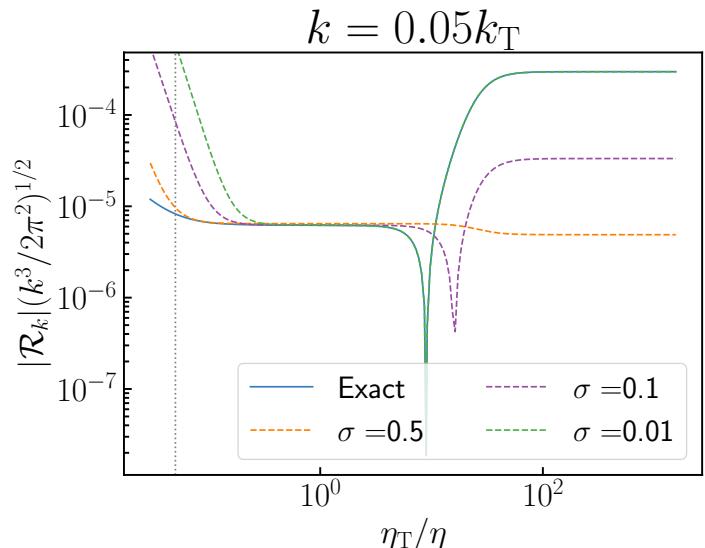
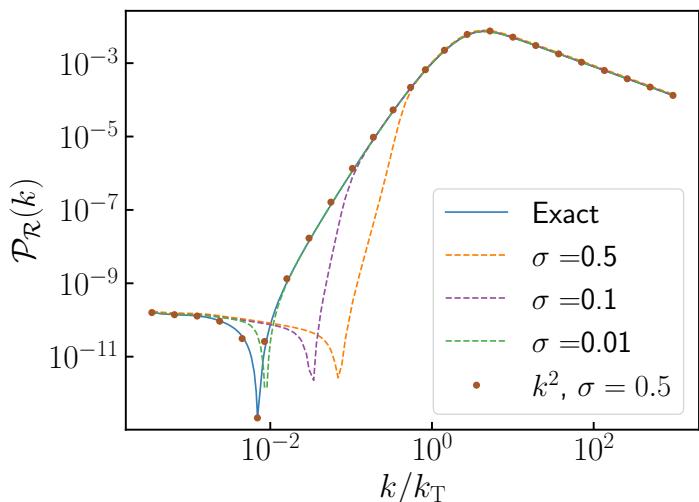
$$\frac{z''}{z} \frac{1}{(aH)^2} = 2 - \epsilon_1 + \frac{3}{2}\epsilon_2 - \frac{1}{2}\epsilon_1\epsilon_2 + \frac{1}{4}\epsilon_2^2 + \frac{1}{2}\epsilon_2\epsilon_3 , \quad (23)$$

$$v_k'' + \left(k^2 - \frac{z''}{z} \right) v_k = 0 \quad \& \quad v_k = \frac{\dot{a\phi}}{H} \mathcal{R}_k = z \mathcal{R}_k . \quad (24)$$

Inflection point - potential



Inflection point - result



η_T corresponds to when $\epsilon_2 < -3$ first occurs.

Implication: classical δN

Apply

$$-\mathcal{R} = \zeta = N(\phi_{\text{in}} + \delta\phi, \dot{\phi}_{\text{in}} + \dot{\delta\phi}) - N(\phi_{\text{in}}), \quad (25)$$

after the transition, using the $\delta\phi/\dot{\delta\phi}$ found from solving

$$\mathcal{R}_k'' + 2\frac{z'}{z}\mathcal{R}_k' + k^2\mathcal{R}_k = 0,$$

and using $\delta\phi = \dot{\phi}\mathcal{R}/H$.

$H/2\pi$ is insufficient.

Implication: stochastic inflation

In the standard case, $\delta\phi_h = H/(2\pi)$ is used. What about sudden transitions?

- Use the unperturbed background evolution to find $\delta\phi_h$ after the transition and then propagate backwards to near horizon crossing time → problems.
- Directly coarse-grain the full numerical $\delta\phi$ with $\sigma \ll e^{-\frac{3}{2}N_{\text{USR}}}$.
- Use $\widetilde{\delta\phi}_h$ for $k < k_T$ and $k < \sigma a_T H_T$. Then apply $\Delta\delta\phi_h$ and $\Delta\delta\phi'_h$ for *all* these modes at the same *instance* → **one kick to rule them all?**

Conclusions

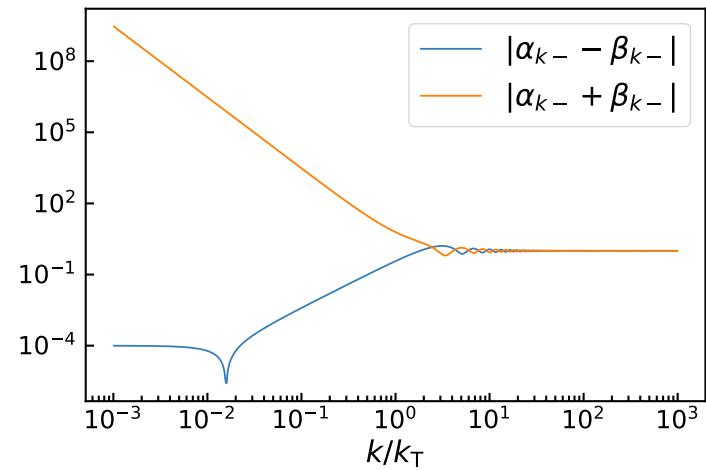
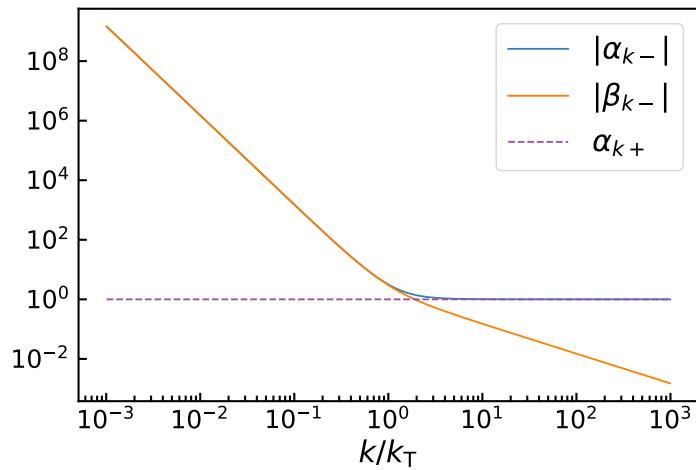
- A sudden transition breaks the separate universe approximation *at the transition*; it applies in a piece-wise manner.
- This is due to mixing between gradient and homogeneous modes at the transition.
- The change of α_k/β_k is associated with particle production on sub-horizon scales and non-adiabatic behaviour on super-horizon scales.
- This has implications for both classical and stochastic δN formalism, as the behaviour of $\delta\phi_h$ is changed by the sudden transition.

α_{k-} and β_{k-}

$$\alpha_{k-} = 1 + \frac{3i}{2} \frac{\Delta A k_T}{A_+ k} \left(1 + \frac{k_T^2}{k^2} \right) , \quad (26)$$

$$\beta_{k-} = -\frac{3i}{2} \frac{\Delta A k_T}{A_+ k} \left(1 + i \frac{k_T}{k} \right)^2 e^{2ik/k_T} . \quad (27)$$

α_{k-} and β_{k-}



Bessel functions

For models with constant

$$\nu^2 = \frac{z''}{z} \frac{1}{(aH)^2} + \frac{1}{4} = \frac{9}{4} - \epsilon_1 + \frac{3}{2}\epsilon_2 - \frac{1}{2}\epsilon_1\epsilon_2 + \frac{1}{4}\epsilon_2^2 + \frac{1}{2}\epsilon_2\epsilon_3 , \quad (28)$$

then

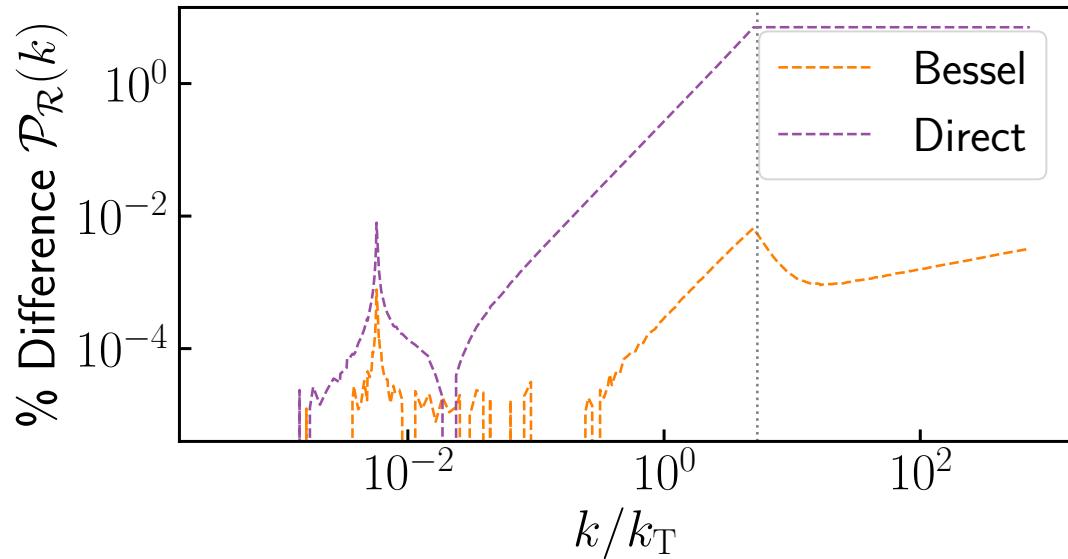
$$\mathcal{R}_k = \text{sign}(d\phi/dN) \frac{\sqrt{-\eta}}{a\sqrt{2\epsilon_1}} [A_k J_\nu(-k\eta) + B_k Y_\nu(-k\eta)] , \quad (29)$$

where

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^{2n} \frac{(-1)^n}{n! \Gamma(\nu + n + 1)} , \quad (30)$$

$$Y_\nu(x) = \frac{J_\nu(x) \cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} . \quad (31)$$

The Bessel matching



$$\begin{aligned} \mathcal{G}_0(\eta) &\approx -\sqrt{\frac{2}{kz^2(\eta)}} \left[B_k \frac{\Gamma(\nu)}{\pi} \right] \left(\frac{-k\eta}{2} \right)^{\frac{1}{2}-\nu}, \\ \mathcal{D}_0(\eta) &\approx \sqrt{\frac{2}{kz^2(\eta)}} \left[\frac{A_k}{\Gamma(\nu+1)} - B_k \frac{\Gamma(-\nu) \cos(\nu\pi)}{\pi} \right] \left(\frac{-k\eta}{2} \right)^{\frac{1}{2}+\nu}. \end{aligned} \quad (32)$$

k^2 expansion

In general we can decompose \mathcal{R}_k into two modes

$$\mathcal{G}(\eta) = \sum_n \mathcal{G}_n(\eta) k^{2n} \quad \text{and} \quad \mathcal{D}(\eta) = \sum_n \mathcal{D}_n(\eta) k^{2n} \quad (33)$$

where

$$\mathcal{G}_0 \equiv C_k \quad \text{and} \quad \mathcal{D}_0(\eta) \equiv D_k \int_{\eta}^0 \frac{d\tilde{\eta}}{z^2(\tilde{\eta})}. \quad (34)$$

The E.O.M can be solved iteratively

$$\mathcal{G}_n'' + 2\frac{z'}{z}\mathcal{G}_n' = -\mathcal{G}_{n-1}, \quad (35)$$

giving

$$\mathcal{G}_n(\eta) = \int_{\eta}^{\eta_3} \frac{d\tilde{\eta}}{z^2(\tilde{\eta})} \int_{\eta_2}^{\tilde{\eta}} z^2(\tilde{\eta}) \mathcal{G}_{n-1}(\tilde{\eta}) d\tilde{\eta}. \quad (36)$$

k^2 in de Sitter

Let us perform the k^2 -expansion in the case where ϵ_1 can be considered as constant. We use the ansatz

$$g_n = -\frac{g_{n-1}}{2n(2n-3)} = \frac{(-1)^n(1-2n)}{(2n)!}. \quad (37)$$

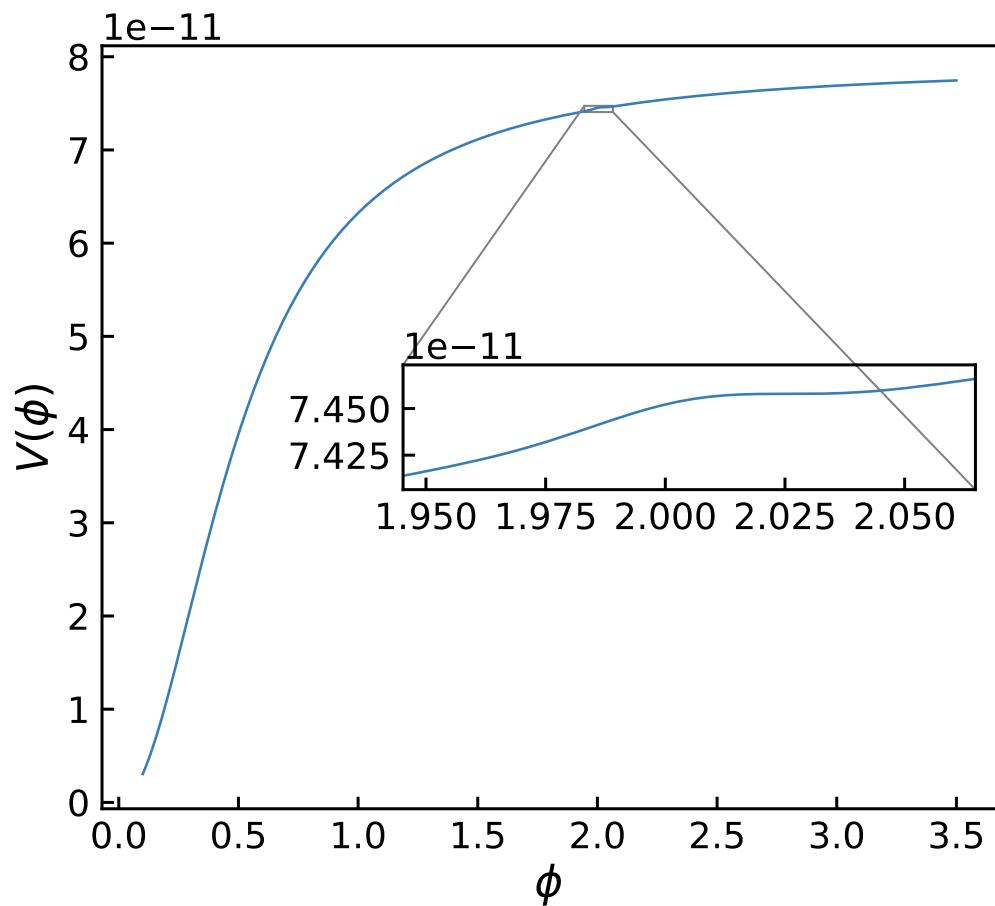
We need to be careful with η_1, η_2 etc.

As a consequence, the power series form leads to

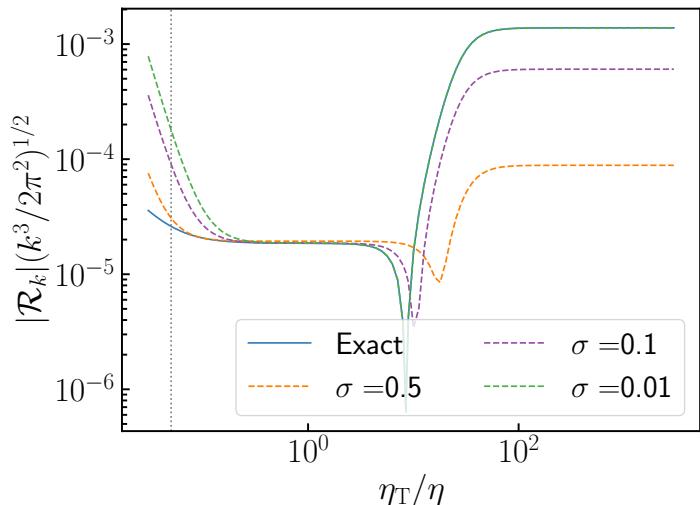
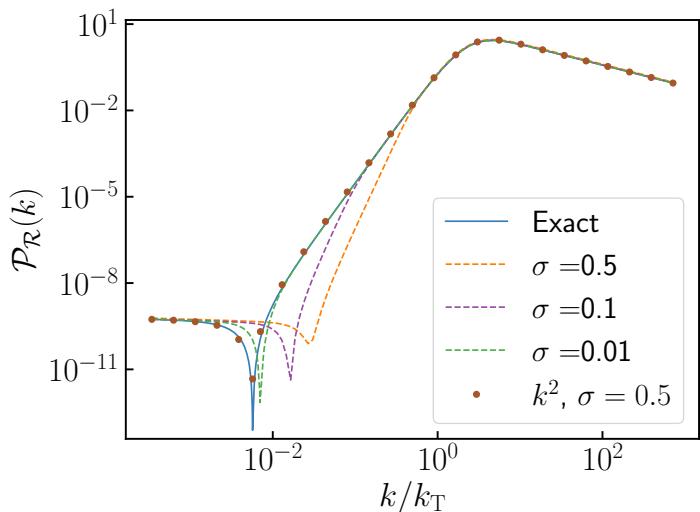
$$\mathcal{G}(\eta) = C_k \sum_{n=0}^{\infty} g_n (k\eta)^{2n} = C_k [\cos(k\eta) + k\eta \sin(k\eta)]. \quad (38)$$

$$\mathcal{D}(\eta) = -D_k \frac{H^2}{6\epsilon_1 k^3} \sum_{n=0}^{\infty} d_n (k\eta)^{2n+3} = -D_k \frac{H^2}{2\epsilon_1 k^3} [\sin(k\eta) - k\eta \cos(k\eta)]. \quad (39)$$

Gaussian Bump - potential



Gaussian Bump - result



$$k = 0.05k_T$$

η_T corresponds when $\epsilon_2 < -3$ first occurs.