Dynamical systems and analytic cosmological bounds

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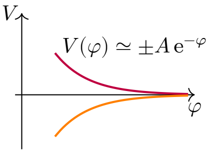
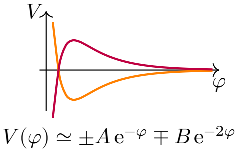
based on works in collaboration with G. Shiu and H.V. Tran

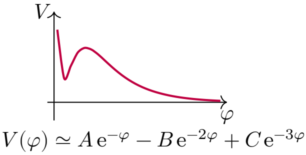
- *Accelerating universe at the end of time* [hep-th/2303.03418]
- *Late-time attractors and cosmic acceleration* [hep-th/2306.07327]
- *Collapsing universe before time* [gr-qc/2312.06772]
- to appear soon [hep-th/2406.XXXXX]

- observations:
  - present-day accelerated cosmic expansion
    also, see recent DESI analysis: the best fit to data might be for a time-dependent dark-energy background!
  - huge scale hierarchies

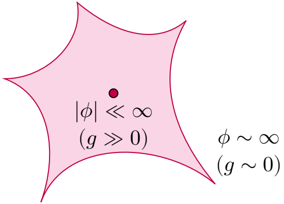
- string-theoretic considerations: all couplings are dynamical ($g = g(\phi)$)
  - Dine-Seiberg problem: hard to find weakly-coupled vacua







  - small coupling constants:
    possibly natural at field-space boundary



Context: dark energy and string theory, pt. 2

- long-standing proposals for 4-dimensional de Sitter vacua:
  - cosmological constant $\Lambda > 0$ in KKLT- and LVS-scenarios
    Kachru, Kallosh, Linde, Trivedi [hep-th/0301240]
    Balasubramanian, Berglund, Conlon, Quevedo [hep-th/0502058]
  - however, fully explicit 10-dimensional realizations remain elusive

- (refined) de Sitter conjecture:
  - effective scalar potentials V consistent with quantum gravity asymptotically bounded as $\partial V/V \geq \kappa_d c$, or $\partial^2 V/V \leq -\kappa_d^2 c'$, with $c, c' \sim 1$
    Obied, Ooguri, Spodyneiko, Vafa [hep-th/1806.08362]
    Ooguri, Palti, Shiu, Vafa [hep-th/1810.05506]
  - generally speaking, strong obstacle to accelerating cosmologies
    - de Sitter minima: ruled out
    - rolling-scalar solutions: admitted,
      but acceleration hard to get if c is not small enough

Our program

our fundamental goal:

to look for cosmic acceleration in scalar-field cosmological solutions
that asymptotically approach the field-space boundary (quintessence-like theories)

FLRW-metric: $d\tilde{s}^2_{1,d-1} = -\mathrm{d}t^2 + a^2(t)\,\mathrm{d}l^2_{\mathbb{E}^{d-1}}$; Hubble parameter: $H = \dfrac{\dot{a}}{a}$

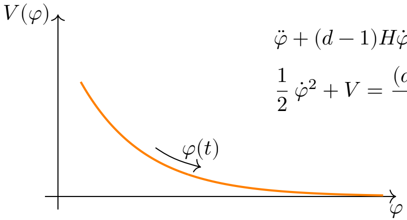
- ϵ-parameter: $\epsilon = -\dfrac{\dot{H}}{H^2}$
  - deceleration parameter: $q = -\dfrac{a\ddot{a}}{\dot{a}^2} = \epsilon - 1$
  - equation of state: $w = \dfrac{p}{\rho} = -1 + \dfrac{2\epsilon}{d-1}$
  - note: we are *not* referring to the slow-roll parameter $\epsilon_V = \dfrac{d-2}{4}\,\dfrac{(\partial V)^2}{\kappa_d^2 V^2}$
- accelerated expansion if $\epsilon < 1$, i.e. $\ddot{a} > 0$

disclaimers:

- no string-theoretic/swampland assumptions employed:
  we study general field theories and obtain model-independent constraints;
  then, we assess what such constraints imply for string compactifications
- we look for a proof of principle that (semi-)eternal cosmic acceleration is possible in string compactifications; we do not try to make close contact with observations

0. Overview and outline: cosmic acceleration with scalar fields

- cosmological equations for a scalar φ rolling down a potential $V = V(\varphi)$:



$$\ddot{\varphi} + (d-1)H\dot{\varphi} + \frac{\partial V}{\partial \varphi} = 0$$

$$\frac{1}{2}\,\dot{\varphi}^2 + V = \frac{(d-1)(d-2)}{2\kappa_d^2}\,H^2$$

- cosmic acceleration possible if V is not too steep:

  rolling is not too quick and "resembles" a de Sitter vacuum

- e.g. $V(\varphi) = \Lambda\,\mathrm{e}^{-\kappa_d \gamma \varphi}$, with $\gamma \leq 2\sqrt{d-1}/\sqrt{d-2}$
  - power-law scale factor: $a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{4}{d-2}\,\frac{1}{\gamma^2}}$
  - $\epsilon = \frac{d-2}{4}\,\gamma^2$: acceleration if $\gamma \leq \frac{2}{\sqrt{d-2}}$

Copeland, Liddle, Wands [gr-qc/9711068]

- note: similar, but more complicated for multiple scalars!

Scaling cosmologies

- cosmologies with multi-field multi-exponential potentials admit solutions with power-law scale factor (these are called scaling solutions: more details later on)

$$a(t) = a_0 \left(\frac{t}{t_0} \right)^{\frac{1}{\epsilon}}$$

Collinucci, Nielsen, Van Riet [hep-th/0407047]

- a shallow exponential provides cosmic acceleration;
  the main challenge in string embeddings is to find shallow-enough potentials

these observations give us the motivation to study multi-field multi-exponential potentials for realizations of late-time cosmic acceleration

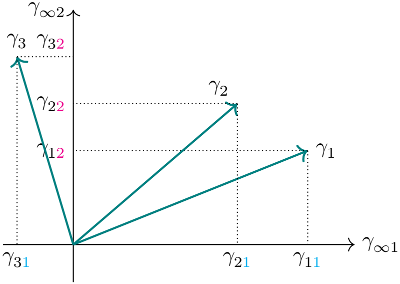
note:

- we are talking about the late-time behavior,
  i.e. we are not looking for cosmic inflation
- at finite times, the solutions can look very different
  (and easily provide cosmic acceleration, e.g. if the field is initially running uphill!
  but typically the accelerating phase is too short to be viable for cosmic inflation)

see e.g. Russo [hep-th/0403010]

field content:

- canonically-normalized scalars ϕ^a, $a = 1, \dots, n$
  - e.g. string compactifications: dilaton, radions (field space is generally curved, but *for now* we may assume that axions are stabilized non-perturbatively)
  - typical in phenomenological constructions (no UV-completion)
- multi-exponential potential $V = \sum_{i=1}^{m} \Lambda_i \, \mathrm{e}^{-\kappa_d \gamma_{ia} \phi^a}$
  - e.g. string compactifications:
    non-trivial curvature, fluxes, localized sources and generic Casimir-energy terms
    for general arguments, see e.g. Ooguri, Palti, Shiu, Vafa [hep-th/1810.05506]
    Hebecker, Wrase [hep-th/1810.08182]
  - typical in phenomenological constructions (no UV-completion)



$$V = \sum_{i=1}^{3} \Lambda_i \, \mathrm{e}^{-\kappa_d \gamma_{ia} \phi^a}$$

$$\phi^a = \phi^1, \phi^2$$

$$(\gamma_i)_a = \gamma_{ia} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \\ \gamma_{31} & \gamma_{32} \end{pmatrix}$$

in this talk, we will see:

1. a *universal bound on late-time cosmic acceleration for flat field spaces*, without any need to find an explicit solution (like a scaling solution)

2. a class of theories with a *universal cosmological attractor* solution, i.e. the scaling solution, with the possibility to compute explicitly any quantity of interest

3. bonus: a *universal bound on cosmic contraction*

4. *universal bounds on late-time cosmic acceleration for certain curved field spaces*, without any need to find an explicit solution

in all cases, we get model-independent results

- independently of initial conditions
- with no approximations in the field equations

1. Bounds on cosmic acceleration

Standard field equations

field-content: canonical scalars ϕ^a, potential $V = \sum_{i=1}^{m} \Lambda_i \, \mathrm{e}^{-\kappa_d \gamma_{ia} \phi^a}$

- cosmological equations:

$$\ddot{\phi}^a + (d-1) H \dot{\phi}^a + \frac{\partial V}{\partial \phi_a} = 0 \qquad \text{(scalar field eqs.)}$$

$$H^2 = \frac{2\kappa_d^2}{(d-1)(d-2)} \left[\frac{1}{2} \, \dot{\phi}_a \dot{\phi}^a + V \right] \qquad \text{(Friedmann eq. 1)}$$

- also imply:

$$\dot{H} = -\frac{\kappa_d^2}{d-2} \, \dot{\phi}_a \dot{\phi}^a \qquad \text{(Friedmann eq. 2)}$$

notes:

- remember that $\epsilon = -\dot{H}/H^2$: more kinetic energy means less cosmic acceleration
- finding general time-dependent solutions is extremely hard, but we can look for universal properties without restricting to explicit solutions

Copeland, Liddle, Wands [gr-qc/9711068]

new variables:

$$x^a = \frac{\kappa_d}{\sqrt{d-1}\sqrt{d-2}} \frac{\dot{\phi}^a}{H}$$
$$y_i = \frac{\kappa_d \sqrt{2}}{\sqrt{d-1}\sqrt{d-2}} \frac{1}{H} \sqrt{\Lambda_i \mathrm{e}^{-\kappa_d \gamma_{ia} \phi^a}}$$

with:

$$f = (d-1)H$$
$$c_{ia} = \frac{1}{2} \frac{\sqrt{d-2}}{\sqrt{d-1}} \gamma_{ia}$$

- cosmological equations (autonomous system: $\underline{\dot{\alpha}} = \underline{\phi}(\underline{\alpha})$, $\underline{\alpha} = \underline{\alpha}(t)$):

$$\dot{x}_a = \left[-x_a (y)^2 + \sum_{i=1}^{m} c_{ia} (y^i)^2 \right] f \qquad \text{(scalar field eqs.)}$$
$$\dot{f} = -(x)^2 f^2 \qquad \text{(Friedmann eq. 2)}$$

with:

$$(x)^2 + (y)^2 = 1 \qquad \text{(Friedmann eq. 1)}$$

- if $(y^i)^2 > 0$, we can write the differential inequality

$$\dot{x}_a = \left[-x_a (y)^2 + \sum_{i=1}^{m} c_{ia} (y^i)^2\right] f \geq \left[-x_a (y)^2 + \overbrace{c_a}^{\min_i c_{ia}} (y)^2\right] f$$

i.e.

$$\dot{x}_a \geq (-x_a + c_a)(y)^2 f$$

- given $\varphi(t) = \int_{t_0}^{t} \mathrm{d}s \, [y(s)]^2 f(s)$, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\mathrm{e}^{\varphi(t)} \, x_a(t)\right] = \mathrm{e}^{\varphi(t)} \left[x_a(t) \, [y(t)]^2 f(t) + \dot{x}_a(t)\right] \geq c_a \, \mathrm{e}^{\varphi(t)} \, [y(t)]^2 f(t)$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\mathrm{e}^{\varphi(t)} \, x_a(t)\right] \geq c_a \, \frac{\mathrm{d}}{\mathrm{d}t} \, \mathrm{e}^{\varphi(t)}$$

which integrates to

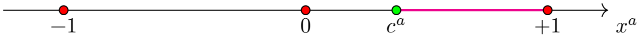
$$\mathrm{e}^{\varphi(t)} \, x_a(t) - x_a(t_0) \geq c_a \left[\mathrm{e}^{\varphi(t)} - 1\right]$$

- if $\lim_{t \to \infty} \varphi(t) = \infty$, we find

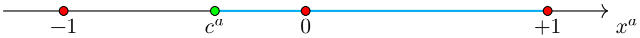
$$\liminf_{t \to \infty} x_a(t) \geq c_a$$

Late-time behaviours: an idea, pt. 2

- as $\epsilon = (d-1)(x)^2$, we want to see how far from $\epsilon = 0$ we are at late times:
  - $c^a \geq 0$: useful bound, as $x^a \geq c^a$ implies $(x^a)^2 \geq (c^a)^2$



  - $c^a < 0$: useless bound, as $x^a \geq c^a$ still only implies $(x^a)^2 \geq 0$



  so, including only the terms $c^a_\infty = c^a > 0$, at late times we have $\epsilon \geq (d-1)(c_\infty)^2$

- if $\lim_{t\to\infty} \varphi(t) < \infty$, we can show that

$$\lim_{t\to\infty} [x(t)]^2 = 1$$

  which means that at late times we have $\epsilon = d - 1$

[all mathematical proofs in the papers]

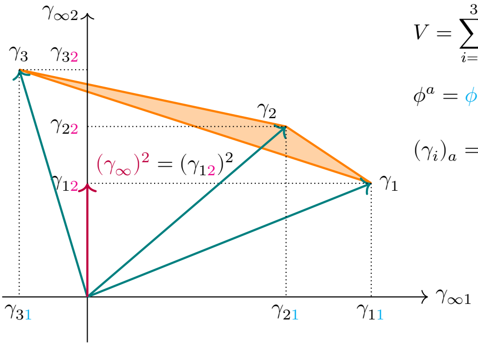
if all $\Lambda_i > 0$, let $\gamma_\infty^a = \begin{cases} \gamma^a, & \gamma^a = \min_i {\gamma_i}^a > 0 \\ 0, & \gamma^a \le 0 \end{cases}$

then, we have the analytic late-time bounds

$$d - 1 \ge \epsilon \ge \frac{d-2}{4} \left(\gamma_\infty\right)^2$$

note: $\epsilon = d - 1$ if $(\gamma_\infty)^2 \ge 4\, \dfrac{d-1}{d-2}$

example:



$$V = \sum_{i=1}^{3} \Lambda_i \, \mathrm{e}^{-\kappa_d \gamma_{ia} \phi^a}$$

$$\phi^a = \phi^1, \phi^2$$

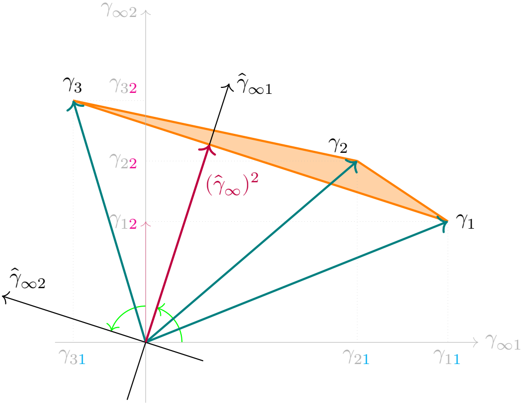
$$(\gamma_i)_a = \gamma_{ia} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \\ \gamma_{31} & \gamma_{32} \end{pmatrix}$$

- the bound can be optimized by a field-space basis $\mathrm{O}(n)$-transformation:

$$d-1 \geq \epsilon \geq \frac{d-2}{4}\,(\hat{\gamma}_\infty)^2$$

- $(\hat{\gamma}_\infty)^2$ corresponds to the squared distance of the origin from the coupling convex hull

example:



$$V = \sum_{i=1}^{3} \Lambda_i \, \mathrm{e}^{-\kappa_d \gamma_{ia} \phi^a}$$

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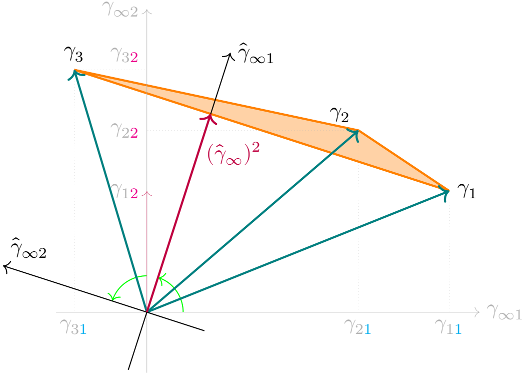
Cosmic-acceleration bound: physical interpretation



$$V = \sum_{i=1}^{3} \Lambda_i \, \mathrm{e}^{-\kappa_d \gamma_{ia} \phi^a}$$

$$\phi^a = \phi^1, \phi^2$$

$$(\gamma_i)_a = \gamma_{ia} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \\ \gamma_{31} & \gamma_{32} \end{pmatrix}$$

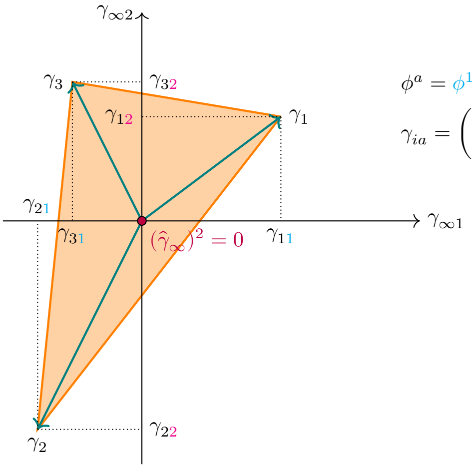
- in the optimal basis:

$$V = \left[\sum_{\sigma=1}^{\hat{m}} \Lambda_\sigma \, \mathrm{e}^{-\kappa_d \hat{\gamma}_{\sigma \check{a}} \hat{\phi}^{\check{a}}} \right] \mathrm{e}^{-\kappa_d \hat{\gamma}_\infty \hat{\phi}^1} + \sum_{\iota=\hat{m}+1}^{m} \Lambda_\iota \, \mathrm{e}^{-\kappa_d \hat{\gamma}_{\iota \hat{1}} \hat{\phi}^1 - \kappa_d \hat{\gamma}_{\iota \check{a}} \hat{\phi}^{\check{a}}}$$

  - the 1-field 1-term potential $\hat{V}_\infty = \hat{\Lambda}_\infty \, \mathrm{e}^{-\kappa_d \hat{\gamma}_\infty \hat{\phi}^1}$ would give $\epsilon = \dfrac{d-2}{4} \, \hat{\gamma}_\infty^2$
  - the presence of other fields creates further steepness

observation 1:

on paper, there are plenty of possibilities for late-time acceleration!



$$\phi^a = \phi^1, \phi^2$$

$$\gamma_{ia} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$$

observation 2:

however, it seems hard to find coupling like these, in string theory

Dilaton obstruction to cosmic acceleration, pt. 1

simple scaling argument:

- d-dim. potentials come from dim. reduction of terms like $(K_{10,r}(\sigma) = \Lambda_{10,r}\,\mathrm{e}^{-k\sigma})$

$$S = -\frac{1}{2\kappa_{10}^2}\int_{\mathrm{X}_{1,9}} [A_r \wedge \star_{1,9} A_r]\, K_{10,r}(\sigma)\, \mathrm{e}^{-\chi_{\mathrm{E}}\Phi}$$

  where $g_s(\Phi) = \mathrm{e}^{\Phi}$ and $\mathrm{vol}_s\, \mathrm{K}_{10-d} = \mathrm{e}^{(10-d)\sigma}\, l_s^{10-d}$

- reduction to d-dimensional Einstein frame:
  - d-dim. dilaton δ, string-frame radion σ, with $\mathrm{e}^{\delta} = \mathrm{e}^{\Phi - \frac{10-d}{2}\sigma}$
  - the canonically-normalized fields $\tilde{\delta}$ and $\tilde{\sigma}$ see the potential

$$V(\tilde{\delta},\tilde{\phi}) = \Lambda\, \mathrm{e}^{\left[\frac{d}{\sqrt{d-2}} - \frac{1}{2}\chi_{\mathrm{E}}\sqrt{d-2}\right]\kappa_d\tilde{\delta} - \left[\left(1-\frac{1}{2}\chi_{\mathrm{E}}\right)\sqrt{10-d} - \frac{2r+k}{\sqrt{10-d}}\right]\kappa_d\tilde{\sigma}}$$

we observe:

- model-dependent $\tilde{\sigma}$-coupling
- for worldsheet Euler character χ_{E}, universal $\gamma_{\tilde{\delta}} = \dfrac{d}{\sqrt{d-2}} - \dfrac{\chi_{\mathrm{E}}}{2}\sqrt{d-2}$

Dilaton obstruction to cosmic acceleration, pt. 2

remember: for worldsheet Euler character χ_{E}, universal $\gamma_{\tilde{\delta}} = \frac{d}{\sqrt{d-2}} - \frac{\chi_{\mathrm{E}}}{2}\sqrt{d-2}$

general string-theoretic considerations:

- upper bound on $\gamma_{\tilde{\delta}}$: $\chi_{\mathrm{E}} \leq \chi_{\mathrm{E}}(\mathrm{S}^2) = 2$, so $\gamma_{\tilde{\delta}} \geq \frac{2}{\sqrt{d-2}}$
- lower bound on ϵ: $\epsilon \geq \frac{d-2}{4}(\gamma_\infty)^2 \geq \frac{d-2}{4}\gamma_{\tilde{\delta}}{}^2 \geq 1$

different argument, but related conclusion, in Rudelius [hep-th/2101.11617]

possible ways out (besides field-space curvature):

- theory not at weak string coupling
- stabilized dilaton
- presence of negative-definite potential terms:
  bound takes a different form, less obvious but still restrictive!

[preliminary discussion of potentials with terms of both signs in the papers]

[more in upcoming work]

2. Scaling cosmologies

Scaling cosmologies: general characterization

- scaling cosmologies are solutions with constant positive ϵ
  - power-law scale factor: $a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{1}{\epsilon}}$

- complete analytic characterization, if $\operatorname{rank} \gamma_{ia} = m$ (where $M_{ij} = \gamma_{ia}\gamma_j{}^a$):
  - field-space trajectory $\phi_*^a(t) = \phi_0^a + \frac{2}{\kappa_d} \left[\sum_{i=1}^m \sum_{j=1}^m \gamma_i{}^a (M^{-1})^{ij}\right] \ln \frac{t}{t_0}$
  - ϵ-parameter $\epsilon = \frac{d-2}{4} \left[\sum_{i=1}^m \sum_{j=1}^m (M^{-1})^{ij}\right]^{-1}$

Copeland, Liddle, Wands [gr-qc/9711068]
Collinucci, Nielsen, Van Riet [hep-th/0407047]

- notes:
  - no slow roll: $w = \frac{T_* - V_*}{T_* + V_*} = -1 + \frac{2\epsilon}{d-1}$, $\ddot{\phi}_*^a \propto H\dot{\phi}_*^a \propto \frac{\partial V}{\partial \phi_{*a}} \propto \frac{1}{t^2}$
  - all scalar-potential terms decay identically: $V_i[\phi_*^a(t)] = V_i(t_0) \left(\frac{t_0}{t}\right)^2$

- also: scaling solutions correspond to critical points of the autonomous system relevance:

- late-time scale factor is bounded by power-law behaviors, remember: $d-1 \geq \epsilon \geq [(d-2)/4]\,(\hat{\gamma}_\infty)^2$
- if $\Lambda_i > 0$, one of the scaling solutions is the unique perturbative late-time attractor

see e.g. Hartong, Ploegh, Van Riet, Westra [gr-qc/0602077]

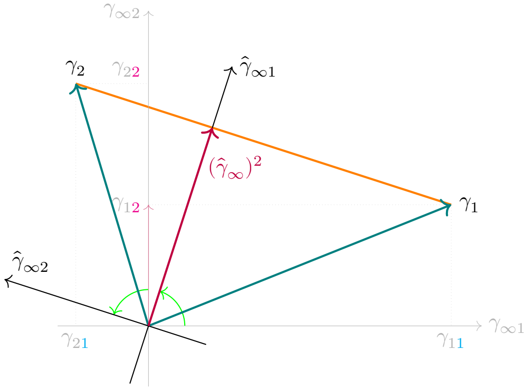
▶ new result:

if all terms in the potential are positive-definite, i.e. if $\Lambda_i > 0$, then we can analyically prove that one of the scaling solutions is the unique universal late-time attractor, irrespectively of the initial conditions, and that such a solution saturates the universal bound on cosmic acceleration, i.e.

$$\epsilon = \frac{d-2}{4}\,(\hat{\gamma}_\infty)^2$$

[all mathematical proof in the papers]

example:



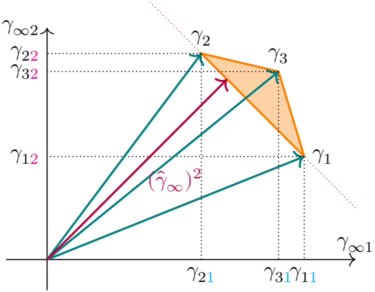
$$V = \sum_{i=1}^{2} \Lambda_i \, \mathrm{e}^{-\kappa_d \gamma_{ia} \phi^a}$$

$$\phi^a = \phi^1, \phi^2$$

$$\gamma_{ia} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$$

- in the optimal basis: $V = \left[\sum_{\sigma=1}^{m} \Lambda_\sigma \, \mathrm{e}^{-\kappa_d \hat{\gamma}_{\sigma \breve{a}} \hat{\phi}^{\breve{a}}} \right] \mathrm{e}^{-\kappa_d \hat{\gamma}_\infty \hat{\phi}^1}$
  - all fields but $\hat{\phi}^1$ have positive exponential potentials with couplings of both signs, and therefore get asymptotically stabilized
  - in the field-space asymptotics, effectively we only have the 1-field 1-term potential $\hat{V}_\infty = \hat{\Lambda}_\infty \, \mathrm{e}^{-\kappa_d \hat{\gamma}_\infty \hat{\phi}^1}$, which gives $\epsilon = \dfrac{d-2}{4} \, \hat{\gamma}_\infty^2$

more examples:



- if the distance vector from the origin to the convex-hull coupling hyperplane intersects the convex hull itself too, we analytically find the late-time ϵ-parameter

$$\epsilon = \frac{d-2}{4}\,(\hat{\gamma}_\infty)^2 = \frac{d-2}{4}\left[\sum_{i=1}^{m}\sum_{j=1}^{m}(M^{-1})^{ij}\right]^{-1}$$

  where some of the potential terms might be truncated

- else, the potential is also formally truncated, still leaving $\epsilon = \dfrac{d-2}{4}\,(\hat{\gamma}_\infty)^2$

in all cases, the universal bound is saturated field-space trajectory is a straight line

original field basis:

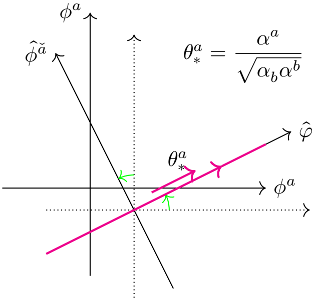
$$\phi^a_*(t) = \phi^a_\infty + \frac{\alpha^a}{\kappa_d} \ln \frac{t}{t_\infty}$$

$\mathrm{O}(n)$-transformed field basis:

$$\hat{\phi}^{\check{a}}_*(t) = \hat{\phi}^{\check{a}}_\infty$$

$$\hat{\varphi}_*(t) = \hat{\varphi}_\infty + \frac{1}{\kappa_d} \frac{2}{\hat{\gamma}_\infty} \ln \frac{t}{t_\infty}$$

note: on-shell potential $\hat{V}_* = \hat{\Lambda}\, \mathrm{e}^{-\kappa_d \hat{\gamma}_\infty \hat{\varphi}_*}$



- exact relationship between scalar-potential slope and ϵ-parameter:

$$-\frac{1}{V}\, \theta^a_* \frac{\partial V}{\kappa_d\, \partial \phi^a}(\phi_*) = \frac{1}{\kappa_d V} \sqrt{\frac{\partial V}{\partial \phi^a} \frac{\partial V}{\partial \phi_a}}(\phi_*) = \frac{2\sqrt{\epsilon}}{\sqrt{d-2}} = \hat{\gamma}_\infty$$

to compare with Obied, Ooguri, Spodyneiko, Vafa [hep-th/1806.08362]
Ooguri, Palti, Shiu, Vafa [hep-th/1810.05506]

false for generic solution ϕ^a, unless $\eta, \Omega = 0$ ($\eta = -\dot{\epsilon}/(\epsilon H)$, Ω: non-geodesity):

$$\gamma_\star(\epsilon, \eta)[\phi] = -\frac{1}{V} \frac{\dot{\phi}^a}{\sqrt{\dot{\phi}_b \dot{\phi}^b}} \frac{\partial V}{\kappa_d\, \partial \phi^a}(\phi) \leq \frac{1}{\kappa_d V} \sqrt{\frac{\partial V}{\partial \phi_a} \frac{\partial V}{\partial \phi^a}}(\phi) = \gamma(\epsilon, \eta, \Omega)[\phi]$$

Achúcarro, Palma [hep-th/1807.04390]
see also Andriot, Horer, Tringas [hep-th/2212.04517]

Scaling cosmologies: non-slow-roll gradient flow

- in a scaling cosmology (no slow roll), we have $\phi_*^a(t) = \phi_\infty^a + \frac{\alpha^a}{\kappa_d} \ln \frac{t}{t_\infty}$, so
  - $\ddot{\phi}_*^a(t) = -\frac{1}{\kappa_d t^2}\, \alpha^a$
  - $(d-1)H\dot{\phi}_*^a(t) = \frac{1}{\kappa_d t^2}\, \frac{d-1}{\epsilon}\, \alpha^a$

  we can plug these into the scalar-field eq. to find the trajectory, i.e.

$$\ddot{\phi}_*^a + (d-1)H\dot{\phi}_*^a = \left[1 - \frac{\epsilon}{d-1}\right](d-1)H\dot{\phi}_*^a = -\frac{\partial V}{\partial \phi_a}(\phi_*)$$

  as the proportionality factor between $\dot{\phi}_*^a$ and $\partial V/\partial \phi_{*a}$ is universal, this happens to give a non-slow-roll gradient-flow trajectory (note: irrespectively of ϵ)

- in the slow-roll regime, one approximates

$$\ddot{\phi}_{\rm sr}^a + (d-1)H\dot{\phi}_{\rm sr}^a \overset{\frac{|\ddot{\phi}_{\rm sr}^a|}{H|\dot{\phi}_{\rm sr}^a|} \ll 1}{\simeq} (d-1)H\dot{\phi}_{\rm sr}^a = -\frac{\partial V}{\partial \phi_a}(\phi_{\rm sr})$$

  conclusion:

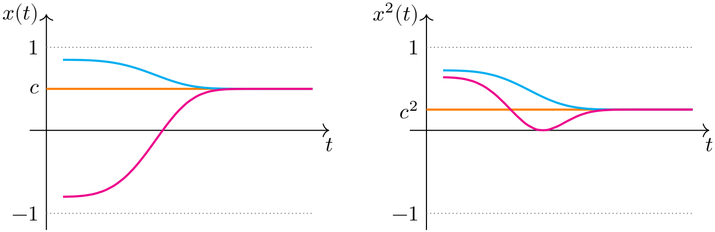
  although the field-space trajectory is accidentally the same,
  the physics (i.e. the time dependence of fields) is completely different

- if the late-time behavior has $\epsilon = 1$, it may still take an infinite time to reach it
- if the solutions approach the asymptotic value $\epsilon = 1$ (not accelerated!) from below, there is an infinite time where the solutions are at $\epsilon < 1$

  mechanism recently exploited, if $k = -1$, by Andriot, Tsimpis, Wrase [hep-th/2309.03938]

- e.g. single-field single-potential theory

  analytic solution: $x(t) = c + \mathrm{e}^{-\varphi(t)}\left[x(t_0) - c\right]$



- however, we have not found a single string-theoretic potential such that $(\hat{\gamma}_\infty)^2 = 4/(d-2)$, which would give $\epsilon = 1$!

3. Bonus: bounds on cosmic contraction

- relevance of contracting universes:
  - with FLRW-space curvature k, a set of perfect fluids with equation-of-state parameters w_α gives the Friedmann equation

$$H^2 = \frac{2\kappa_d^2}{(d-1)(d-2)} \sum_\alpha \rho_{\alpha,0} \left(\frac{a_0}{a}\right)^{(d-1)(1+w_\alpha)} - \frac{k}{a^2},$$

  - if the universe is contracting, all energy-density contributions blow up, with the fluid with the maximal w of all dominating

this the conceptual basis of ekpyrotic scenarios, i.e. proposed solutions to the flatness and horizon problems, alternative to cosmic inflation: a fundamental element is a cosmological fluid with a large w-parameter $w \gg 1$

Khoury, Ovrut, Steinhardt, Turok [hep-th/0103239]
Khoury, Ovrut, Seiberg, Steinhardt, Turok [hep-th/0108187]

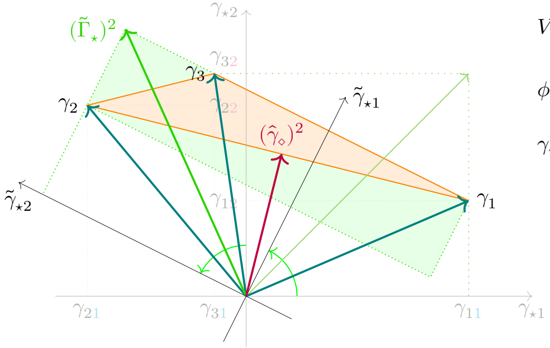
- scalar fields with a negative potential realize a contracting universe: we are agnostic about the viability of ekpyrosis; yet, our methods allow us to find analytic bounds on w for a universe composed of canonical scalars ϕ^a under the potential $V = -\sum_{i=1}^m K_i \, \mathrm{e}^{-\kappa_d \gamma_{ia} \phi^a}$, with $K_i > 0$

[all mathematical proofs in the papers]

- let $(\tilde{\Gamma}_\infty)^2$ be the largest squared distance of the origin from the rectangular box with the smallest area that contains all the coupling vectors and let $(\hat{\gamma}_\diamond)^2$ be the squared distance of the origin from the coupling convex hull
- after a finite time, the w-parameter is universally bounded as

$$-1 + \frac{1}{2}\,\frac{d-2}{d-1}\,(\hat{\gamma}_\diamond)^2 \leq w \leq -1 + \frac{1}{2}\,\frac{d-2}{d-1}\,(\tilde{\Gamma}_\star)^2$$

example:



$$V = -\sum_{i=1}^{3} K_i\, e^{-\kappa_d \gamma_{ia} \phi^a}$$

$$\phi^a = \phi^1, \phi^2$$

$$\gamma_{ia} = (\gamma_i)_a = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \\ \gamma_{31} & \gamma_{32} \end{pmatrix}$$

4. Adding field-space curvature

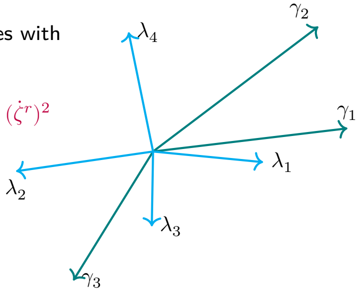
from now on, we consider (pseudo)scalar theories with

- kinetic energy with multi-exponential couplings:

$$T[\phi,\zeta] = \frac{1}{2}\sum_{a=1}^{n}(\dot{\phi}^a)^2 + \frac{1}{2}\sum_{r=1}^{p} \mathrm{e}^{-\kappa_d \sum_a \lambda_{ra}\phi^a}\,(\dot{\zeta}^r)^2$$

- multi-exponential scalar potentials:

$$V[\phi] = \sum_{i=1}^{m} \Lambda_i\, \mathrm{e}^{-\kappa_d \sum_a \gamma_{ia}\phi^a}$$



some perspectives:

- not yet the most general action arising in string compactifications, but it now has a field space with negative curvature, which may be expected in the asymptotics

  see e.g. Ooguri, Vafa [hep-th/0605264]

  for instance, any 4-dim. theory with $N_4 = 1$ supersymmetry with Kähler potential $\kappa_4^2 K = -n\ln[-\mathrm{i}(\xi-\bar{\xi})]$ for a chiral multiplet $\xi = \theta + \mathrm{i}\,\mathrm{e}^{l\varphi}$ gives the kinetic action

$$T = \frac{n}{4\kappa_4^2}\left[l^2\dot{\varphi}^2 + \mathrm{e}^{-2l\varphi}\dot{\theta}^2\right] = \frac{1}{2}\,\dot{\phi}^2 + \frac{1}{2}\mathrm{e}^{-\kappa_d \frac{2\sqrt{2}}{\sqrt{n}}\phi}\,\dot{\zeta}^2$$

  e.g. STU-models ($n = p$, diagonal λ_{ra}-matrix)

  e.g. complex-structure moduli asymptotics
  in type-II compactifications on Calabi-Yau orientifolds

  e.g. Grimm, Li, Valenzuela [hep-th/1910.09549]

- this action can be used for richer model-building, independently of UV-completions

State of the art and our results

- existing phase-space analyses for 1 or 2 scalars and 1 axion

Sonner, Townsend [hep-th/0608068]
Russo, Townsend [hep-th/2203.09398]

- existing studies of the critical points for
  diagonal kinetic couplings and single-term potentials

Cicoli, Dibitetto, Pedro [hep-th/2002.02695]
Cicoli, Dibitetto, Pedro [hep-th/2007.11011]
Brinkmann, Cicoli, Dibitetto, Pedro [hep-th/2206.10649]
Revello [hep-th/2311.12429]

extremely concise summary ($d = 4$): the perturbatively-stable attractor has

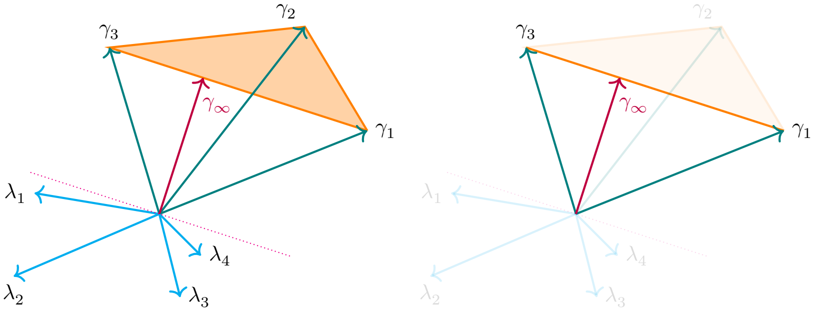
- $\epsilon = \dfrac{3\gamma}{\gamma + \lambda}$, if $\sqrt{\dfrac{\lambda^2}{4} + 6} - \dfrac{\lambda}{2} < \gamma < \lambda$ and $\lambda > 0$
- $\epsilon = \dfrac{\gamma^2}{2}$, else

▶ we can prove bounds on the late-time ϵ-parameter,
beyond the analysis of linear stability,
for non-diagonal kinetic couplings and multi-term potentials

▶ for certain arrangements of the coupling vectors, we can prove that
the same kinds of bounds as for flat field spaces are still in place

[all mathematical proofs in the papers]

if $\gamma_\infty \cdot \lambda_r < 0$, then at late-enough times we have $\epsilon = \frac{d-2}{4}(\gamma_\infty)^2$

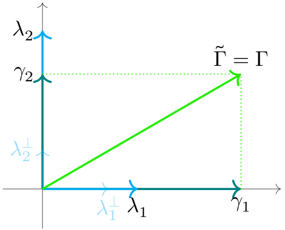


in particular:

late-time attractor is the scaling solution associated to the truncated potential fixing γ_∞, with zero axion kinetic energy

note: if $(\gamma_\infty)^2 \geq 4\frac{d-1}{d-2}$, then at late times we have $\epsilon = d-1$

- pedagogical case: diagonal coupling matrices in one (hyper)quadrant



for $\Gamma_a = \max_i \gamma_{ia}$, at late-enough times:

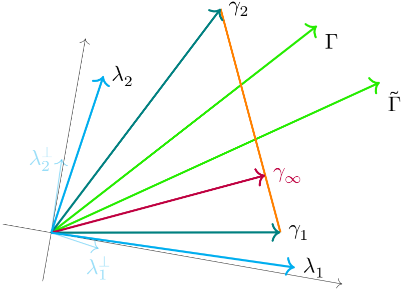
$$\epsilon \leq \frac{d-2}{4}\,(\Gamma)^2$$

- general statement:
  - $\lambda_{1,2}$ in same quadrant
  - $\gamma_{1,2}$ in the convex cone $\mathrm{C}(\lambda_1, \lambda_2)$

  - define $\lambda_{1,2}^\perp$ such that:
    - $\lambda_{1,2} \cdot \lambda_{1,2}^\perp = 1$
    - $\lambda_{1,2} \cdot \lambda_{2,1}^\perp = 0$

  - define $\tilde{\Gamma}_a = \sum_{r=1}^{2} (\max_i (\gamma_i \cdot \lambda_r^\perp)) \lambda_{ra}$

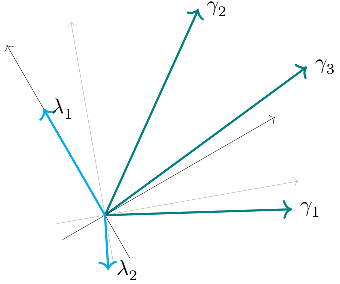
    (note: $\tilde{\Gamma}_a = \Gamma_a$ for orthogonal couplings, as above)



at late-enough times:

$$\epsilon \leq \frac{d-2}{4}\,(\Gamma \cdot \tilde{\Gamma})$$

- let $\gamma_a = \min_i \gamma_{ia}$ and $\Lambda_a = \max_r \lambda_{ra}$
- if $\Lambda_a \leq 0$ and $\gamma_a + \Lambda_a \geq 2\sqrt{\dfrac{d-1}{d-2}}$ for at least one field ϕ^a, then at late-enough times we have $\epsilon = d - 1$



note:
to sum up, we singled out several geometric configurations for the potential and kinetic couplings in which the universal bounds for flat field spaces are still in place

5. Conclusions

Recap

- analytic model-independent results, independently of the initial conditions and with no approximations in the field equations:
  - universal bound for cosmic acceleration and convex-hull criterion
    1-field 1-potential case: see also Rudelius [hep-th/2208.08989]
  - string-theoretic dilaton obstruction to cosmic acceleration and ways out
    compatible with strong de Sitter conjecture in Rudelius [hep-th/2101.11617]
    related conclusions in Hertzberg, Kachru, Taylor, Tegmark [hep-th/0711.2512]
    Hebecker, Skrzypek, Wittner [hep-th/1909.08625]
    Cicoli, Cunillera, Padilla, Pedro [hep-th/2112.10779]
    Andriot, Horer [hep-th/2208.14462]
  - side result: universal bound on cosmic contraction
    see also Bernardo, Brandenberger [hep-th/2104.00630]
    Andriot, Horer, Tringas [hep-th/2212.04517]
  - scaling cosmologies: universal proof of convergence
  - bounds on cosmic acceleration with negatively-curved field spaces, for specific arrangements of the coupling vectors

- analytic handle on swampland conjectures:
  - relationship between cosmic acceleration and scalar-potential derivatives
    to compare with Obied, Ooguri, Spodyneiko, Vafa [hep-th/1806.08362]
    Ooguri, Palti, Shiu, Vafa [hep-th/1810.05506]
  - precise handle on attempts for string-theoretic accelerated expansion
    Calderón-Infante, Ruiz, Valenzuela [hep-th/2209.11821]
    Cremonini, Gonzalo, Rajaguru, Tang, Wrase [hep-th/2306.15714]
  - asymptotic acceleration implies higher-dimensional de Sitter spacetime
    Hebecker, Schreyer, Venken [hep-th/2306.17213]

we saw that, sometimes, a study of geometric relations of the coupling space is enough to assess whether late-time acceleration is possible in certain corners of string compactifications: this is easier than a study of the solutions to the field equations, and it can be done in a fully analytic way

further steps include:

- incorporating into our analytic description:
  - more general kinetic couplings
  - barotropic fluids with constant equation of state

- improving our analytic description in the simultaneous presence of both positive and negative potential terms:

  we aim to look for cosmic acceleration with the help of negative potentials, or to prove more general no-go conclusions

  related: Van Riet [hep-th/2308.15035]

- discussing with our language and looking for new accelerated solutions in FLRW-backgrounds with negative spatial curvature

  Marconnet, Tsimpis [hep-th/2210.10813]
  Andriot, Tsimpis, Wrase [hep-th/2309.03938]
  Andriot, Parameswaran, Tsimpis, Wrase, Zavala [hep-th/2405.09323]
  Alestas, Delgado, Ruiz, Akrami, Montero, Nesseris [hep-th/2406.09212]

Thank you!

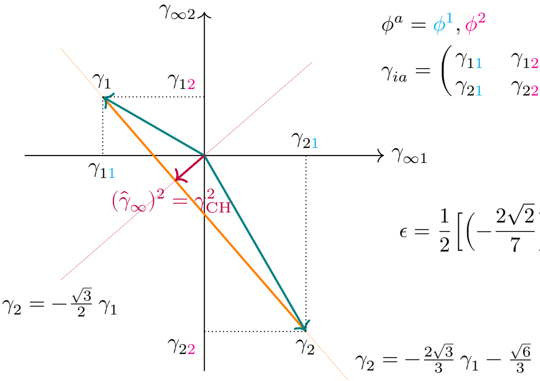
6. Backup material

Example

4-dimensional potential: $V = \Lambda_1\, \mathrm{e}^{\kappa_4 \sqrt{2}\, \tilde{\phi}^1 - \kappa_4 \sqrt{\frac{2}{3}}\, \tilde{\phi}^2} + \Lambda_2\, \mathrm{e}^{-\kappa_4 \sqrt{2}\, \tilde{\phi}^1 + \kappa_4 \sqrt{6}\, \tilde{\phi}^2}$

Calderón-Infante, Ruiz, Valenzuela [hep-th/2209.11821]

- convex-hull hyperplane: $\gamma_2 = -\frac{2\sqrt{3}}{3}\,\gamma_1 - \frac{\sqrt{6}}{3}$
- orthogonal line: $\gamma_2 = \frac{\sqrt{3}}{2}\,\gamma_1$, intersection at $(\gamma_1, \gamma_2) = \left(-\frac{2\sqrt{2}}{7}, -\frac{\sqrt{6}}{7}\right)$



$$\phi^a = \phi^1, \phi^2$$

$$\gamma_{ia} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} = \begin{pmatrix} -\sqrt{2} & \sqrt{\frac{2}{3}} \\ \sqrt{2} & -\sqrt{6} \end{pmatrix}$$

$$\epsilon = \frac{1}{2}\left[\left(-\frac{2\sqrt{2}}{7}\right)^2 + \left(-\frac{\sqrt{6}}{7}\right)^2\right] = \frac{1}{7}$$

- exact relationship between scalar-potential slope and ϵ-parameter:

$$-\frac{1}{V}\,\theta_*^a\,\frac{\partial V}{\kappa_d\,\partial\phi^a}(\phi_*) = \frac{1}{\kappa_d V}\sqrt{\frac{\partial V}{\partial\phi^a}\frac{\partial V}{\partial\phi_a}}(\phi_*) = \frac{2\sqrt{\epsilon}}{\sqrt{d-2}} = \hat{\gamma}_\infty$$

to compare with Obied, Ooguri, Spodyneiko, Vafa [hep-th/1806.08362]
Ooguri, Palti, Shiu, Vafa [hep-th/1810.05506]

- the potential directional derivative and gradient norm are not necessarily related to ϵ for non-scaling solutions

  a generic solution $\phi^a(t)$, gives ($\eta = -\dot{\epsilon}/(\epsilon H)$, Ω: non-geodesity factor)

$$\gamma_\star(\phi) = -\frac{1}{V}\,\frac{\dot{\phi}^a}{\sqrt{\dot{\phi}_b\dot{\phi}^b}}\,\frac{\partial V}{\kappa_d\,\partial\phi^a}(\phi) = \frac{2\sqrt{\epsilon}}{\sqrt{d-2}}\left[1 - \frac{\eta}{(d-1)-\epsilon}\right]$$

$$\gamma(\phi) = \frac{1}{\kappa_d V}\sqrt{\frac{\partial V}{\partial\phi_a}\frac{\partial V}{\partial\phi^a}}(\phi) = \sqrt{\gamma_\star^2 + \frac{4\epsilon}{d-2}\,\frac{1}{[(d-1)-\epsilon]^2}\,\frac{\Omega^2}{H^2}}$$

Achúcarro, Palma [hep-th/1807.04390]