MHV Amplitudes and Cutting Methods for NLO Multi-Leg Processes

Simon Badger (DESY, Zeuthen)

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Outline

On-shell techniques for one-loop amplitudes

- Generalised unitarity with complex momenta
- On-shell recursion relations
  - Tree-level and one-loop
- $\mathrm{Higgs} + 2j$: Compact analytic formulae
- Extensions to massive particles

Computations of Virtual Corrections

- A lot of recent progress in computational methods for virtual corrections:

  Bern,Dixon,Dunbar,Kosower,Britto,Cachazo,Feng,Mastrolia,
  Ossola,Papadopoulos,Pittau,Ellis,Giele,Kunszt,Melnikov,Forde,. . .

- Automated numerical approaches:

  [`BlackHat`, `Rocket`, `CutTools/Helac-NLO`, `GOLEM`, Denner et al.,`samurai`, . . .]

  - Growing number of phenomenological studies
  - Continuing improvements: colour dressing, GPU's

  [see talks of Denner, Dixon,Forde,Giele,Papadopoulos]

- Efficiency:
  - Numerical stability
  - Fast numerical evaluation
  - Complexity of processes with additional jets
  - Portability $\Rightarrow$ Public codes
- Analytical computations good for numerical stability and speed

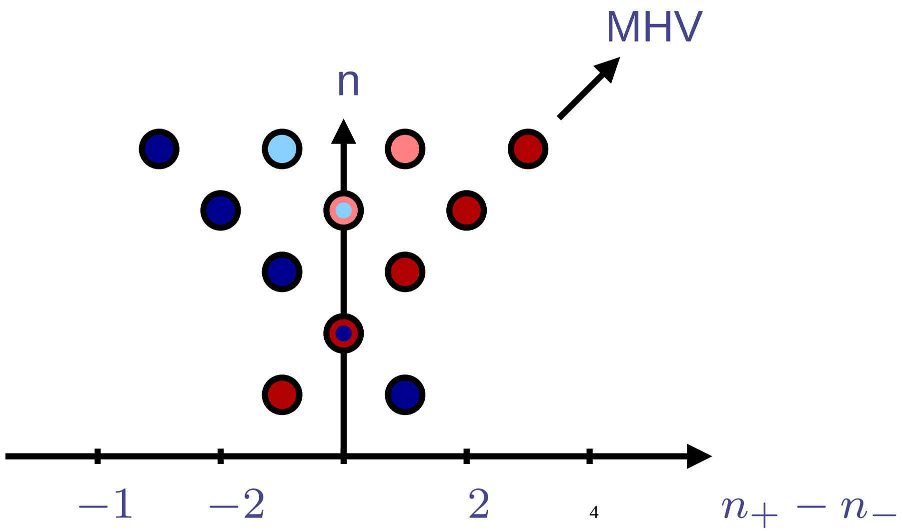
QCD Helicity Amplitudes

- On-shell helicity amplitudes have simple structure
- Example: $gg \to n(g)$ [Parke,Taylor (1985)]

$$A_n^{(0)}(1^+, 2^+, \ldots, n^+) = 0 \qquad \text{"all-plus"}$$

$$A_n^{(0)}(1^-, 2^+, \ldots, n^+) = 0 \qquad \text{"one-minus"}$$

$$A_n^{(0)}(1^-, 2^-, \ldots, n^+) = \frac{\langle 12 \rangle^4}{\prod_{k=1}^{n} \langle k(k+1) \rangle} \qquad \text{MHV}$$



Amplitudes in Twistor Space

- New insight from weak-weak Gauge/String Duality [Witten (2003)]

  $\mathcal{N} = 4$ SYM $\leftrightarrow$ Topological Strings on $\mathbb{CP}^{3|4}$

- MHV amplitudes supported on straight lines in Twistor space
- non-MHV amplitudes $=$ intersecting straight lines $\Rightarrow$ MHV rules
  - Effective tool for tree-level amplitudes

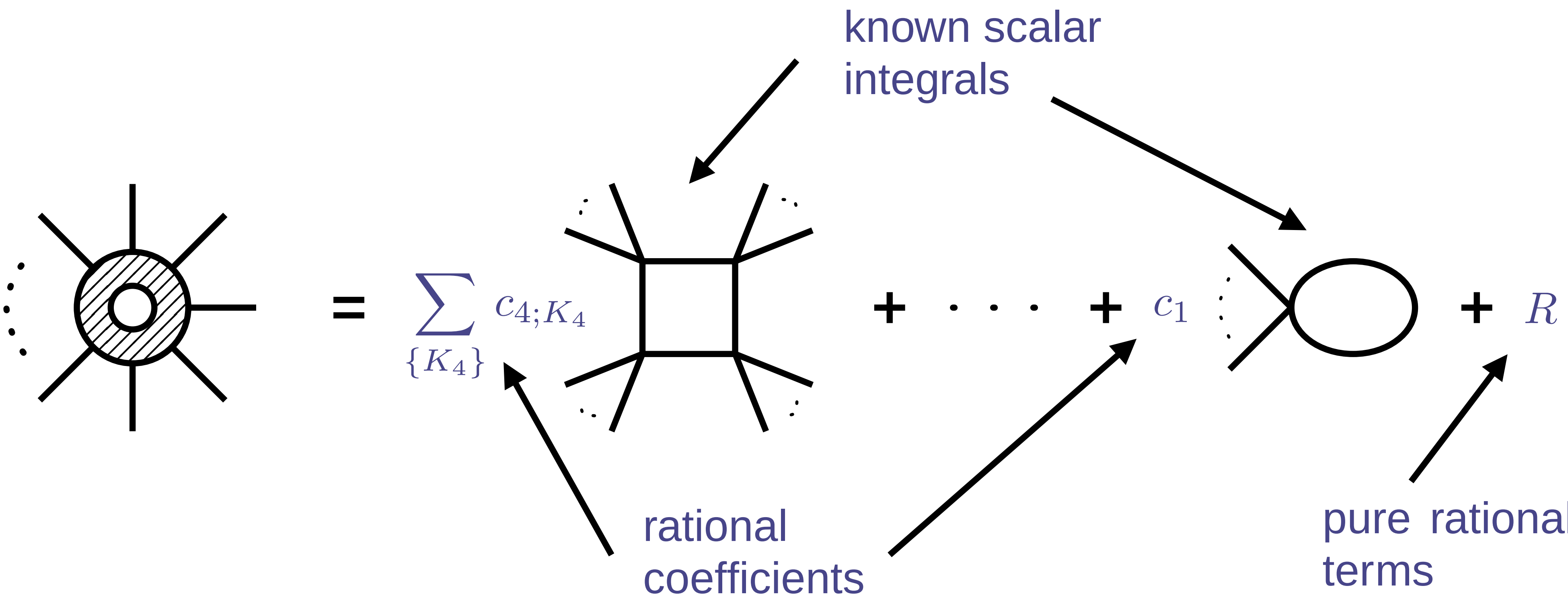
    Recursive construction for QCD, Higgs bosons, Vector Bosons
  - One-loop extensions proved complicated. . .
- Practical lessons : complex momenta $\mathbb{CP}^3 \leftrightarrow \mathbb{M}^c$

  $\Rightarrow$ MHV amplitudes localised in momentum space

  $\Rightarrow$ Three-point amplitudes defined on-shell in $\mathbb{M}^c$

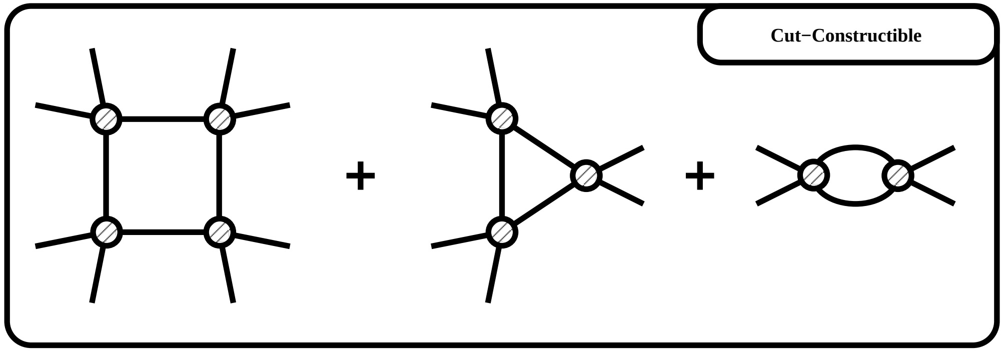
Generalised Unitarity, On-Shell Recursion

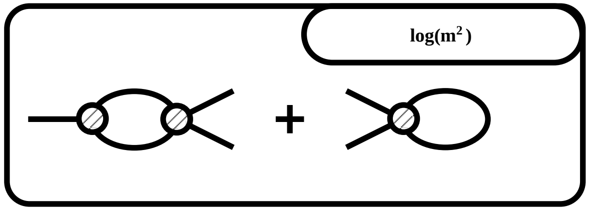
Structure of One-Loop Amplitudes



- Gauge theory amplitudes reduce to boxes or simpler [Passarino,Veltman][Melrose]
- Isolate logarithms with cuts [Bern,Dixon,Kosower (1994)]
  - Exploit on-shell simplifications
- General cutting principle:
  - apply δ-functions to L and R sides
  - generate and solve the linear system for the coefficients

Generalised Unitarity for One-Loop Amplitudes





trees

Britto,Cachazo,Feng,Witten

Berends,Giele

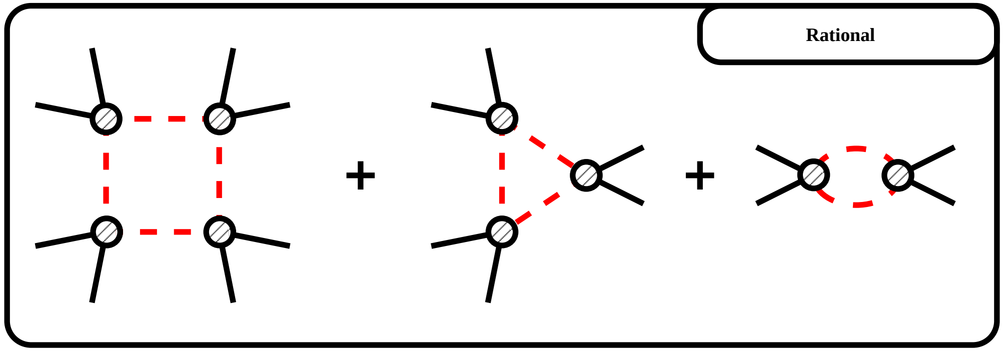
trees $\Rightarrow$ loops

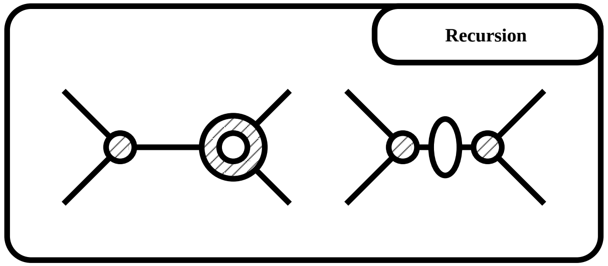
Bern,Dixon,Dunbar,Kosower,Berger,Forde

Britto,Cachazo,Feng,Mastrolia,Yang

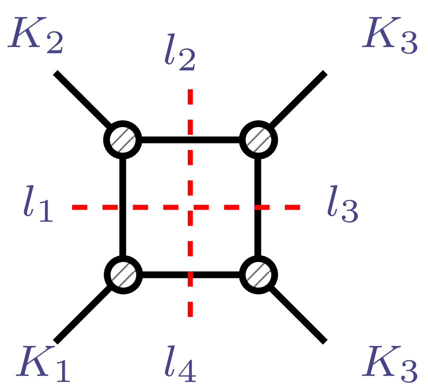
Ossola,Papadopoulos,Pittau

Ellis,Giele,Kunszt,Melnikov





Quadruple Cuts



- Quadruple cut$\rightarrow$ 4 on-shell δ-functions
- $C_4 = \frac{1}{2}\sum_{\sigma=\pm} A_1 A_2 A_3 A_4(l_1^\sigma)$

[Britto,Cachazo,Feng (2004)]

- Two complex solutions

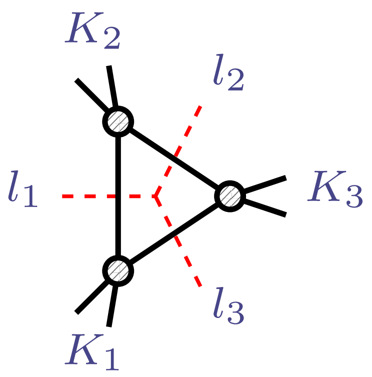
$$l_\pm^\mu = aK_1^{\flat,\mu} + bK_2^{\flat,\mu} + \frac{c}{2}\langle K_1^\flat|\gamma^\mu|K_2^\flat] + \frac{d}{2}\langle K_2^\flat|\gamma^\mu|K_1^\flat]$$

- Complete amplitude in $\mathcal{N}=4$ SYM

Triple Cuts

- Triple cut$\rightarrow$ 3 on-shell δ-functions
- Parametrise free integration

[Ossola,Papadopoulos,Pittau][Forde]

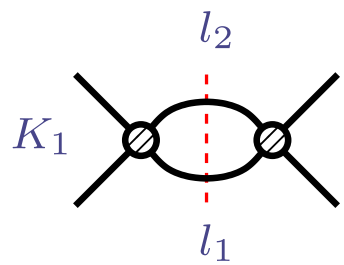


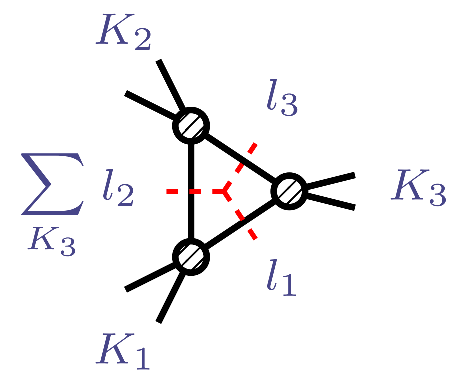
$$\oint J_t dt A_1 A_2 A_3 = \oint J_t dt \mathrm{Inf}_t [A_1 A_2 A_3(t)] + \sum_k \frac{\mathrm{Res}_{\mathrm{t=t_k}}(A_1 A_2 A_3)}{\xi_k (t - t_k)}$$

- $C_3 = \frac{1}{2} \sum_{\sigma=\pm} \mathrm{Inf}_t [A_1 A_2 A_3 (l_1^\sigma(t))]|_{t^0}$

- $\mathrm{Inf}_t [f(t)] = \lim_{t\rightarrow\infty} (f(t))\Big|_{\mathrm{pole}} = c_0 + c_1 t + c_2 t^2 + c_3 t^3$

Double Cuts





Bubble coefficients follow from a similar analysis:

$$C_2 = \mathrm{Inf}_t \mathrm{Inf}_y [A_1 A_2(t,y)] - \frac{1}{2} \sum_{\sigma=\pm} \mathrm{Inf}_t [A_1 A_2' A_3'(t, y_\pm)]$$

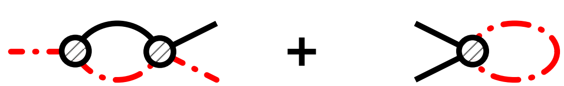
3-cut: Cauchy's Theorem

[Dunbar,Perkins,Warwick]

2-cut: 2D complex integration

$\Rightarrow$ Stokes' Theorem

[Mastrolia]



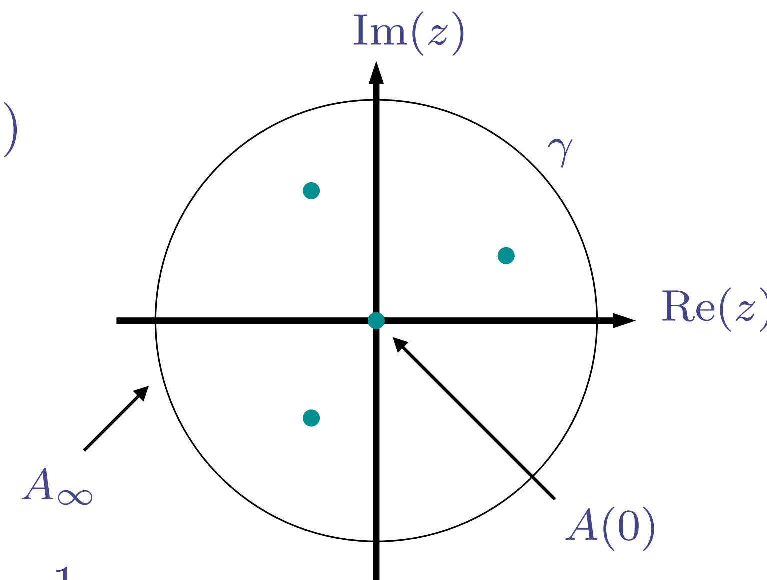
- Wave-function bubbles $\propto \log(m^2)$
- Double cut diverges
- Solution: explicitly remove poles

[Ellis,Giele,Kunszt,Melnikov]

BCFW Recursion

[Britto,Cachazo,Feng,Witten]

n-particle tree amplitude with complex deformation $\langle p_1, p_2] \to \langle p_1(z), p_2(z)]$,



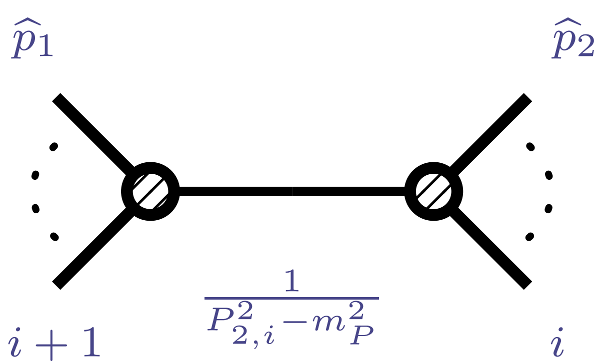
$$A(p_1, p_2, \ldots, p_n) \to A(\widehat{p_1}(z), \widehat{p_2}(z), \ldots, p_n) = A(z)$$

Momentum conservation: $\widehat{p_1}(z) + \widehat{p_2}(z) = p_1 + p_2$

On-shell conditions: $\widehat{p_1}(z)^2 = 0, \widehat{p_2}(z)^2 = 0$

$$0 = \frac{1}{2\pi i} \oint_\gamma dz \frac{A(z)}{z} = A(0) + \sum_{i=3}^{n-1} A_L(z_i) \frac{1}{P_i^2 - m_i^2} A_R(z_i) + A_\infty$$

$$\sum_{i=3}^{n}$$



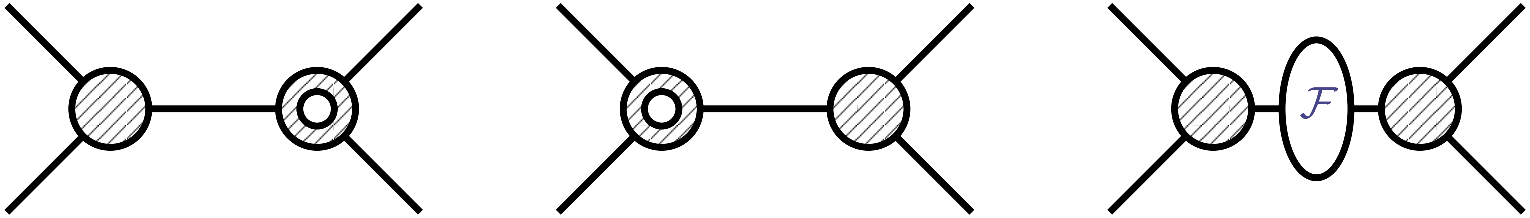
Massive particles [SB,Glover,Khoze,Svrček]

On-shell Recursion at One-Loop

- Generalisation of BCFW to one-loop [Bern,Dixon,Kosower]

$$A^{(1)} = A^{cc} + R = (A^{cc} + CR) + (R - CR)$$

$$R - CR = R^{\rm rec} - \sum {\rm Res}_k(CR)$$



- Extremely compact expressions for rational terms
  - $gg \to n(g)$, $W + 3/4j$ (leading colour), $gg \to H + n(g)$ (MHV), $q\bar{q} \to H + n(g)$ (MHV+finite)
- Subtleties in complex factorisation:
  - Double shifts needed for 6 gluon NMHV [Berger,Bern,Dixon,Forde,Kosower]
  - Some NMHV amplitudes not currently possible [see $H + 2j$]

D-dimensional Cuts

- Rational terms detected by cuts in higher dimensions

  Bern,Dixon,Kosower,Anastasiou, Britto,Feng,Kunszt,Mastrolia

- Higher integer dimensions well suited for numerical implementations

  [Giele,Kunszt.Melnikov (2008)]

$$\mathcal{A}^{FDH} = \frac{D_2 - 4}{D_2 - D_1} \mathcal{A}_{D,D_s=D_1} - \frac{D_1 - 4}{D_2 - D_1} \mathcal{A}_{D,D_s=D_2}$$

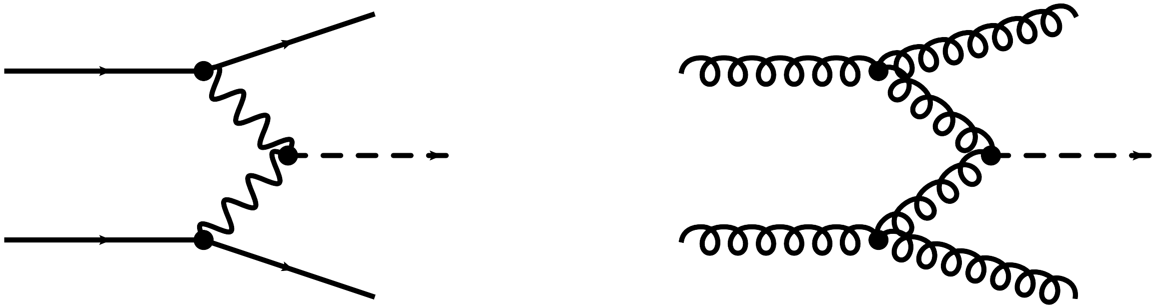
- Massive cuts with additional series expansion [OPP (2008)][SB (2008)]

$$l^2_{[D]} = l^2_{[4]} - \mu^2, \qquad R \sim \lim_{\mu^2 \to \infty} A^{(1)}(\mu^2)$$

  - Easy to automate analytically: all-helicities for $gg \to 4g$
  - In general expressions can be very long (on-shell recursion better)
  - Numerical version used in `BlackHat` for $W + 3/4j$

Hadronic Production of $H + 2j$

- Initial studies using semi-numerical approach [Ellis,Giele,Zanderighi (2006)] [Campbell,Ellis,Zanderighi (2006)]
- Dominant background to Higgs production via VBF



- Compact analytic expressions allow for fast evaluation of distributions
  - Detailed phenomenological studies with decays [Campbell,Ellis,Williams (2010)]
  - Publicly available as part of MCFM v5.7

The Effective Higgs Model

- Effective Higgs-gluon coupling $m_t \to \infty$

$$\mathcal{L}_{\rm eff} = C H \,{\rm tr}\left(G^{\mu\nu} G_{\mu\nu}\right)$$

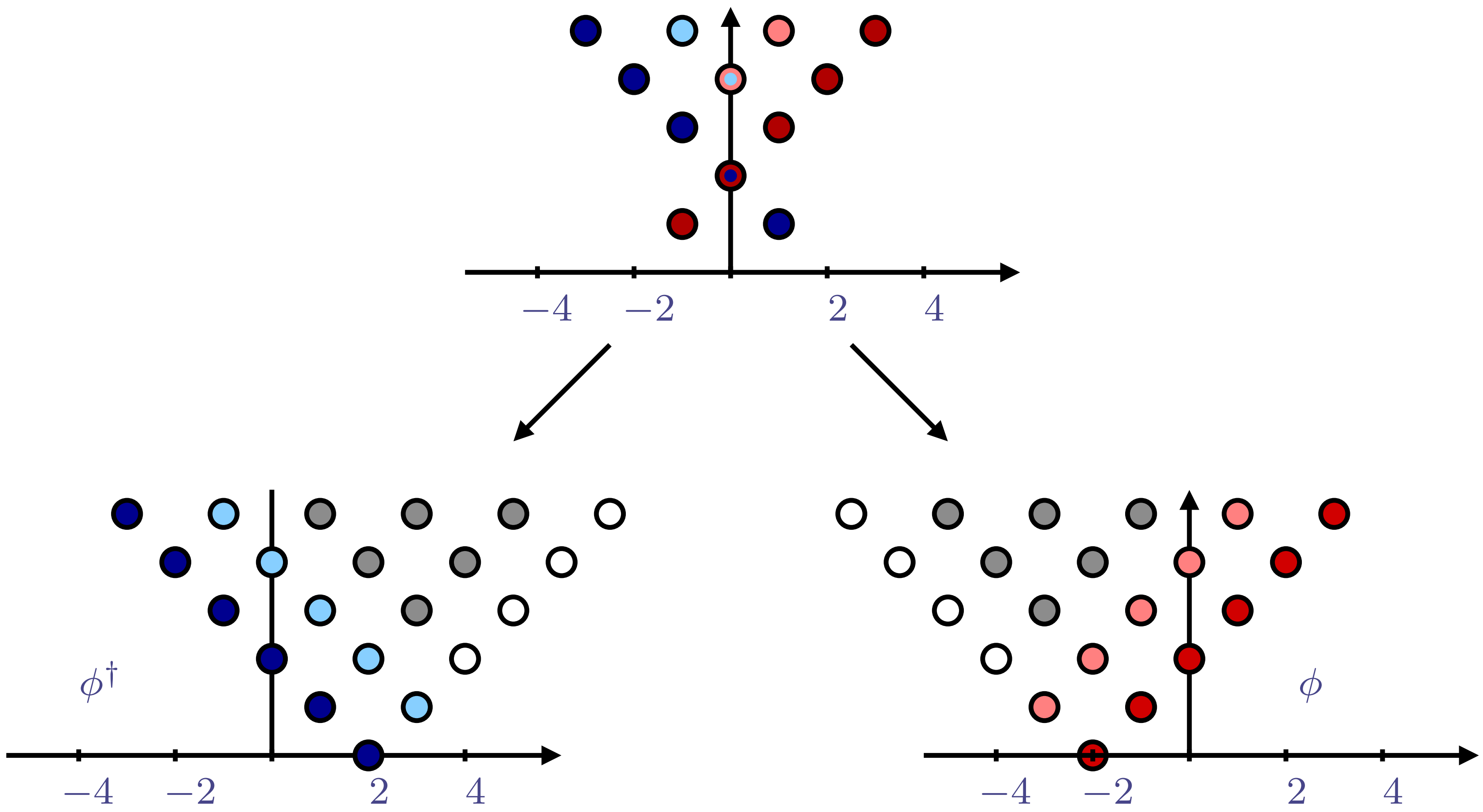
- Detailed analysis up to NNLO proves to be accurate approximation

  [Krämer,Laenen,Spira][Harlander,Ozeren][Pak,Rogal,Steinhauser]

- Uncovering the MHV structure [Dixon,Glover,Khoze (2004)]
  - Complex Higgs doublet $(H, A) \to (\phi, \phi^\dagger)$
  - $\phi(\phi^\dagger)$ couples to (anti)self-dual components of the gluons
  - On-shell simplicity for ϕ+QCD amplitudes

    [Dixon,Glover,Khoze `gluons`][SB,Glover,Khoze `fermions`]
  - $\phi^\dagger$ amplitudes via parity symmetry

ϕ-QCD Helicity Amplitudes



Analytic Expressions for $p\bar{p} \to H + 2j$

- Calculation of all helicity amplitudes completed

$gg \to Hgg$

H Amplitude	ϕ-amplitude	$\phi^\dagger$ amplitude
$\mathcal{A}(H,+,+,+,+)$	$\mathcal{A}(\phi,+,+,+,+)$ [Berger,Del Duca,Dixon]	$\mathcal{A}(\phi^\dagger,+,+,+,+)$ [SB,Glover]
$\mathcal{A}(H,-,+,+,+)$	$\mathcal{A}(\phi,-,+,+,+)$ [Berger,Del Duca,Dixon]	$\mathcal{A}(\phi^\dagger,-,+,+,+)$ [SB,Glover,Mastrolia,Williams]
$\mathcal{A}(H,-,-,+,+)$	$\mathcal{A}(\phi,-,-,+,+)$ [SB,Glover,Risager]	$\mathcal{A}(\phi^\dagger,-,-,+,+)$ [SB,Glover,Risager]
$\mathcal{A}(H,-,+,-,+)$	$\mathcal{A}(\phi,-,+,-,+)$ [Glover,Mastrolia,Williams]	$\mathcal{A}(\phi^\dagger,-,+,-,+)$ [Glover,Mastrolia,Williams]

$q\bar{q} \to Hgg$

H Amplitude	ϕ-amplitude	$\phi^\dagger$ amplitude
$\mathcal{A}(H,-_{\bar{q}},+_q,+,+)$	$\mathcal{A}(\phi,-_{\bar{q}},+_q,+,+)$ [Berger,Del Duca,Dixon]	$\mathcal{A}(\phi^\dagger,-_{\bar{q}},+,+,+)$ [SB,Campbell,Ellis,Williams]
$\mathcal{A}(H,-_{\bar{q}},+_q,-,-)$	$\mathcal{A}(\phi,-_{\bar{q}},+_q,-,-)$ [SB,Campbell,Ellis,Williams]	$\mathcal{A}(\phi^\dagger,-_{\bar{q}},+,-,-)$ [Berger,Del Duca,Dixon]
$\mathcal{A}(H,-_{\bar{q}},+_q,+,-)$	$\mathcal{A}(\phi,-_{\bar{q}},+_q,+,-)$ [Dixon,Sofianatos]	$\mathcal{A}(\phi^\dagger,-_{\bar{q}},+,+,-)$ [Dixon,Sofianatos]
$\mathcal{A}(H,-_{\bar{q}},+_q,-,+)$	$\mathcal{A}(\phi,-_{\bar{q}},+_q,-,+)$ [Dixon,Sofianatos]	$\mathcal{A}(\phi^\dagger,-_{\bar{q}},+,-,+)$ [Dixon,Sofianatos]

$\mathcal{A}(H,-_{\bar{q}},+_q,-_{\bar{Q}},+_Q)$ [Ellis,Giele,Zanderighi][Dixon,Sofianatos]

Rational Terms

- On-shell recursion for MHV and finite-$q\bar{q}$
- NMHV amplitudes via Feynman diagrams
  - Reduced complexity via SUSY decomposition

$$R^{[L]} + R^{[R]} + R^{[f]} = 2\left(A^{(0)}(\phi) - A^{(0)}(\phi^\dagger)\right)$$

  - $H + 4g$: full rational contribution from fermion loop
    749 diagrams $\to$ 136 diagrams, rank-5 pentagon $\to$ rank-4 box
  - Remarkably simple results after simplification

$$R_4(H, 1^+, 2^-, 3^-, 4^-) = \left\{ \left(1 - \frac{N_f}{N_c} + \frac{N_s}{N_c}\right)\frac{1}{2} \right.$$
$$\left(\frac{\langle 23\rangle\langle 34\rangle\langle 4|p_H|1][31]}{3s_{123}\langle 12\rangle[21][32]} - \frac{\langle 3|p_H|1]^2}{s_{124}[42]^2} + \frac{\langle 24\rangle\langle 34\rangle\langle 3|p_H|1][41]}{3s_{124}s_{12}[42]} - \frac{[12]^2\langle 23\rangle^2}{s_{14}[42]^2} \right.$$
$$\left. \left. - \frac{\langle 24\rangle(s_{23}s_{24} + s_{23}s_{34} + s_{24}s_{34})}{3\langle 12\rangle\langle 14\rangle[23][34][42]} + \frac{\langle 2|p_H|1]\langle 4|p_H|1]}{3s_{234}[23][34]} - \frac{2[12]\langle 23\rangle[31]^2}{3[23]^2[41][34]} \right) \right\} + \left\{ (2 \leftrightarrow 4) \right\}$$

Sub-leading Colour Simplifications

- Sub-leading colour for $Hq\bar{q}gg$
- New colour structure at NLO:

$$A_4^{(1)} = (T^2T^3)_{14}A_{4;1}(\phi; 1_q, 2, 3, 4_{\bar{q}}) + (T^3T^2)_{14}A_{4;1}(\phi; 1_q, 3, 2, 4_{\bar{q}}) + \delta_{14}\delta^{23}A_{4;3}(\phi; 1_q, 2, 3, 4_{\bar{q}})$$

- Improved UV constraints $\Rightarrow$ large cancellations
  - c.f. no-triangle in $\mathcal{N} = 8$ super-gravity, QED
- Only boxes remain in $A_{4;3}$! (c.f. $N = 4$ SYM)

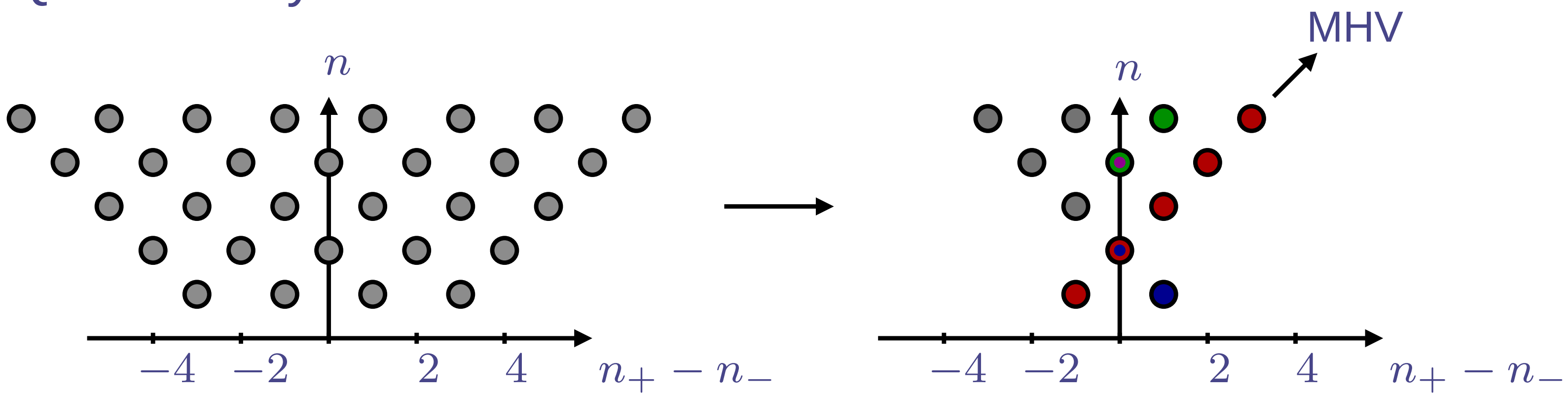
Helicity independent

Extensions to Massive Particles

- Addition of mass scales leads to:
  - New tree amplitude structures : no MHV amplitude
  - Mass renormalisation : $\log(m^2)$ terms $\Rightarrow 2 \to n+2$ trees
  - Rapid growth in analytic complexity
- Generalised unitarity is simple to extend [Britto,Feng,Mastrolia][Kilgore] [Ellis,Giele,Kunszt,Melnikov]
- Rapid recent progress with state-of-the-art numerical codes
  - $pp \to t\bar{t}b\bar{b}$ [Bredenstein,Denner,Dittmaier,Pozzorini] [Czakon et al. `Helac-NLO`]
  - $pp \to t\bar{t} + j$ [Dittmaier,Uwer,Weinzierl] [Czakon et al. `Helac-NLO`][Schulze,Melnikov]
  - $pp \to t\bar{t} + 2j$ [Czakon et al. `Helac-NLO`]

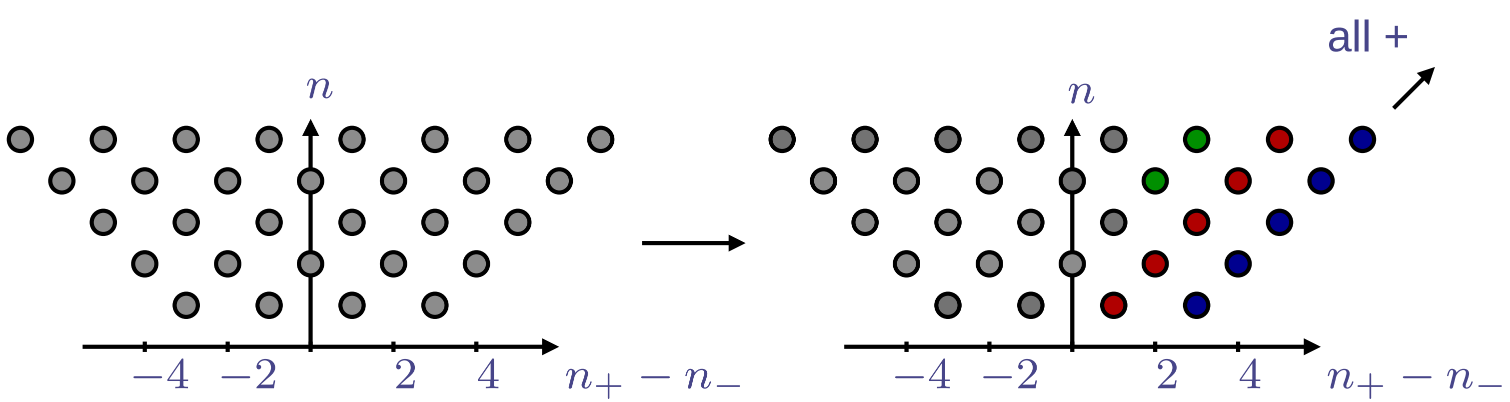
Massive Helicity Amplitudes

Massless QCD helicity structure:



Symmetry relates massive quark helicity states:

$$u_-(p,m;p^\flat,\eta) = \frac{\langle p^\flat \eta\rangle}{m} u_+(p,m;\eta,p^\flat)$$



Helicity Amplitudes for $p\bar{p} \to t\bar{t}$

- Compact helicity amplitudes for $gg \to t\bar{t}$ and $q\bar{q} \to t\bar{t}$

  [SB,Sattler,Yundin (in preparation)]
- Rational terms from Feynman diagrams
- Full agreement with much lengthier Feynman diagram calculations

  [Körner,Merebashvili (2002)][Anastasiou,Aybat (2008)]
- More simplifications in $gg \to t\bar{t}$ sub-leading colour
  - No bubbles or rational terms
- Should prove faster and more flexible than previous implementations

  [MCFM (2010)][Melnikov,Schulze (2009)][Bernreuther et al. (2004)]

$$A_4^{[L]}(1^+,2^+,3^+,4^+) = -I_4\left(s_{12},s_{23},0,m^2,m^2,0;0,0,m^2,0\right)\frac{\langle\eta_1\eta_4\rangle[32]^2m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle}$$

$$+ F_2\left(s_{12},0,m^2\right)\frac{\left(\langle\eta_1|K_{12}K_{32}|\eta_4\rangle - 2\langle\eta_1\eta_4\rangle s_{12}\right)[32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]^2} + \left(I_2\left(m^2,0,m^2\right) - 2\right)A_4^{(0)}$$

$$- \frac{\left(\langle\eta_1 K_{12}K_{32}\eta_4\rangle + \langle\eta_1\eta_4\rangle\langle 2|1|2]\right)[32]m}{2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]} - \frac{\left(\langle\eta_1\eta_4\rangle\langle 2|1|2] + \langle\eta_1 2\rangle\langle\eta_4 3\rangle[32]\right)m}{3\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2}$$

Conclusions and Outlook

- On-shell simplifications essential for NLO multi-jet cross-sections
- Generalised unitarity reduces computation of tree amplitudes with complex momenta
  - First $2 \to 5$ cross section $W + 4j$ [see talks of Dixon and Forde]
- Analytic results for $H + 2j$: fast, flexible public code [Campbell,Ellis,Williams `MCFM`]
- Full mass dependence at NLO : $t\bar{t} + 2j$ [`Helac-NLO`]
  - New analytic structure for $t\bar{t}$ production
- Open problems : Complex masses, EW corrections, NNLO?
  - Exciting new developments in $N = 4$ SYM [talk of Dixon]
- Automated NLO MC not too far away...

$\log(m^2)$ Terms in $pp \to t\bar{t}$

- Problematic $\log(m^2)$ dependence is in $I_2(m^2;0;m^2), I_1(m^2)$
- $\frac{1}{\epsilon}$ poles fully constrained by universal poles [Catani,Dittmaier,Trocsanyi]
- $\log(m^2)$ directly connected to $\frac{1}{\epsilon}$ pole
  - Missing information is rational

$$c_{2;m^2} I_2(m^2;0,m^2) + c_1 I_1(m^2) = (c_{2;m^2} + m^2 c_1) I_2(m^2;0,m^2) + m^2$$

- Massless limit fully under control with universal structure [Moch,Mitov]
- $m^2 \log(m^2)$ controls the additional rational terms [Bern,Morgan (1995)]
  - Works nicely for $++++$ and heavy fermion loop
  - More integral structure in $++-+$: needs generalisation

Simplicity in the Sub-Leading Colour

- Take some lessons from amplitudes in Gravity (and QED):

  Bern,Dixon,Kosower,Carrasco,Johansson,Roiban,Spradlin,Volovich,
  Ita,Forde,Bjerrum-Bohr,Dunbar,Perkins,... (many more)

  - Colour unordered contain additional gauge cancellations
  - One-loop amplitudes only have box structure
    $\mathcal{N}=8$ SUGRA [Arkani-Hamed,Cachazo,Kaplan]

    [Bjerrum-Bohr,Vanhove]

    n-photon, $n>8$ [SB,Bjerrum-Bohr,Vanhove]
  - Proof via Generalised Unitarity : $A(z) \underset{z\to\infty}{\to} \frac{1}{z^2}$
- Sub-leading colour amplitudes from permutation sums of primitive amplitudes $\Rightarrow$ Simpler integral structure
- How general is this structure?