Mock modularity and refinement: from BPS black holes to Vafa-Witten theory

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S.A., B.Pioline arXiv:1808.08479
S.A., J.Manschot, B.Pioline arXiv:1910.03098

S.A. arXiv:2005.03680
arXiv:2006.10074

S.A., S.Nampuri in progress

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Motivation

The main object of interest:

BPS indices $\Omega(\gamma)$ — (signed) number of BPS states in theories with extended SUSY

- degeneracies of BPS black holes
- spectrum of states in supersymmetric gauge theories
- weights of instanton corrections to the effective action
- Donaldson-Thomas invariants of Calabi-Yau manifolds
- Vafa-Witten invariants of complex surfaces – topologically twisted SYM

It is useful to study *generating functions*

$$h_{\dots}(\tau) = \sum_{q_0>0} \Omega(\gamma)\, e^{2\pi \mathrm{i} q_0 \tau}$$

Sometimes they possess non-trivial *modular* properties:

$$\tau \mapsto \frac{a\tau+b}{c\tau+d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

they can be *modular forms,*
mock modular forms,
higher depth mock modular forms…

$$h\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w h(\tau)$$

Motivation

The goal: understand modular properties of the generating functions of BPS indices $\Omega(\gamma)$

- Can be used to extract the asymptotic behavior of BPS indices to compare with the macroscopic entropy of BPS black holes

- Can be used to find them exactly!

- They encode a non-trivial geometry of the quantum corrected moduli spaces affected by BPS instantons

The plan of the talk

1. D4-D2-D0 black holes in Type IIA/CY and their BPS indices

2. Modularity of the generating functions of (*refined*) BPS indices and their modular completions

3. Applications:

   a) BPS dyons in N=4 string theory

   b) Vafa-Witten theory on projective surfaces

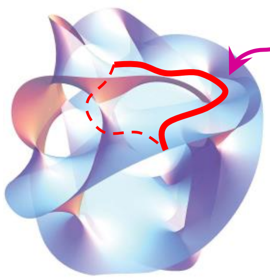
   c) Holomorphic anomaly for BPS partition function

   d) Quantization of the moduli space

4. Conclusions

D4-D2-D0 black holes in Type IIA/CY

These are ½ BPS black holes in 4d N=2 SUGRA
with electro-magnetic charge



$$\gamma = (0, p^a, q_a, q_0) \qquad\qquad a = 1, \ldots, b_2(CY)$$

label 4- and 2-dim cycles
wrapped by D4 and D2-branes

BPS index $\Omega(\gamma)$ — black hole degeneracy = generalized Donaldson-Thomas invariant of CY

Natural generating function $h^{\mathrm{DT}}_{p^a,q_a}(\tau) = \sum_{q_0>0} \Omega(\gamma)\, e^{2\pi \mathrm{i} q_0 \tau}$

but:
- the generating function depends on too many charges
- DT invariants depend on CY moduli: $\Omega(\gamma; z^a)$ — *wall-crossing*

(BPS *bound* states can form or decay)

no nice modular properties expected

MSW invariants

Solution: consider *MSW invariants*
count states in SCFT constructed
in Maldacena,Strominger,Witten '97

$$\boxed{\Omega_\gamma^{\rm MSW} = \Omega(\gamma, z^a_\infty(\gamma))}$$

large volume attractor point

$$z^a_\infty(\gamma) = \lim_{\lambda\to\infty}(-q^a + \mathrm{i}\lambda p^a)$$

Properties:

- independent of CY moduli
- invariant under *spectral flow symmetry*

$$\Omega_\gamma^{\rm MSW} = \Omega_p(\hat q_0)$$

$\hat q_0 \equiv q_0 - \frac{1}{2}\,\kappa^{ab} q_a q_b$ – invariant charge bounded from above

spectral flow

$$q_a \mapsto q_a - \kappa_{ab}\epsilon^b$$

$$q_0 \mapsto q_0 - \epsilon^a q_a + \tfrac{1}{2}\,\kappa_{ab}\epsilon^a\epsilon^b$$

$\kappa_{ab} = \kappa_{abc}p^c$ – quadratic form, given by intersection numbers of 4-cycles, of *indefinite* signature $(1, b_2-1)$

generating function of MSW invariants

$$h_p(\tau) = \sum_{\hat q_0 \le \hat q_0^{\rm max}} \bar\Omega_p(\hat q_0)\, e^{-2\pi \mathrm{i}\hat q_0 \tau}$$

where

$$\bar\Omega(\gamma) := \sum_{d|\gamma} \frac{1}{d^2}\,\Omega(\gamma/d)$$

rational invariants

The origin of modular symmetry

modular group

S-duality

M-theory / CY × T^2 ⟷ **Type IIA / CY × S^1** ⟷ **Type IIB / CY × S^1**

T-duality

M5-brane / divisor ⟷ **D4-D2-D0 bl.h.** ⟷ **D3-D1-D(-1) inst.**

affect the metric on the QK moduli space $\mathcal{M}$

$\mathcal{M}$ carries an isometric action of $SL(2,\mathbb{Z})$ preserved by non-pert. corrections

Twistor description of D-instantons
S.A.,Pioline,Saueressig,Vandoren '08
S.A. '09

A function $\mathcal{G}$ on $\mathcal{M}$ (called "*instanton generating potential*") constructed from DT-invariants is *modular* of weight $\left(-\frac{3}{2},\frac{1}{2}\right)$

Restriction on (the generating function of) BPS indices $\Omega(\gamma)$

The functions $h_p(\tau)$ have a modular anomaly, but one can construct an explicit expression for a non-holomorphic modular completion $\widehat{h}_p(\tau,\bar\tau)$

Refinement

There is a *refined* version of BPS indices $\Omega(\gamma,y) \sim \mathrm{Tr}_{\mathcal{H}_\gamma}(-y)^{2J_3}$

y – refinement parameter conjugate to the angular momentum

generating function of refined MSW invariants

$$h_p^{\mathrm{ref}}(\tau,y) = \sum_{\hat{q}_0 \le \hat{q}_0^{\max}} \frac{\bar{\Omega}_p(\hat{q}_0,y)}{y-y^{-1}}\, e^{-2\pi \mathrm{i} \hat{q}_0 \tau}$$

single order pole at $y \to 1$

The claim: the unrefined construction of the modular completion $\widehat{h}_p(\tau,\bar{\tau})$ has a natural generalization to the refined case provided the (log of the) refinement parameter transforms as Jacobi elliptic variable:

If $y = e^{2\pi \mathrm{i} w}$ then $w \mapsto \dfrac{w}{c\tau + d}$

or $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ for $w = \alpha - \tau\beta$

The unrefined construction can be obtained by taking the limit $y \to 1$

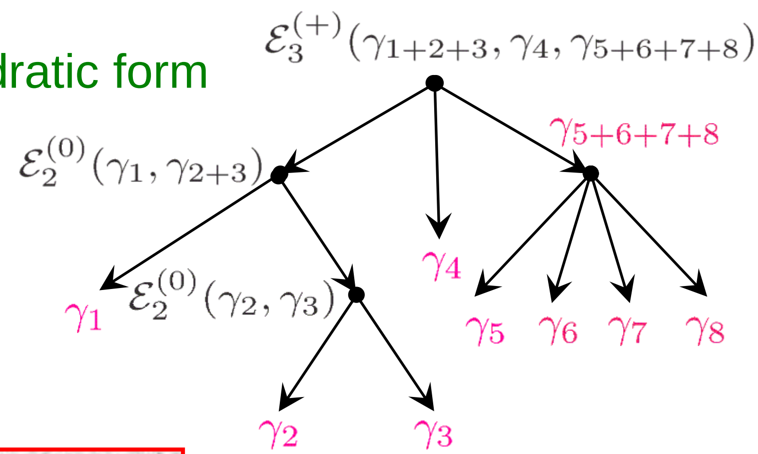
S.A.,Manschot,Pioline '19

Modular completion

$$\widehat{h}_p^{\rm ref} = h_p^{\rm ref} + \sum_{n=2}^{\infty} \sum_{\sum_{i=1}^n \gamma_i=\gamma} R_n^{\rm ref}(\{\gamma_i\};\tau_2)\,(-y)^{\sum_{i<j}\gamma_{ij}}\,e^{\pi \mathrm{i}\tau Q_n(\{\gamma_i\})} \prod_{i=1}^{n} h_{p_i}^{\rm ref}$$

Jacobi modular form of weight $-\frac{1}{2}\,b_2$

- $\gamma_{ij} = \langle \gamma_i, \gamma_j\rangle$ — Dirac skew-symmetric product
- $Q_n = \kappa^{ab} q_a q_b - \sum_{i=1}^{n} \kappa_i^{ab} q_{i,a} q_{i,b}$ — indefinite quadratic form
- $R_n^{\rm ref}$ — sum over (*Schröder)* trees of products of signs and generalized error functions of order *n*-1 assigned to vertices of the trees, with parameters defined by charges



Generalized error functions

$$\Phi_n^E(\{\boldsymbol{v}_i\},\boldsymbol{x}) = \int_{\mathrm{Span}\{\boldsymbol{v}_i\}} \mathrm{d}\boldsymbol{x}'\, e^{\pi(P(\boldsymbol{v})\cdot\boldsymbol{x}-\boldsymbol{x}')^2} \prod_{i=1}^{n} \mathrm{sgn}(\boldsymbol{v}_i\cdot\boldsymbol{x}')$$

$P(\boldsymbol{v})$ — projector on $\mathrm{Span}\{\boldsymbol{v}_i\}$

For *n*=1 reduces to the usual error function

$$\Phi_1^E(\boldsymbol{v},\boldsymbol{x}) = \mathrm{Erf}\left(\sqrt{\pi}\,\frac{\boldsymbol{x}\cdot\boldsymbol{v}}{|\boldsymbol{v}|}\right)$$

Modular completion – unrefined limit

The unrefined limit $y \to 1$

It is *well-defined* because the sum over charges produces a zero of order *n*-1 which cancels the poles of the refined generating functions

The unrefined construction differs only by the factors assigned to vertices of the trees which now involve sums of *derivatives* of the generalized error functions

Example: $n = 2$

$$R_2^{\rm ref} = -\frac{1}{2}\,\mathrm{sgn}(\gamma_{12})\,\mathrm{Erfc}\left(\sqrt{\frac{2\pi\tau_2}{(pp_1p_2)}}\,|\gamma_{12}|\right) \qquad R_2 = -\frac{|\gamma_{12}|}{8\pi}\,\beta_{\frac{3}{2}}\left(\frac{2\tau_2\gamma_{12}^2}{(pp_1p_2)}\right)$$

where $(pp_1p_2) = \kappa_{abd}p^a p_1^b p_2^c$

$$\beta_{\frac{3}{2}}(x^2) = \frac{2}{|x|}\,e^{-\pi x^2} - 2\pi\mathrm{Erfc}(\sqrt{\pi}|x|)$$

The origin of the completion

These results follow from

indefinite theta series of signature

$(nb_2-n+1,n-1)$

$$\mathcal{G}\sim\frac{1}{\sqrt{\tau_2}}\sum_{n=1}^{\infty}\left[\prod_{i=1}^{n}\sum_{p_i}h_{p_i}(\tau)\right]\theta_{\boldsymbol{p}}^{(n)}(\tau,t^a,b^a,\dots)$$

Modular properties of the theta series determine the properties of $h_p(\tau)$

for $n\geq 2$ there is a modular *anomaly*

rearrange the expansion

$$=\frac{1}{\sqrt{\tau_2}}\sum_{n=1}^{\infty}\left[\prod_{i=1}^{n}\sum_{p_i}\widehat{h}_{p_i}(\tau,\bar\tau)\right]\widehat{\theta}_{\boldsymbol{p}}^{(n)}(\tau,t^a,b^a,\dots)$$

modular completion

modular invariant

How to construct modular completions of indefinite theta series?

Indefinite theta series

$$\vartheta_{\boldsymbol{p}}(\tau,\boldsymbol{z}) = \sum_{\boldsymbol{q}\in\boldsymbol{\Lambda}+\frac{1}{2}\boldsymbol{p}} (-1)^{\boldsymbol{q}\cdot\boldsymbol{p}}\, e^{\pi\mathrm{i}\tau\boldsymbol{q}^2+2\pi\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{z}}$$

d-dim. lattice

If the quadratic form $\boldsymbol{q}^2$ is *positive definite*, the theta series is known to be a *Jacobi modular form*

$$\vartheta_{\boldsymbol{p}}\left(\frac{a\tau+b}{c\tau+d}, \frac{\boldsymbol{z}}{c\tau+d}\right) \sim \vartheta_{\boldsymbol{p}}(\tau,\boldsymbol{z})$$

What if the quadratic form is *indefinite*?

$\longrightarrow$ the theta series diverges!

But one can make it convergent by restricting to the wedge where $\boldsymbol{q}^2>0$

Example: Lorentzian signature $(d-1,1)$

$$\vartheta_{\boldsymbol{p}}(\tau,\boldsymbol{z}) = \sum_{\boldsymbol{q}\in\boldsymbol{\Lambda}+\frac{1}{2}\boldsymbol{p}} (-1)^{\boldsymbol{q}\cdot\boldsymbol{p}}\Big[\mathrm{sgn}((\boldsymbol{q}+\boldsymbol{b})\cdot\boldsymbol{v}) - \mathrm{sgn}((\boldsymbol{q}+\boldsymbol{b})\cdot\boldsymbol{v}')\Big]\, e^{\pi\mathrm{i}\tau\boldsymbol{q}^2+2\pi\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{z}}$$

$$\boldsymbol{z}=\boldsymbol{c}-\tau\boldsymbol{b}$$

Converges if $\boldsymbol{v}^2,\ \boldsymbol{v}'^2,\ \boldsymbol{v}\cdot\boldsymbol{v}'<0$. But the sign functions spoil modularity!

Can it be cured? — **Yes!**

The modular completion of $\vartheta_{\boldsymbol{p}}(\tau,\boldsymbol{z})$ is obtained by replacement

$$\boxed{\begin{aligned}&\mathrm{sgn}(\boldsymbol{x}\cdot\boldsymbol{v}) \to \mathrm{Erf}\left(\sqrt{\pi}\,\frac{\boldsymbol{x}\cdot\boldsymbol{v}}{|\boldsymbol{v}|}\right)\\ &\boldsymbol{x}=\sqrt{2\tau_2}(\boldsymbol{q}+\boldsymbol{b})\end{aligned}}$$

Important property:

$$\mathrm{Erf}\left(\sqrt{\pi}\,\frac{\boldsymbol{x}\cdot\boldsymbol{v}}{|\boldsymbol{v}|}\right) \underset{|v|\to 0}{\to} \mathrm{sgn}(\boldsymbol{x}\cdot\boldsymbol{v})$$

$\longrightarrow$ *null vectors don't require completion*

Mock modularity

$\vartheta_{\boldsymbol{p}}(\tau, \boldsymbol{z})$ **is an example of *mock modular form***

Zwegers. '02

holomorphic function which is "almost" modular with an anomaly controlled by another modular form (*shadow*)

$$h(\tau) \;\mapsto\; (c\tau+d)^w \left(h(\tau) - \int_{-d/c}^{-i\infty} \frac{\overline{g(\bar z)}}{(\tau-z)^w}\, dz \right)$$

modular form of weight $2-w$

first 17 examples: Ramanujan (lost notebook)

and (*non-holomorphic*) modular *completion* given by $\widehat{h}(\tau,\bar\tau) = h(\tau) - \int_{\bar\tau}^{-i\infty} \frac{\overline{g(\bar z)}}{(\tau-z)^w}\, dz$ — Eichler integral

In our case: $\mathrm{Erf}(u\sqrt{\pi}) = \mathrm{sgn}(u) - \mathrm{sgn}(u)\,\mathrm{Erfc}(|u|\sqrt{\pi})$ ⟵ can be written as Eichler integral

Problem: what are the modular properties and the modular completion for generic signature?

Mock modularity of higher depth

Solution of this problem was found from the twistorial formulation of D-instantons S.A.,Banerjee,Manschot,Pioline; Nazaroglu '16

The kernel for signature (*d-n,n*) is given by combinations of products of *n* signs. The completion is obtained by

$$\prod_{i=1}^{n} \mathrm{sgn}(\boldsymbol{x}\cdot\boldsymbol{v}_i) \to \Phi_n^E(\{\boldsymbol{v}_i\},\boldsymbol{x})$$

generalized error functions

Indefinite theta series of signature $(d-n,n)$ are examples of mock modular forms of higher depth (equal to n)

$\partial_{\bar\tau}\widehat{h}(\tau)$ is expressed via mock modular forms of lower depth and $\widehat{h}(\tau)$ is constructed using *n* iterated Eichler integrals

The generating function of (refined) MSW invariants is a higher depth mock Jacobi form

$$\mathcal{D}=\mathcal{D}_1+\cdots+\mathcal{D}_n$$

irreducible divisors

Depth = n-1

N=4 dyons

Appear in Type IIA/K3×T^2 or in Het/T^6 and labelled by charge $\Gamma = \begin{pmatrix} Q^I \\ P^I \end{pmatrix}$
transforming in **(2,28)** of $SL(2,\mathbb{Z}) \times O(22,6,\mathbb{Z})$

→ Degeneracies depend only on duality invariants:

$$n = \frac{1}{2}\, Q^2 \qquad m = \frac{1}{2}\, P^2 \qquad \ell = Q \cdot P$$ — T-duality invariants

$$I = \gcd(Q \wedge P)$$ — U-duality invariant

- **½ BPS states** $(Q^I \parallel P^I)$

Depend only on *n* and counted by $\Delta^{-1}(\tau) = \sum_{n=-1}^{\infty} \Omega_{1/2}(n)\, \mathrm{q}^n = \eta^{-24}(\tau)$ *true modular form*

where $\mathrm{q} = e^{2\pi i \tau}$

- **¼ BPS states**

For $I=1$ counted by *(Igusa) Seigel modular form* Dijkgraaf,Verlinde,Verlinde '96

$$\frac{1}{\Phi_{10}(\tau,z,\sigma)} = \sum_{m=-1}^{\infty} \psi_m(\tau,z)\, e^{2\pi i m \sigma}$$

meromorphic Jacobi form

Dabholkar,Murthy,Zagier '12

$$\boxed{\psi_m = \psi_m^P + \psi_m^F}$$

polar part counting 2-particle bound states

mock Jacobi forms

finite part counting immortal dyons ↔ $\Omega_\gamma^{\mathrm{MSW}}$

Holomorphic anomaly

S.A.,Nampuri
in progress

Holomorphic anomaly for the modular completion

Dabholkar,Murthy,Zagier '12

$$\boxed{\tau_2^{3/2}\partial_{\bar\tau}\widehat{\psi}^F_m = \frac{\sqrt{m}}{8\pi\mathrm{i}}\,\frac{\Omega_{1/2}(m)}{\Delta(\tau)}\sum_{\ell=0}^{2m-1}\overline{\theta_{m,\ell}(\tau,0)}\,\theta_{m,\ell}(\tau,z)}$$

where

$$\theta_{m,\ell}(\tau,z)=\sum_{r\in 2m\mathbb{Z}+\ell}\mathrm{q}^{\frac{r^2}{4m}}\,y^r$$

$$\boxed{(y-y^{-1})\partial_{\bar\tau}\widehat{h}^{\mathrm{ref}}_p \sim \sum_{n=2}^{\infty}\sum_{\sum_{i=1}^n\gamma_i=\gamma}\Theta_n(\{\gamma_i\};y)\prod_{i=1}^n\Omega(\gamma_i,y)}$$

Do they agree?

Holomorphic anomaly

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Dabholkar,Murthy,Zagier '12

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They do agree!

helicity supertraces	*helicity generating function*	*refined BPS index*
$B_{2n}(\gamma)=\mathrm{Tr}_{\mathcal{H}_\gamma}\left[(-1)^{2J_3}J_3^{2n}\right]$	$B(\gamma,y)=\mathrm{Tr}_{\mathcal{H}_\gamma}\left[(-y)^{2J_3}\right]$	$\Omega(\gamma,y)\sim\frac{yB(\gamma,y)}{2(1-y)^2}$
• $\Omega(\gamma)\sim B_2(\gamma)$	$\sim\partial_y^2 B(\gamma,y)\vert_{y=1}$	$=\Omega(\gamma,y)\vert_{y=1}$
• $\Omega_{1/2}(n)\sim B_4(\gamma)$	$\sim\partial_y^4 B(\gamma,y)\vert_{y=1}$	$\sim\partial_y^2\Omega(\gamma,y)\vert_{y=1}$
• $\Omega_{1/4}(n,m,\ell)\sim B_6(\gamma)$	$\sim\partial_y^6 B(\gamma,y)\vert_{y=1}$	$\sim\partial_y^4\Omega(\gamma,y)\vert_{y=1}$

The refined construction allows to go beyond the case $I=1$

S.A.,Manschot,Pioline '19

Relation to Vafa-Witten

Consider a CY given by an elliptic fibration over a projective surface *S* and take a *local limit* where the elliptic fiber becomes large

- local CY – the canonical bundle over *S*

non-compact!

(refined) DT invariant of local CY	**=**	**(refined) VW invariant of *S***

- The only surviving divisor is the base of the fibration [*S*] → $p^a = N p_0^a$ ← charge corresponding to [*S*]

All D4-brane charges are collinear!

degree of reducibility of the divisor — N — rank of the VW gauge group *U(N)*

Prediction for all ranks and surfaces with
$b_2^+(S)=1,\ b_1(S)=0$

$$\boxed{\widehat{h}^{\mathrm{VW,ref}}_{N,J=-K_S} = \widehat{h}^{\mathrm{ref}}_{Np_0}}$$

canonical polarization (attractor chamber)

Check: for $S=\mathbb{P}^2$ the modular completions have been explicitly computed for *N*=2 and 3 (*only!*). They perfectly coincide with our predictions!

Rank *N* Vafa-Witten invariants

The formula for the completion allows to find the VW invariants themselves!

Example: $N=2$

$$\widehat{h}_2 = h_2 + \frac{h_1^2}{2} \sum_{k\in\Lambda_S} \left(\mathrm{Erf}\left(2\sqrt{\frac{\pi\tau_2}{K_S^2}}\,(K_S\cdot k + \beta K_S^2)\right) - \mathrm{sgn}(K_S\cdot k)\right) \mathrm{q}^{-k^2}\, y^{2K_S\cdot k}$$

where for all surfaces

$$h_1 = \frac{\mathrm{i}}{\theta_1(\tau,2z)\,\eta(\tau)^{b_2(S)-1}}$$

modularity requires the kernel

$$\mathrm{Erf}(\sqrt{\tau_2}v\cdot(k+\beta K_S)) - \mathrm{Erf}(\sqrt{\tau_2}v'\cdot(k+\beta K_S))$$

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modularity requires the kernel

$$\mathrm{Erf}(\sqrt{\tau_2}v\cdot(k+\beta K_S)) - \mathrm{sgn}(v_0\cdot(k+\beta K_S))$$

holomorphic! $v_0^2=0$

$$\boxed{h_{2,-K_S} = H_2 + \frac{h_1^2}{2}\sum_{k\in\Lambda_S}\left(\mathrm{sgn}(K_S\cdot k)-\mathrm{sgn}(v_0\cdot(k+\beta K_S))\right)\mathrm{q}^{-k^2}\,y^{2K_S\cdot k}}$$

holomorphic & modular

H_2 is found by requiring well-defined unrefined limit

$\longrightarrow$ $h_1^{-2}h_2$ must have a zero at $y=\pm 1$

for *Hirzebruch* and *del Pezzo*

$$H_2 \sim \frac{\mathrm{i}\eta(\tau)}{\theta_1(\tau,4z)\theta_1(\tau,2z)^2}$$

Explicit expressions
for generating functions of refined
VW invariants and their completions
for all *N*

S.A. 2005.03680
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Rank *N* Vafa-Witten invariants

The formula for the completion allows to find the VW invariants themselves!

Example: $N=2$

$$\widehat{h}_2 = h_2 + \frac{h_1^2}{2}\sum_{k\in\Lambda_S}\left(\mathrm{Erf}\left(2\sqrt{\frac{\pi\tau_2}{K_S^2}}\,(K_S\cdot k+\beta K_S^2)\right) - \mathrm{sgn}(K_S\cdot k)\right)\mathrm{q}^{-k^2}\,y^{2K_S\cdot k}$$

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modularity requires the kernel

$$\mathrm{Erf}(\sqrt{\tau_2}v\cdot(k+\beta K_S)) - \mathrm{sgn}(v_0\cdot(k+\beta K_S))$$

holomorphic! $v_0^2=0$

$$\boxed{h_{2,J} = H_2 + \frac{h_1^2}{2}\sum_{k\in\Lambda_S}\left(\mathrm{sgn}((-J)\cdot k) - \mathrm{sgn}(v_0\cdot(k+\beta K_S))\right)\mathrm{q}^{-k^2}\,y^{2K_S\cdot k}}$$

holomorphic & modular

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for *Hirzebruch* and *del Pezzo*

$$H_2 \sim \frac{\mathrm{i}\eta(\tau)}{\theta_1(\tau,4z)\theta_1(\tau,2z)^2}$$

Explicit expressions for generating functions of refined VW invariants and their completions for all *N* and all *J*

Comments

- This construction requires only two ingredients:
  1) a *unimodular* charge (second homology) lattice Λ_S
  2) a *null* vector $v_0 \in \Lambda_S$

What if there are several null vectors in the lattice?

- The null vectors satisfying $v_0 \cdot v_0' = 1$ & $v_0 \cdot K_S = v_0' \cdot K_S$
  give the *same* generating functions ⟵ example of *fiber-base duality*

  Katz,Mayr,Vafa '97

  Requires very non-trivial identities between theta functions!

  simplest example $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$

- Other null vectors give different generating functions ⟶ **new invariants?**
  simplest example $S = \mathbb{F}_1$

- What if the lattice does not have a null vector? simplest example $S = \mathbb{P}^2$
  Then one can extend the lattice by multiplying the formula for the completion by a theta series

Explicit results at all *N* for $S = \mathbb{P}^2$

An improved version of the *blow-up formula*

Holomorphic anomaly of VW partition function

There is a conjecture for *U(N)* Vafa-Witten theory
(checked for $S=\frac{1}{2}K3$ and $\mathbb{P}^2$ up to $N\le 3$)

Minahan,Nemeschansky, Vafa,Warner '98

$$\boxed{\partial_{\bar\tau}\widehat{Z}_N^{\rm VW}\sim\sum_{N=N_1+N_2}N_1N_2\,\widehat{Z}_{N_1}^{\rm VW}\,\widehat{Z}_{N_2}^{\rm VW}}$$

where $\widehat{Z}_N^{\rm VW}=\widehat{h}_N^{\rm VW}(\tau)\,\vartheta_N^{\rm SN}(\tau,v)$ — partition function

$\vartheta_N^{\rm SN}(\tau,v)$ — Siegel-Narain theta series

Our formula for the modular completion implies:

$$\partial_{\bar\tau}\widehat{h}_p^{\rm ref}(\tau,y)=\sum_{n=2}^{\infty}\sum_{\sum_{i=1}^n\gamma_i=\gamma}\mathcal{J}_n^{\rm ref}(\{\gamma_i\},\tau_2)\,(-y)^{\sum_{i<j}\gamma_{ij}}\,e^{\pi\mathrm{i}\tau Q_n(\{\gamma_i\})}\prod_{i=1}^n\widehat{h}_{p_i}^{\rm ref}(\tau,y)$$

How can these two anomalies be consistent?

Holomorphic anomaly: unrefined case

The key restriction: D4-brane charges are collinear $p^a = N p_0^a$

$$\mathcal{J}_n^{\mathrm{ref}} = 0, \;\; n > 2$$

$$\partial_{\bar\tau} \widehat{h}_N = \sum_{\substack{N_1+N_2=N \\ q_{1,a}+q_{2,a}=q_a}} \frac{(-1)^{\gamma_{12}}}{8\pi\mathrm{i}(2\tau_2)^{3/2}} \sqrt{p_0^3 N N_1 N_2}\, e^{-\frac{2\pi\tau_2\gamma_{12}^2}{p_0^3 N N_1 N_2}}\, e^{\pi\mathrm{i}\tau Q_2(\gamma_i,\gamma_2)} \widehat{h}_{N_1} \widehat{h}_{N_2}$$

The crucial property: $\vartheta_N^{\mathrm{SN}} = \sum_{q_a} (-1)^{p^a q_a} \mathcal{X}_{N,q}^{\mathrm{SN}}$ exponential factor defining the Siegel-Narain theta series

Then $\mathcal{X}_{N,q}^{\mathrm{SN}} = e^{\frac{2\pi\tau_2\gamma_{12}^2}{p_0^3 N N_1 N_2}}\, e^{-\pi\mathrm{i}\tau Q_2(\gamma_1,\gamma_2)} \mathcal{X}_{N_1,q_1}^{\mathrm{SN}} \mathcal{X}_{N_2,q_2}^{\mathrm{SN}}$

$$\boxed{\overline{\mathcal{D}} \widehat{\mathcal{Z}}_N = \frac{\sqrt{2\tau_2}}{32\pi\mathrm{i}} \sum_{N_1+N_2=N} N_1 N_2\, \widehat{\mathcal{Z}}_{N_1} \widehat{\mathcal{Z}}_{N_2}}$$

$$\widehat{Z}_N = \frac{\sqrt{p_0^3}}{\sqrt{N}} \widehat{h}_N(\tau) \vartheta_N^{\mathrm{SN}}(\tau, v)$$

$$\overline{\mathcal{D}} = \tau_2^2 \left(\partial_{\bar\tau} - \frac{\mathrm{i}}{4\pi} (\partial_{c_+} + 2\pi\mathrm{i} b_+)^2 \right)$$

such that $\overline{\mathcal{D}} \vartheta_N^{\mathrm{SN}} = 0$

Holomorphic anomaly: refined case

Can one write a similar anomaly equation for the refined partition function?

$$\partial_{\bar\tau}\widehat{h}^{\rm ref}_N = \sum_{\substack{N_1+N_2=N\\ q_{1,a}+q_{2,a}=q_a}} \frac{\mathrm{i}(-y)^{\gamma_{12}}}{2\sqrt{2\tau_2}}\,\frac{\gamma_{12}-\beta p_0^3 N N_1 N_2}{\sqrt{p_0^3 N N_1 N_2}}\, e^{-2\pi\tau_2\frac{(\gamma_{12}+\beta p_0^3 N N_1 N_2)^2}{p_0^3 N N_1 N_2}}\, e^{\pi\mathrm{i}\tau Q_2(\gamma_i,\gamma_2)}\widehat{h}^{\rm ref}_{N_1}\widehat{h}^{\rm ref}_{N_2}$$

The product property of $\mathcal{X}^{\rm SN}_{N,q}$ is not enough to absorb all factors.
This can be done using the *non-commutative star product* :

$$f\star g = f\exp\left[\frac{1}{2\pi\mathrm{i}}\left(\overleftarrow{\mathscr{D}_a}\,\overrightarrow{\partial}_{\tilde c_a}-\overleftarrow{\partial}_{\tilde c_a}\,\overrightarrow{\mathscr{D}_a}\right)\right] g$$

$$\begin{aligned}\mathscr{D}_a &= w\partial_{v^a}+\bar w\partial_{\bar v^a}\\ &= \alpha\partial_{c^a}+\beta\partial_{b^a}\end{aligned}$$

$\tilde c_a, c^a$ – RR-fields coupled to D4 and D2-brane charges

$$\overline{\mathcal{D}}\widehat{\mathcal{Z}}^{\rm ref}_N = \frac{\mathrm{i}(2\tau_2)^{3/2}}{32\pi^2 p_0^3 N}\sum_{N_1+N_2=N}\left(\partial_{\bar v^a}\widehat{\mathcal{Z}}^{\rm ref}_{N_2}\star\partial_{\tilde c_a}\widehat{\mathcal{Z}}^{\rm ref}_{N_1}-\partial_{\tilde c_a}\widehat{\mathcal{Z}}^{\rm ref}_{N_2}\star\partial_{\bar v^a}\widehat{\mathcal{Z}}^{\rm ref}_{N_1}\right)$$

where $\widehat{Z}^{\rm ref}_N = \frac{1}{\sqrt{Np_0^3}}\widehat{h}^{\rm ref}_N(\tau,y)\vartheta^{\rm SN}_N(\tau,v)$

Refined instanton generating potential

The refined construction lacks justification from the S-duality constraint

Unrefined case:

$$\mathcal{G} = \sum_{\gamma\in\Gamma_+}\int_{\ell_\gamma} \mathrm{d}z\, H_\gamma - \frac{1}{2}\sum_{\gamma_1,\gamma_2\in\Gamma_+}\int_{\ell_{\gamma_1}}\mathrm{d}z_1\int_{\ell_{\gamma_2}}\mathrm{d}z_2\, K_{\gamma_1\gamma_2}\, H_{\gamma_1} H_{\gamma_2} = \frac{1}{\sqrt{\tau_2}}\sum_{n=1}^{\infty}\left[\prod_{i=1}^{n}\sum_{p_i}\widehat{h}_{p_i}(\tau)\right]\widehat{\theta}_{\boldsymbol{p}}(\tau,\boldsymbol{v})$$

where

$$K_{\gamma_1\gamma_2} = 2\pi\left(\kappa_{abc}t^a p_1^b p_2^c + \frac{\mathrm{i}\langle\gamma_1,\gamma_2\rangle}{z_1 - z_2}\right)$$

and $H_\gamma(z)$ encodes *Darboux coordinates* on the twistor space of $\mathcal{M}$ and is determined by a *TBA-like equation*

large volume limit of the integral equation of Gaiotto-Moore-Neitzke for N=2 SYM / S¹ →

$$\boxed{H_\gamma(z) = H_\gamma^{\mathrm{cl}}(z)\,\exp\left[\sum_{\gamma'\in\Gamma_+}\int_{\ell_{\gamma'}}\mathrm{d}z'\, K_{\gamma\gamma'}(z,z')\, H_{\gamma'}(z')\right]}$$

$$H_\gamma^{\mathrm{cl}} = \bar\Omega(\gamma)\, e^{-2\pi\mathrm{i}\hat q_0 - \pi\mathrm{i}\tau(q+b)^2 - 2\pi\tau_2 z^2 + \cdots}$$

Refined analogue of the instanton generating potential:

$$\mathcal{G}^{\mathrm{ref}}(y) = \frac{1}{\sqrt{\tau_2}}\sum_{n=1}^{\infty}\left[\prod_{i=1}^{n}\sum_{p_i}\widehat{h}^{\mathrm{ref}}_{p_i}(\tau,y)\right]\widehat{\vartheta}^{\mathrm{ref}}_{\boldsymbol{p}}(\tau,\boldsymbol{v},y)$$

— by construction, Jacobi modular form of weight $\left(-\frac{1}{2},\frac{1}{2}\right)$

obtained by shifting the elliptic variable $\widehat{\vartheta}^{\mathrm{ref}}_{\boldsymbol{p}}(\tau,\boldsymbol{v},y) = \widehat{\vartheta}_{\boldsymbol{p}}(\tau,\boldsymbol{v}+w\mathfrak{p})$

corresponds to the insertion of $y^{\sum_{i<j}\langle\gamma_i,\gamma_j\rangle}$

Refined TBA equations

It is possible to reverse the derivation and obtain an *integral, non-perturbative* representation of $\mathcal{G}^{\mathrm{ref}}$

$$\mathcal{G}^{\mathrm{ref}} = \sum_{\gamma\in\Gamma_+} \int_{\ell_\gamma} \mathrm{d}z\, H_\gamma^{\mathrm{ref}}$$

where

$$\boxed{H_\gamma^{\mathrm{ref}}(z) = H_\gamma^{\mathrm{ref,cl}}(z) \star \left[1 + \sum_{\gamma'} \int_{\ell_{\gamma'}} \frac{\mathrm{d}z'}{z-z'}\, H_{\gamma'}^{\mathrm{ref}}(z')\right]}$$

$$H_\gamma^{\mathrm{ref,cl}} = \frac{\bar\Omega(\gamma,y)}{(y-y^{-1})}\, e^{-2\pi \mathrm{i}\hat q_0 - \pi \mathrm{i}\tau(q+b)^2 - 2\pi\tau_2 z^2 + \cdots}$$

Non-commutative TBA-like integral equation

The refinement effectively quantizes the moduli space and its twistor space consistently with S-duality

- However, the unrefined limit $y \to 1$ is non-trivial…
- Relation to the non-commutative TBA equation for framed BPS states [Gaiotto-Moore-Neitzke '13] is not clear

Conclusions

Main result: *Explicit* form of the *modular completion* of the generating function of (refined) black hole degeneracies (DT invariants) at large volume attractor point for *arbitrary* divisor of CY

$\longrightarrow$ $h_p(\tau)$ – higher depth mock modular form

Numerous applications: N=4 dyons, VW invariants for arbitrary rank, fiber-base duality, blow-up formula, holomorphic anomaly, quantization of the moduli space consistent with S-duality….

Open problems:

- Extension of this technique to evaluation of DT invariants for *compact* CYs
- Understanding the non-commutative geometry of the refined moduli space and relations to previous constructions

$\longrightarrow$ relation to *twistorial topological string*? [Ceccotti-Neitzke-Vafa '14]

- Geometric or physical meaning of the (refined) instanton generating potential
- Geometric of physical meaning of the invariants generated by "wrong" null vectors
- Understanding the completion from the point of view of the world-volume theory on M5-brane wrapped on a reducible divisor