Future Challenges for (N)NLO Accuracy in QCD

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ECT* Workshop on QCD at the LHC
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Motivation

- "The past few years have seen a revolution in our understanding of scattering amplitudes, particularly in maximally supersymmetric theories [in the limit $N_c \rightarrow \infty$]"

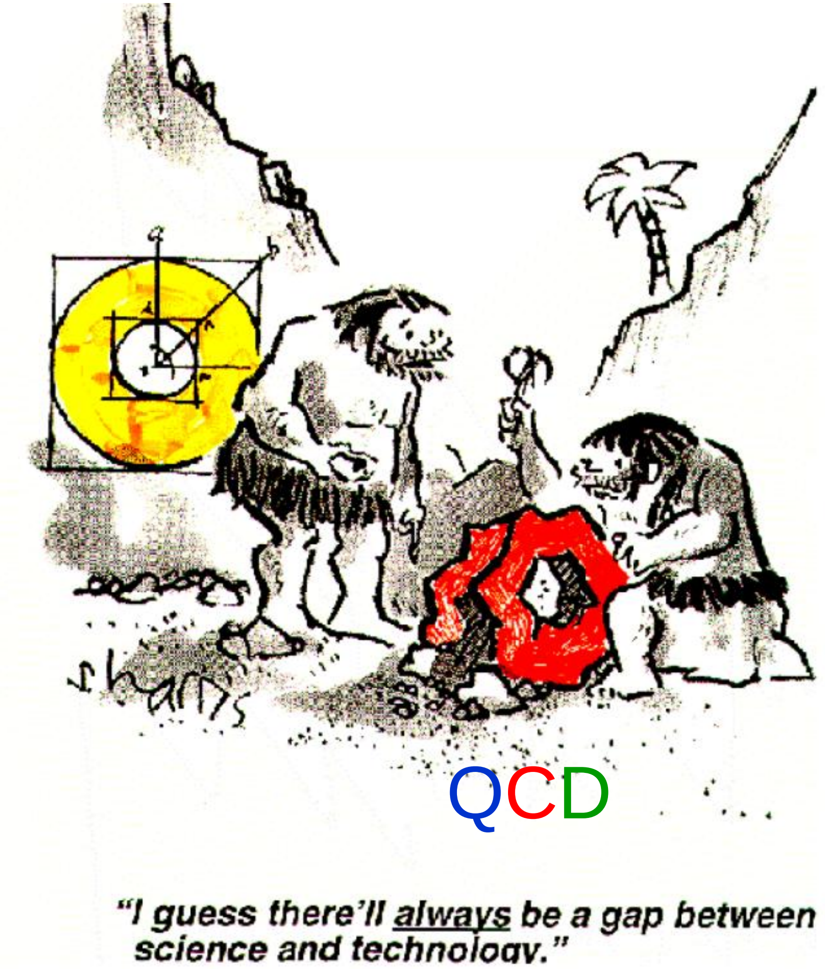
  - M. Spradlin, 22/09/2010; also first sentence of ~100 recent papers

- Can we exploit this progress for plain old, non-supersymmetric, $N_c = 3$, QCD?

- What other results can be expected, and what other novel tools might be developed, in addition to the ones we have already?

From “science” to “technology”

N=4 SYM



Levels of approximations

Increasing accuracy ↓

- Basic Monte Carlo [PYTHIA, HERWIG, Sherpa, …]
- LO QCD parton level
- LO QCD matched to parton showers [MadGraph/MadEvent, ALPGEN/PYTHIA, Sherpa, …]

- NLO QCD at parton level
- NLO matched to parton showers [MC@NLO, POWHEG,…]
- NLO matched to [N](N)NLL accurate resummation
- + NLO electroweak

- NNLO inclusive at parton level
- NNLO with flexible cuts at parton level (V+X, H+X)
- NNLO matched to [N](N)NLL accurate resummation
- NNLO matched to parton showers

- NNNLO QCD at parton level

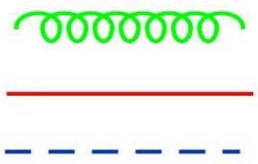
Decreasing availability ↓

How can maximal supersymmetry help?

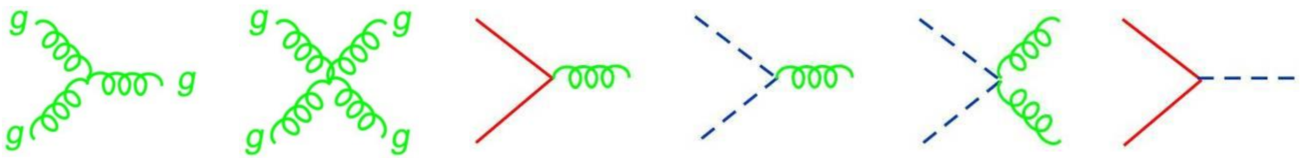
- N=4 super-Yang-Mills theory essentially contains QCD with massless quarks
- Much is known about N=4 SYM scattering amplitudes: the "hydrogen atom" of relativistic scattering in D=4
- Many symmetries, both explicit and subtle
- At tree level, very easy to exploit N=4 SYM results
- At each loop order, becomes progressively more difficult to disentangle unwanted states
- Not trivial to include massive (top) quarks or too many electroweak or BSM particles

N=4 SYM (vs. QCD)

massless spin 1 gluon
4 massless spin 1/2 gluinos
6 massless spin 0 scalars



all states in adjoint representation, all linked by N=4 supersymmetry



Interactions all proportional to same *dimensionless* coupling constant, *g*.
$\beta(g) = 0$ → conformal invariance: SO(1,3) → SO(2,4)

4 supersymmetries give powerful constraints on S matrix (at any loop order): Supersymmetry Ward Identities

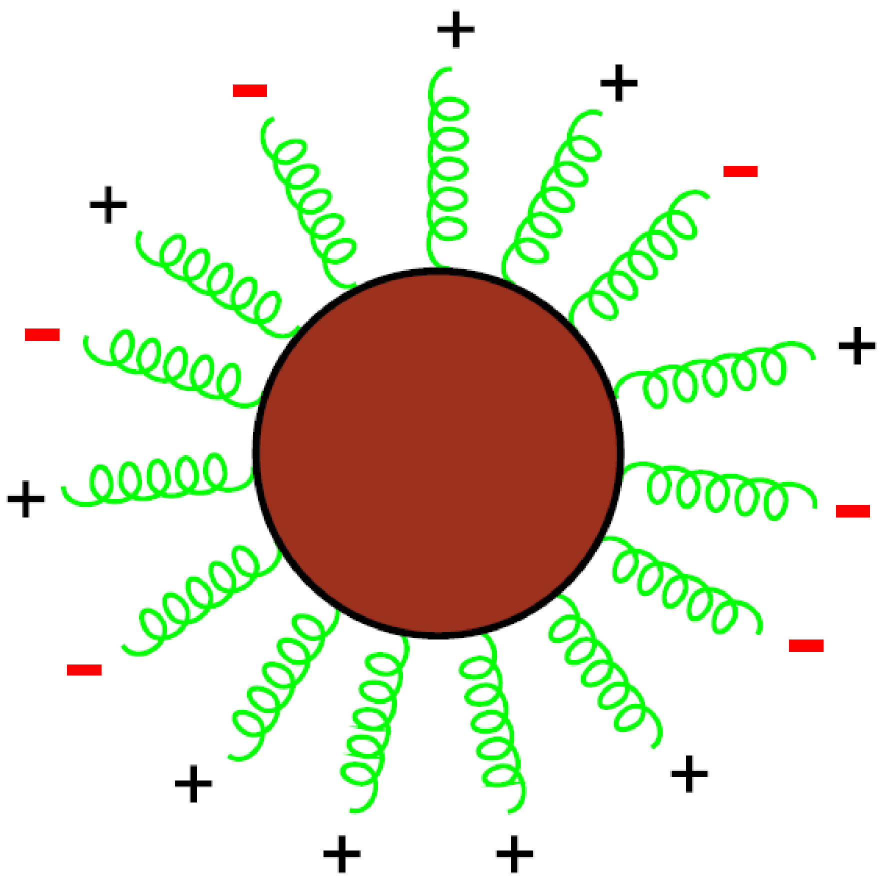
Grisaru, Pendleton, van Nieuwenhuizen (1977)

SWI already applied to QCD at LO in 1980s Parke, Taylor; Kunszt (1985)

N=4 SYM even more amazing for a large number of colors (N_c)

• Described by topological string in twistor space Witten (2004)
• Strong-coupling ←→ strings in 5-dimensional Anti-DeSitter space
Alday, Maldacena (2007)
• Dual (momentum space) conformal invariance
Drummond, Henn, Korchemsky, Smirnov, Sokatchev, … (2006-7)
• Part of a larger "Yangian" symmetry, associated with extra conserved currents: integrability, solvability
Dolan, Nappi, Witten; Drummond, Ferro; Beisert et al., …
• Conjecture that all loop amplitudes are determined by a particular analytic property, "leading singularities" Arkani-Hamed et al. (2008)
• Recursive formula recently proposed, "manifesting full Yangian symmetry", for integrands for all N=4 loop amplitudes. Expressed in Grassmannians (space of complex planes) & momentum-twistor space, related to dual (sector) variables. Arkani-Hamed et al. (2008)

Trees

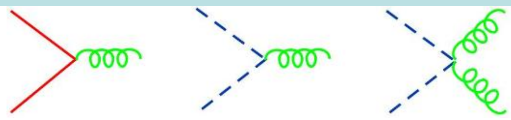


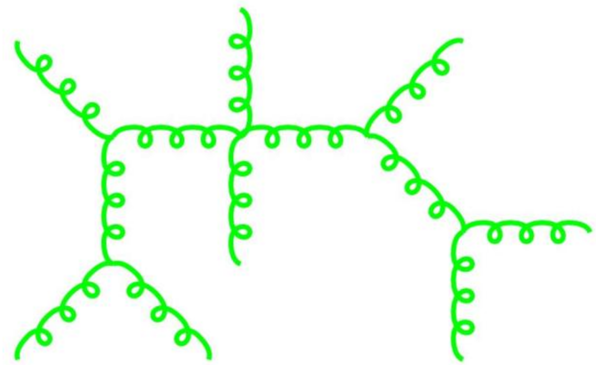
QCD and N=4 SYM at tree level (LO)

They are essentially identical

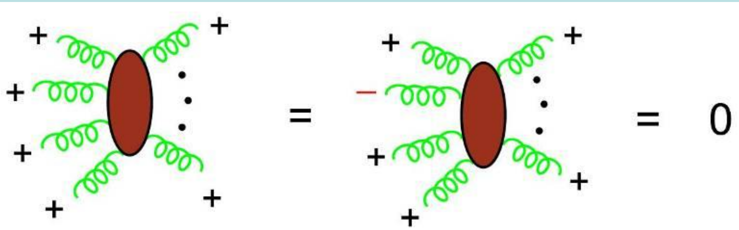
Consider a tree amplitude for *n* gluons

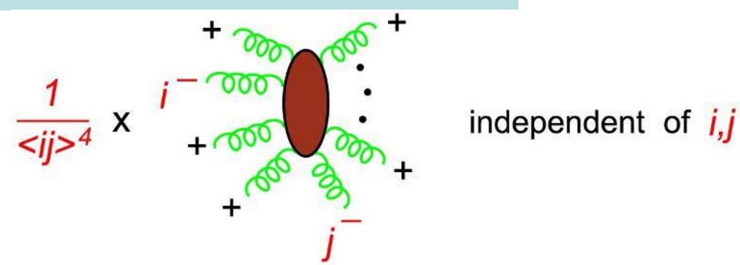
Fermions and scalars cannot appear because they are produced in pairs





Hence the amplitude is the same in QCD and N=4 SYM.
The QCD tree amplitude "secretly" obeys all identities of N=4 supersymmetry:

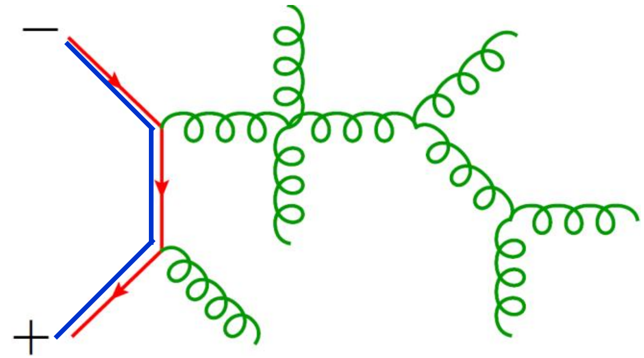




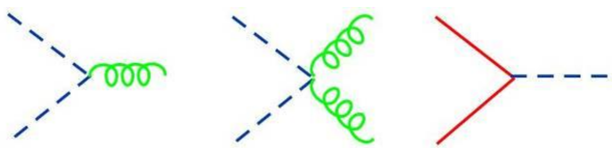
What about fermions?

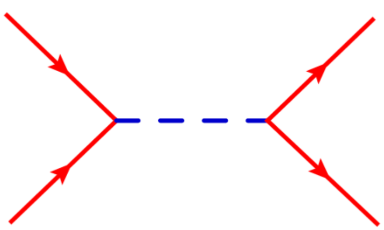
Gluinos are adjoint, quarks are fundamental

Color not a problem because it’s easily manipulated – common to work with “color stripped” amplitudes anyway



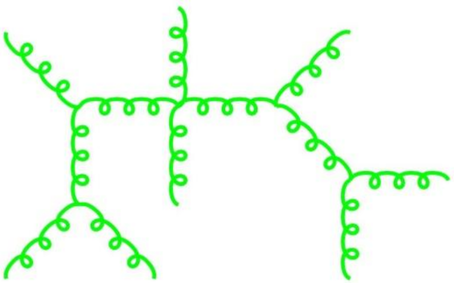
No unwanted scalars can enter amplitude with only one fermion line: Impossible to destroy them once created, until you reach two fermion lines:





Can use flavor of gluinos to pick out desired QCD amplitudes, through (at least) three fermion lines (all color/flavor orderings), and including V+jets trees
LD, Henn, Plefka, Schuster, to appear

All N=4 SYM trees now known in closed form



Drummond, Henn, 0808.2475

For example, for
3 (-) gluon helicities and [n-3] (+) gluon helicities,
extract from:

$$\boxed{\mathcal{P}_n^{\mathrm{NMHV}} = \sum_{2\leq a_1,b_1\leq n-1} R_{n;a_1b_1}}$$

5 (-) gluon helicities and [n-5] (+) gluon helicities:

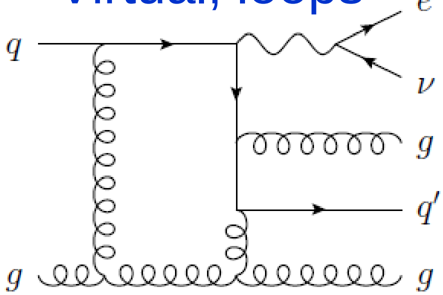
$$\mathcal{P}_n^{\mathrm{N^3MHV}} = \sum_{2\leq a_1,b_1\leq n-1} R_{n;a_1b_1}\Bigg[\sum_{a_1+1\leq a_2,b_2\leq b_1} R^{0;a_1b_1}_{n;b_1a_1;a_2b_2}\Bigg(\sum_{a_2+1\leq a_3,b_3\leq b_2} R^{0;b_1a_1a_2b_2}_{n;b_1a_1;b_2a_2;a_3b_3} + \sum_{b_2\leq a_3,b_3\leq b_1} R^{b_1a_1a_2b_2;a_1b_1}_{n;b_1a_1;a_3b_3} + \sum_{b_1\leq a_3,b_3\leq n-1} R^{a_1b_1;0}_{n;a_3b_3}\Bigg)$$
$$+ \sum_{b_1\leq a_2,b_2\leq n-1} R^{a_1b_1;0}_{n;a_2b_2}\Bigg(\sum_{a_2+1\leq a_3,b_3\leq b_2} R^{0;a_2b_2}_{n;b_2a_2;a_3b_3} + \sum_{b_2\leq a_3,b_3\leq n-1} R^{a_2b_2;0}_{n;a_3b_3}\Bigg)\Bigg].$$

Why “waste time” talking about trees?

- Because numerical (N)NLO programs “waste [computer] time” evaluating trees:
  - directly, to compute real emission terms
  - Inside loop computations, when using unitarity-based methods (at NLO)

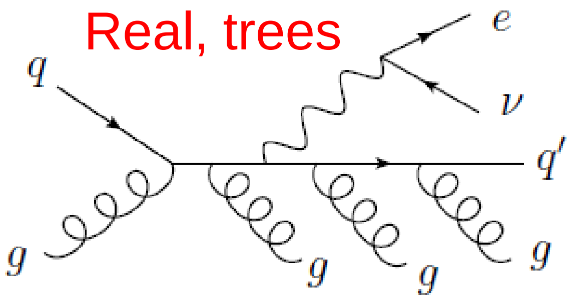
NLO real emission

Virtual, loops



Infrared singularities cancel

Real, trees



- General subtraction methods for integrating real-emission contributions developed in mid-1990s

Frixione, Kunszt, Signer, hep-ph/9512328;
Catani, Seymour, hep-ph/9602277, hep-ph/9605323

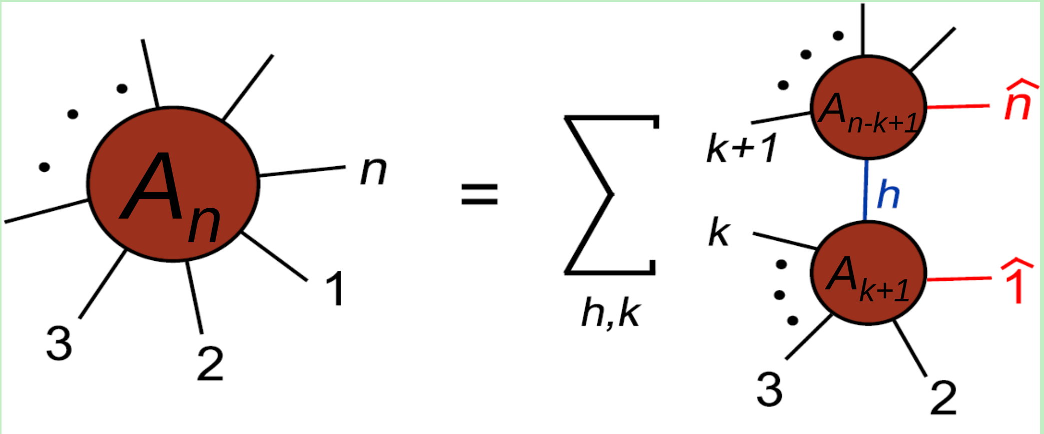
- Automated by several groups in last few years

Gleisberg, Krauss, 0709.2881;
Seymour, Tevlin, 0803.2231;
Hasegawa, Moch, Uwer, 0807.3701;
Frederix, Gehrmann, Greiner, 0808.2128;
Czakon, Papadopoulos, Worek, 0905.0883;
Frederix, Frixione, Maltoni, Stelzer, 0908.4272

BH+Sherpa evaluates trees about 100 times as often as loops

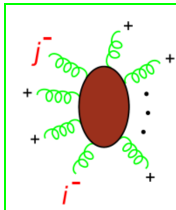
How to use N=4 SYM trees?

- Could use analytic formulae directly for all helicities needed.
- **Or**: use simplest nontrivial formulae (through NMHV), plus on-shell recursion relations. BCFW (2004-5)



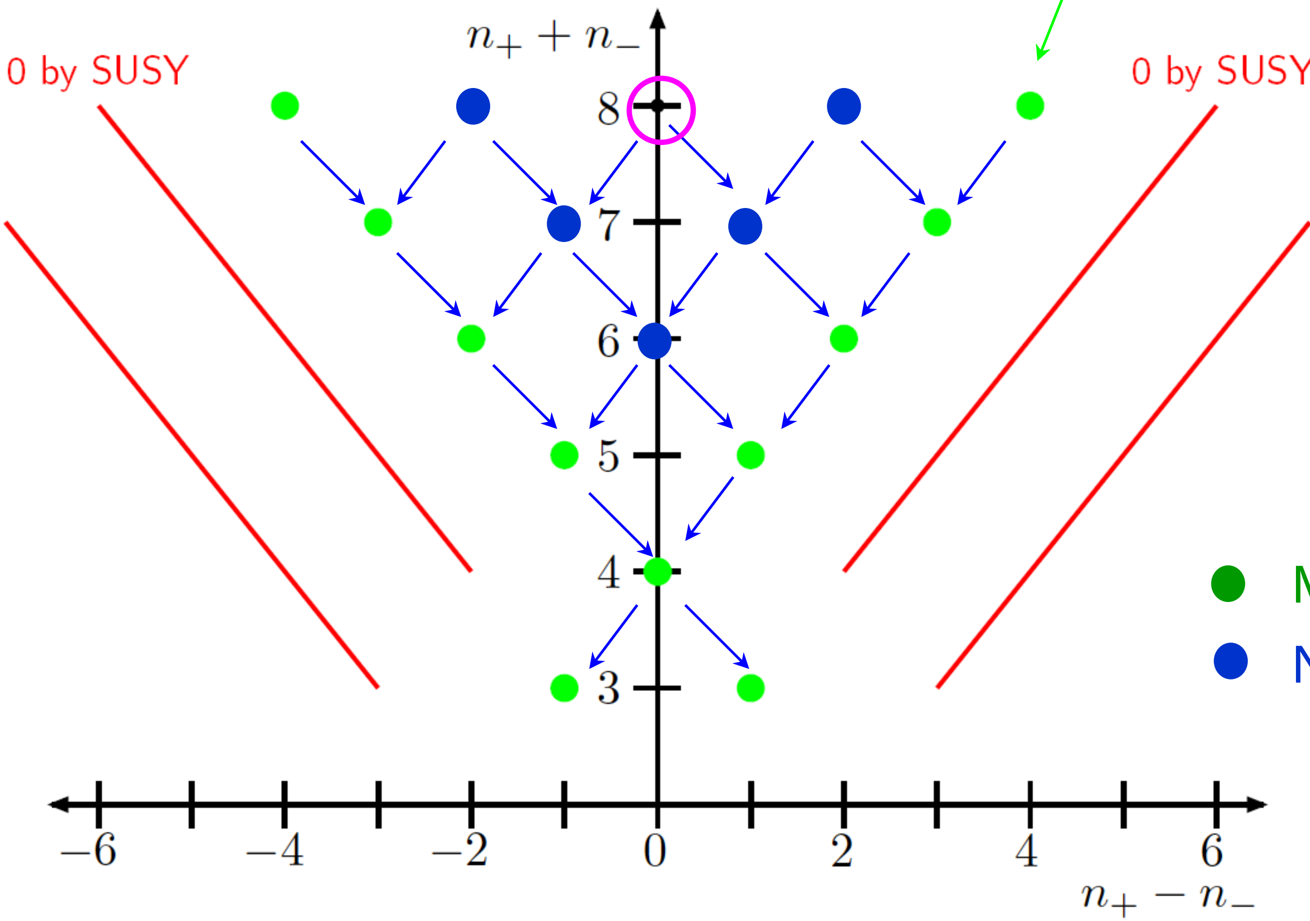
- A_{k+1} and A_{n-k+1} are **on-shell** tree amplitudes with **fewer** legs, and with momenta shifted by a **complex** amount
- NMHV covers every tree amplitude through 7 external massless legs: (- - - + + + +).
- For NLO W + 4 jets, encounter up to 9-point amplitudes ($W \rightarrow l\nu$) → at most 2 recursions needed BlackHat

Improving initial data

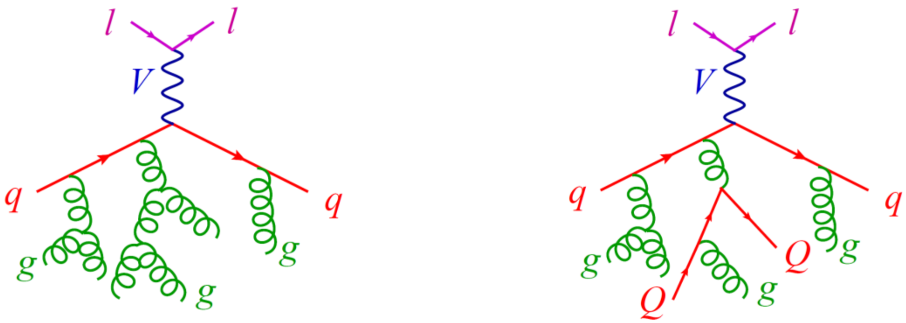


$$= \frac{\langle i\, j\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

Parke-Taylor formula

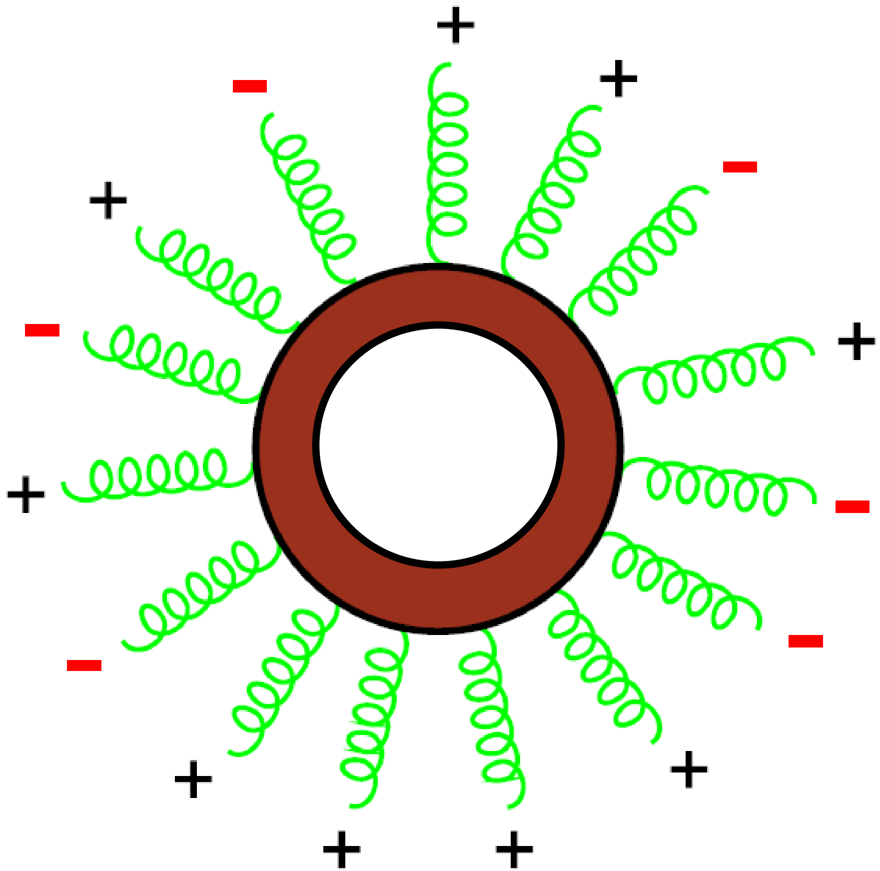


Fast trees from N=4 SYM(?)



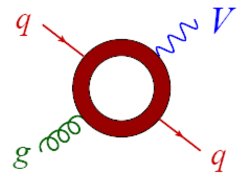
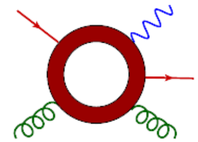
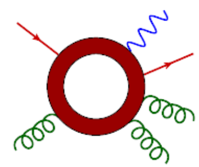
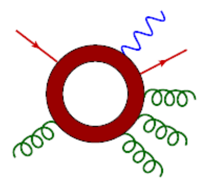
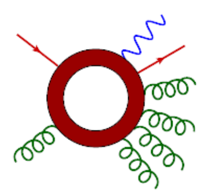
• Within Blackhat (V + jets), N=4 SYM NMHV trees speed up numerical implementation of on-shell (BCFW) recursion relations significantly (~ factor of 4 in sum over all helicities).
• However, there are a lot of fast – and very versatile – LO matrix element generators based on
  - Feynman diagrams (e.g. MadGraph)
  - off-shell recursion relations (e.g. Berends-Giele) or Dyson-Schwinger-type relations (e.g. ALPHA, HELAC).
• Future will tell whether N=4 SYM can help LO generators, for selected processes, such as V + jets…

One loop [NLO]



One-loop QCD amplitudes via Feynman diagrams

For ***V* + *n* jets** (maximum number of external gluons only)

# of jets	# 1-loop Feynman diagrams	
1		11
2		110
3		1,253
4		16,648
5		256,265

Motivates "on-shell" methods, which exploit unitarity to reduce loop amplitudes to products of tree amplitudes

NLO in pretty good shape right now

Les Houches Experimenters' Wish List 2010

process wanted at NLO	background to
1. $pp \to VV +$ jet	$t\bar{t}H$, new physics Dittmaier, Kallweit, Uwer; Campbell, Ellis, Zanderighi
2. $pp \to H + 2$ jets	H in VBF BCDEGMRSW; Campbell, Ellis, Williams Campbell, Ellis, Zanderighi; Ciccolini, Denner Dittmaier
3. $pp \to t\bar{t}b\bar{b}$	$t\bar{t}H$ Bredenstein, Denner Dittmaier, Pozzorini; Bevilacqua, Czakon, Papadopoulos, Pittau, Worek
4. $pp \to t\bar{t} + 2$ jets	$t\bar{t}H$ Bevilacqua, Czakon, Papadopoulos, Worek
5. $pp \to VVb\bar{b}$	VBF $\to H \to VV$, $t\bar{t}H$, new physics
6. $pp \to VV + 2$ jets	VBF $\to H \to VV$ VBF: Bozzi, Jäger, Oleari, Zeppenfeld
7. $pp \to V + 3$ jets	new physics Berger Bern, Dixon, Febres Cordero, Forde, Gleisberg, Ita, Kosower, Maitre; Ellis, Melnikov, Zanderighi
8. $pp \to VVV$	SUSY trilepton Lazopoulos, Melnikov, Petriello; Hankele, Zeppenfeld; Binoth, Ossola, Papadopoulos, Pittau
9. $pp \to b\bar{b}b\bar{b}$	Higgs, new physics GOLEM

Feynman diagram methods

now joined by

on-shell methods

table courtesy of C. Berger

Some On-Shell Programs for NLO QCD

CutTools: Ossola, Papadopolous, Pittau, 0711.3596

NLO *WWW, WWZ*, ... Binoth+OPP, 0804.0350

NLO $t\bar{t}b\bar{b}$, $t\bar{t}$ + 2 jets

Bevilacqua, Czakon, Papadopoulos, Pittau, Worek, 0907.4723; 1002.4009

Rocket: Giele, Zanderighi, 0805.2152

Ellis, Giele, Kunszt, Melnikov, Zanderighi, 0810.2762

NLO *W* + 3 jets (large N_c adjustment) W^+W^+ + 2 jets

EMZ, 0901.4101, 0906.1445; Melnikov, Zanderighi, 0910.3671;
Melia, Melnikov, Rontsch, Zanderighi, 1007.5313

SAMURAI: Mastrolia, Ossola, Reiter, Tramontano, 1006.0710

Blackhat: Berger, Bern, LD, Febres Cordero, Forde, H. Ita,
D. Kosower, D. Maître; T. Gleisberg, 0803.4180, 0808.0941,
0907.1984, 0912.4927, 1004.1659, 1009.2338

+ **Sherpa** → NLO production of *W,Z* + 3 or 4 jets

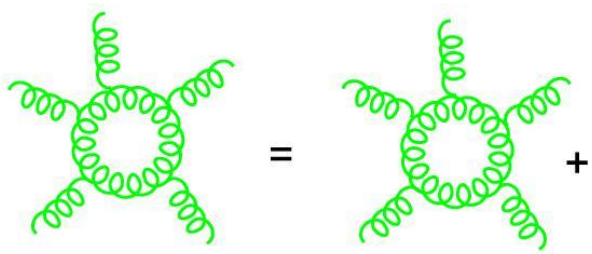
Another indicator of NLO progress

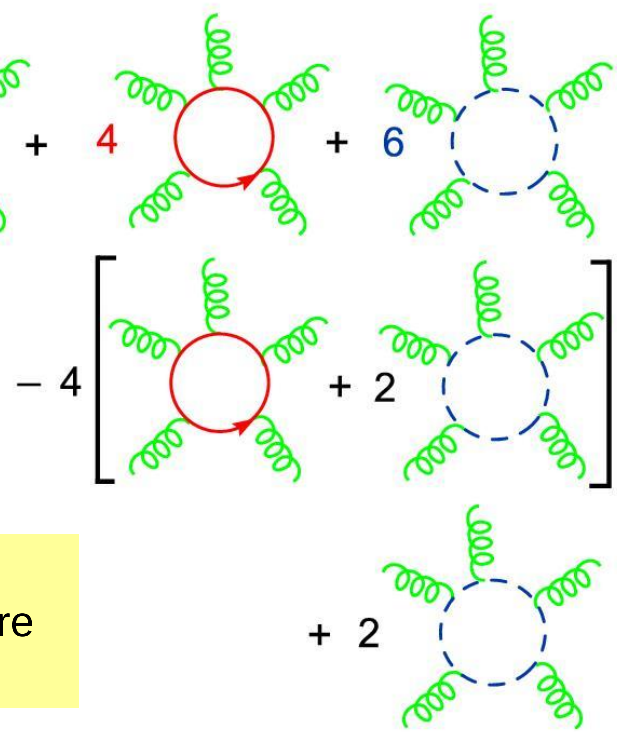
pp → W + 0 jet	1978	Altarelli, Ellis, Martinelli
pp → W + 1 jet	1989	Arnold, Ellis, Reno
pp → W + 2 jets	2002	Campbell, Ellis
pp → W + 3 jets	2009	Berger et al.; Ellis et al.
pp → W + 4 jets	2010	Berger et al.

At loop-level, QCD and N=4 SYM differ

In the early days of multi-parton loop amplitudes (1990s), it was profitable to rearrange the QCD computation to exploit supersymmetry

gluon loop





N=4 SYM

contains most of the $1/\varepsilon$ poles

N=1 multiplet

In modern on-shell methods, this decomposition is not used much any more (gluon loop no harder than scalar loop)

scalar loop – no SUSY, but also no spin tangles in Feynman diagrams

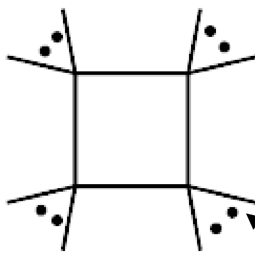
One-loop amplitudes reduced to trees

When all external momenta are in $D = 4$, loop momenta in $D = 4\text{-}2\varepsilon$ (dimensional regularization), one can write (inspired by N=4 SYM):

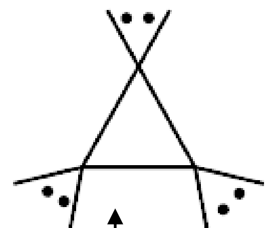
Bern, LD, Dunbar, Kosower (1994)

coefficients are all rational functions – determine algebraically from products of trees using (generalized) unitarity

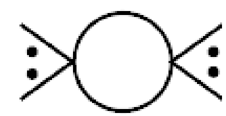
$$A^{1\text{-loop}} = \sum_i d_i$$



$$+ \sum_i c_i$$



$$+ \sum_i b_i$$

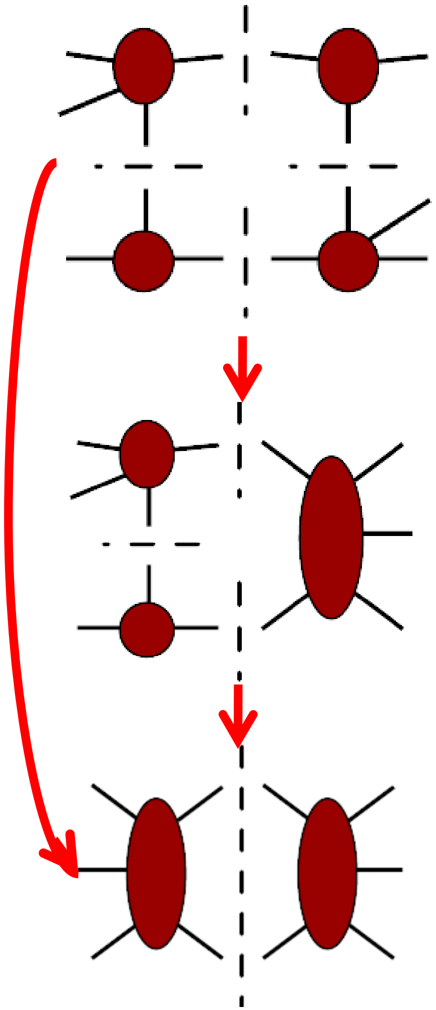


$$+ \; R \; + \mathcal{O}(\epsilon)$$

rational part

well-known **scalar** one-loop integrals, master integrals, same for all amplitudes

Unitarity method (numerical version)



Each box coefficient uniquely isolated by a "quadruple cut" given simply by a product of 4 tree amplitudes

Britto, Cachazo, Feng, hep-th/0412103

Ossola, Papadopolous, Pittau, hep-ph/0609007;
Mastrolia, hep-th/0611091; Forde, 0704.1835;
Ellis, Giele, Kunszt, 0708.2398; Berger et al., 0803.4180;…

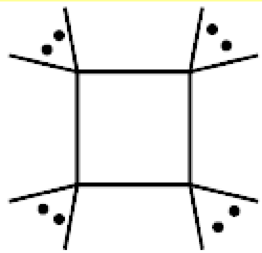
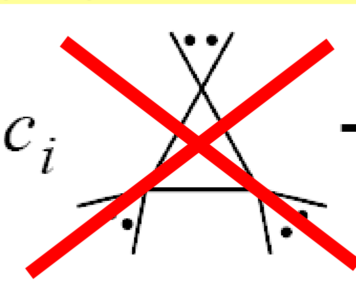
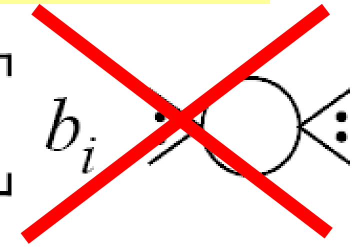
triangle coefficients come from triple cuts, product of 3 tree amplitudes, but these are also "contaminated" by boxes

bubble coefficients come from ordinary double cuts, after removing contributions of boxes and triangles

Generalized unitarity for N=4 SYM

One-loop N=4 amplitudes contain only boxes, due to SUSY cancellations of loop momenta in numerator:
Bern, LD, Dunbar, Kosower (1994)

$$(\ell^\mu)^n \;\Rightarrow\; (\ell^\mu)^{n-4}$$

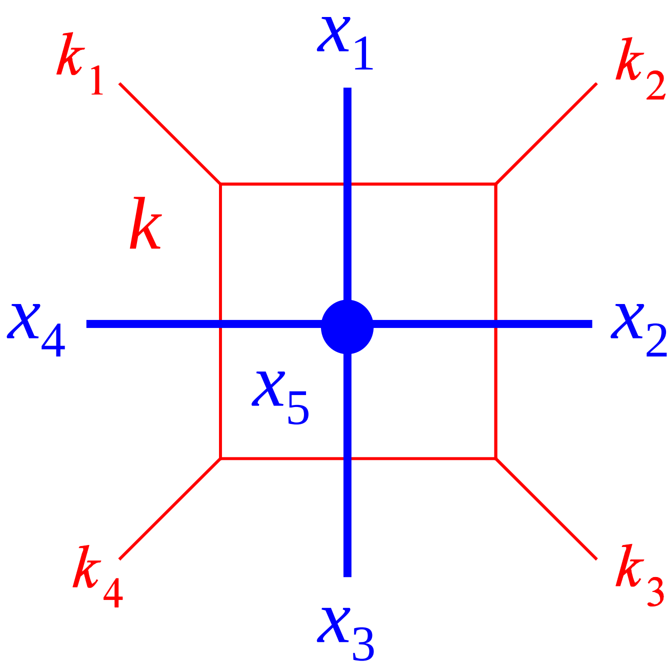
$$A^{1\text{-loop}} = \sum_i d_i$$

$$+ \sum_i c_i$$

$$+ \sum_i b_i$$

$$+ \cancel{R} + \mathcal{O}(\epsilon)$$

L-loop generalization of this property recently conjectured for N=4:
All amplitudes determined by “leading-singularities” – imposing 4*L* cuts on the *L* loop momenta in D=4
Cachazo, Skinner, 0801.4574; Arkani-Hamed, Cachazo, Kaplan, 0808.1446

Dual Conformal Invariance

Broadhurst (1993); Lipatov (1999); Drummond, Henn, Smirnov, Sokatchev, hep-th/0607160

A conformal symmetry acting in momentum space, on dual (sector) variables x_i
First seen in N=4 SYM planar amplitudes within loop integrals
Conformal group can be generated by usual Lorentz group, plus inversion



$$I = \int d^4k \frac{(k_1+k_2)^2(k_2+k_3)^2}{k^2(k-k_1)^2(k-k_1-k_2)^2(k+k_4)^2}$$

$$I = \int d^4x_5 \frac{x_{24}^2 x_{13}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

$$k_1 = x_{41}$$
$$k_2 = x_{12}$$
$$k_3 = x_{23}$$
$$k_4 = x_{34}$$
$$k = x_{45}$$

invariant under inversion:

$$x_i^\mu \to \frac{x_i^\mu}{x_i^2}$$

$$x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}, \qquad \mathrm{d}^4x_i \to \frac{\mathrm{d}^4x_i}{x_i^8}$$

Dual Conformal Invariance (cont.)

- A powerful constraint on integrands of amplitudes

Bern, Czakon, LD, Kosower, Roiban, Smirnov;
Bern, Carrasco, Johansson, Kosower (2006-7)

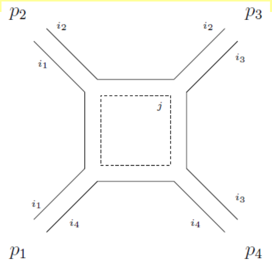
- Apparently only holds for planar contributions (those that dominate in the limit of large N_c).
- Does not hold for N < 4 planar theories (violated by triangle terms in QCD starting at one loop).
- Requires D=4. This means it is violated by IR divergences if regulated by dimensional regularization (D=4-2ε).
- If we are going to make use of it, have to somehow sidestep these difficulties.
- Start with the last one.

Higgs Regulator

- In planar limit, it's possible to regulate non-Abelian gauge amplitudes by spontaneously breaking gauge invariance, but without changing the IR content of the theory

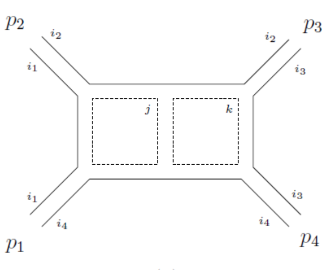
Alday, Henn, Plefka, Schuster, 0908.0684

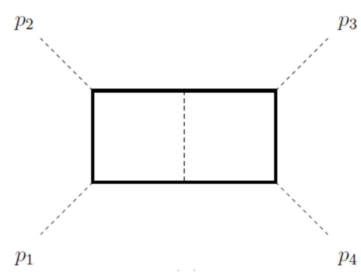
- Mass ~ momentum in 5th dimension.
- Only "outer" propagators get a mass
- Dual conformal symmetry can be retained.



$$I^{(1)}(s,t,m_i) = c_0 \int d^4x_5 \frac{(x_{13}^2 + (m_1-m_3)^2)(x_{24}^2 + (m_2-m_4)^2)}{(x_{15}^2+m_1^2)(x_{25}^2+m_2^2)(x_{35}^2+m_3^2)(x_{45}^2+m_4^2)}$$

Planar 2-loop example:





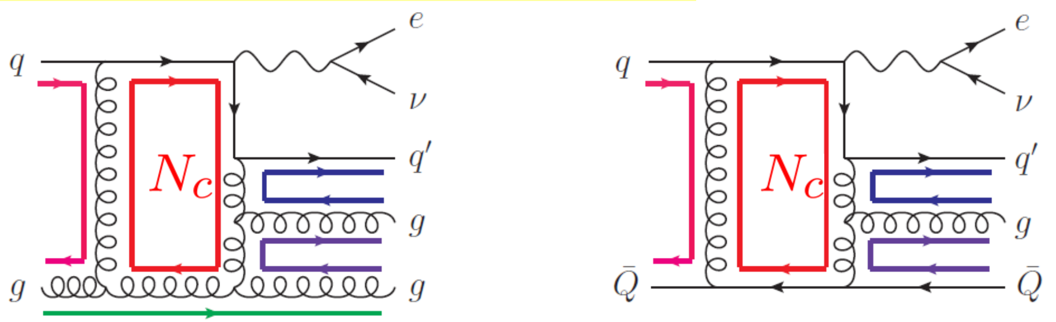
Higgs Regulator for QCD?

- All one-loop graphs are really planar graphs.
- Can we use a Higgs regulator for QCD, independent of dual conformal invariance (which inspired it)?
- $1/\varepsilon \leftarrow\rightarrow \ln(m^2)$
- Keep only log terms as $m^2 \rightarrow 0$.
- Advantage of Higgs regulator over dimensional regularization:
  - At one loop: No need for D-dimensional unitarity
  - At two loops and beyond: No need to expand lower-loop amplitudes to higher order in ε. No such thing as $1/\ln(m^2)$. (Like QED with electron and photon mass.)

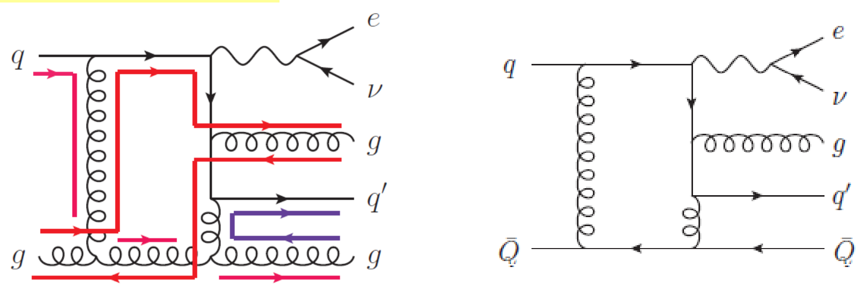
- Phase space integrals for real radiation should be done this way too. Can this be done consistently at one loop without subtraction terms?

Subleading-color terms small at one loop → neglect at two loops?

Leading-color terms in pp → W + 3 jets at NLO:

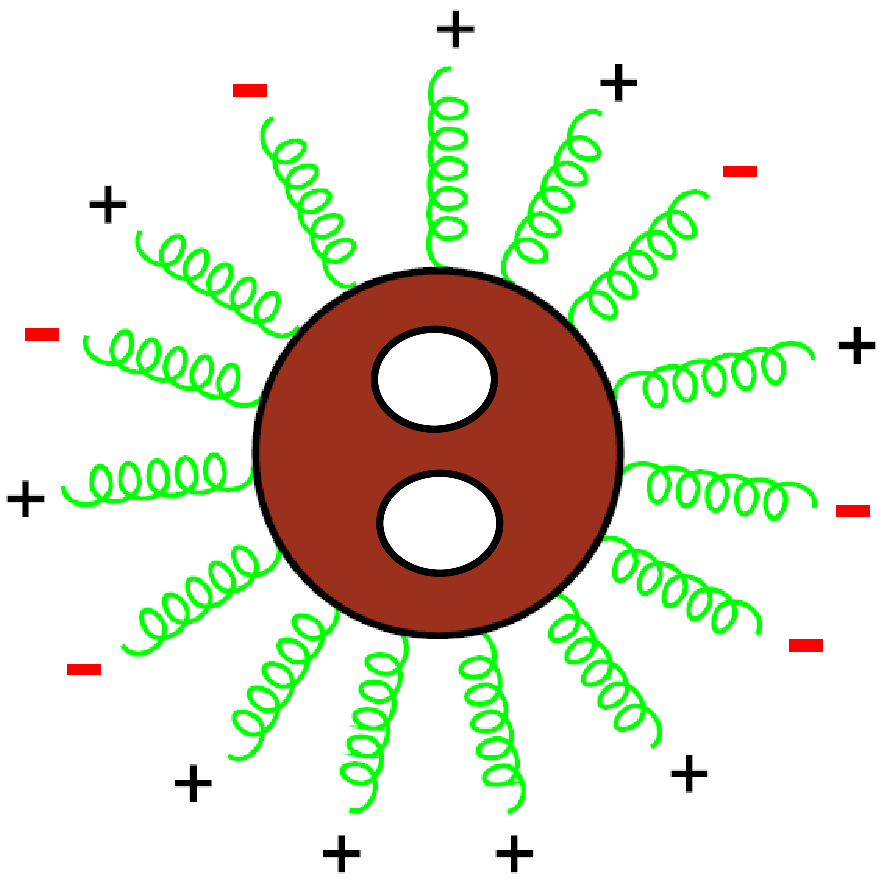


Subleading-color terms:



Latter include many more terms, and are much more time-consuming for computer to evaluate. But they are much smaller, suppressed by $1/N_c^2$ (~ 1/30 of total cross section for W + 3 jets). Drop altogether in some cases…

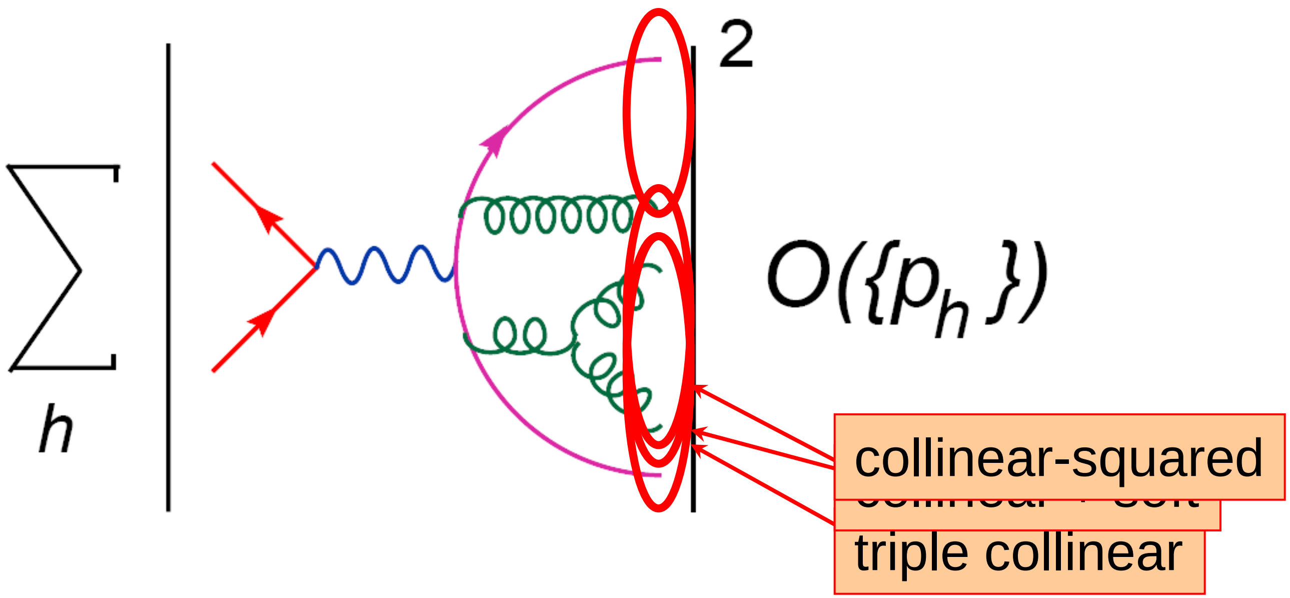
Two loops [NNLO]



NNLO bottlenecks

- Current NNLO bottleneck is doing phase space integrations of double-real emissions, not availability of two-loop amplitudes.
- Worked out already for $e^+e^- \to$ 3 jets

Gehrmann-DeRidder, Gehrmann, Glover, Heinrich, 2006-9; Weinzierl (2008)



Corresponding problems for LHC, NNLO $pp \to V + 1$ jet and $pp \to 2$ jets are a bit more general (ISR as well as FSR) and are being attacked right now.

Two loop amplitudes

- Those needed for $e^+e^- \to$ 3 jets, $pp \to V$ + jet, $pp \to jj$

were all computed around 2001-3

Anastasiou et al., Bern et al., Gehrmann et al.

- In one case (Bern et al.) generalized unitarity was used, in other cases Feynman diagrams.
- Main issues: reducing loop integrands to basis of master integrals, and evaluating the integrals Smirnov, Veretin; Tausk; Anastasiou et al., Gehrmann, Remiddi
- Recent steps taken toward a general basis for two-loop planar integrals Gluza, Kajda, Kosower, 1009.0472
- If integrals can also be evaluated, should make it possible to extend on-shell methods to general processes at two loops
- Also much recent work on two-loop integrals and amplitudes for processes with more massive lines, e.g. for NNLO top-quark production Czakon et al.

"N=4" ideas for evaluating integrals

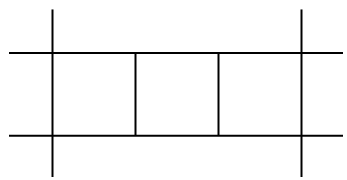
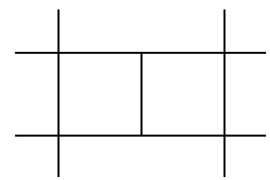
Drummond, Henn, Trnka (to appear):

- Using dual (sector) variables, find second order differential equations for L-loop integrals with (L-1)-loop integrals as source terms.

- Basic idea: use Laplace equation [Drummond, J.H., Smirnov, Sokatchev 2006]

$$\Box_1 \frac{1}{(x_1 - x_i)^2} \propto \delta^{(4)}(x_1^\mu - x_i^\mu)$$

- reduces loop order by one:

$\Box_1$  $\propto$ 

- Momentum twistors Z seem to work even better (do not require off-shell legs next to loop to be removed).

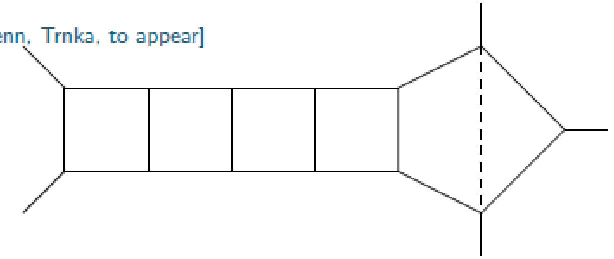
$$p_b^\mu = \sigma^\mu_{\alpha\dot\alpha}\, \lambda_b^\alpha\, \tilde\lambda_b^{\dot\alpha}$$

$$\mathcal{Z}^D = \begin{pmatrix} \lambda \\ \mu \\ \eta \end{pmatrix}$$

$$\tilde\lambda_b = \frac{\langle b+1\, b\rangle \mu_{b-1} + \langle b-1\, b+1\rangle \mu_b + \langle b\, b-1\rangle \mu_{b+1}}{\langle b-1\, b\rangle \langle b\, b+1\rangle}$$

“N=4” ideas for evaluating integrals (cont.)

- five-point example: [Drummond, Henn, Trnka, to appear]
  integral regulated by m^2



- differential equations (variables $y_1 = x_{35}^2/x_{13}^2, y_2 = x_{25}^2/x_{24}^2, y_3 = x_{14}^2/m^2$)

$$y_1\partial_{y_1}\, y_2\partial_{y_2}\, \Psi^{(L)}(y_1,y_2,y_3) = \Psi^{(L-1)}(y_1,y_2,y_3) + \mathcal{O}(m^2)$$

- Solution

$$\Psi^{(1)} = -\frac{1}{2}\log^2(y_1y_2y_3) - 2\,\mathrm{Li}_2(1-y_1) - 2\,\mathrm{Li}_2(1-y_2) + \frac{\pi^2}{6} + \mathcal{O}(m^2)$$

$$\Psi^{(2)} = h(y_1y_2y_3) - 2\log(y_1y_2y_3)[f(y_1)+f(y_2)] + [g(y_1)+g(y_2)] + \mathcal{O}(m^2)$$

$$h(x) = -\frac{7}{40}\pi^4 + 2\zeta_3\log(x) - \frac{1}{4}\pi^2\log^2(x) - \frac{1}{24}\log^4(x)\,,$$

$$g(x) = -\frac{2}{3}\pi^2 H_2(x) + 2H_0(x)H_{2,0}(x) - 4H_{3,0}(x) - 2H_{2,0,0}(x) - 4H_{2,1,0}(x)\,,$$

$$f(x) = H_{0,1,0}(x)\,.$$

First examples have been planar, but that is probably not necessary

Conclusions

- The future of (N)NLO QCD may well benefit from studying properties of amplitudes in N=4 SYM – as it has in the past, both at tree level and one loop.
- This benefit could be technical as well as conceptual
- However, as in the past, the key is to abstract important principles from the most useful and versatile of the properties being uncovered.

Whether or not this is achieved, the future of higher order computation in QCD is very bright!

Extra Slides

Dual variables and strong coupling

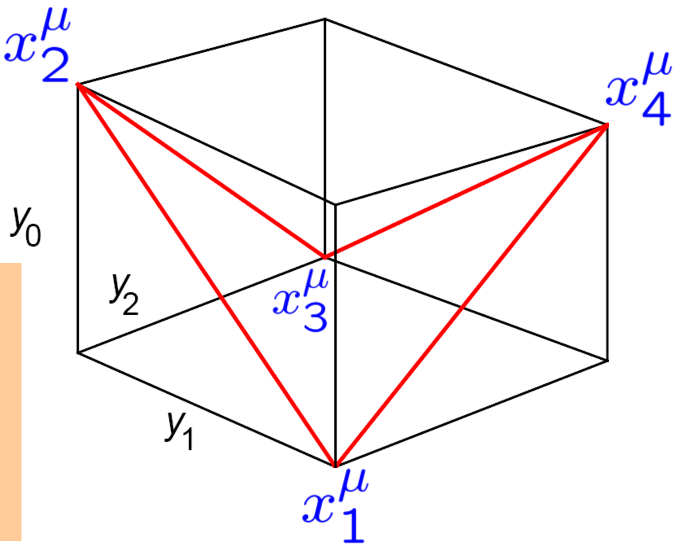
• T-dual momentum variables y^μ introduced by Alday, Maldacena

• Boundary values for world-sheet are light-like segments in y^μ :

$$\Delta y^\mu = 2\pi k^\mu$$

for gluon with momentum k^μ

• For example, for *gg* → *gg* 90-degree scattering, *s = t = -u/2*, the boundary looks like:



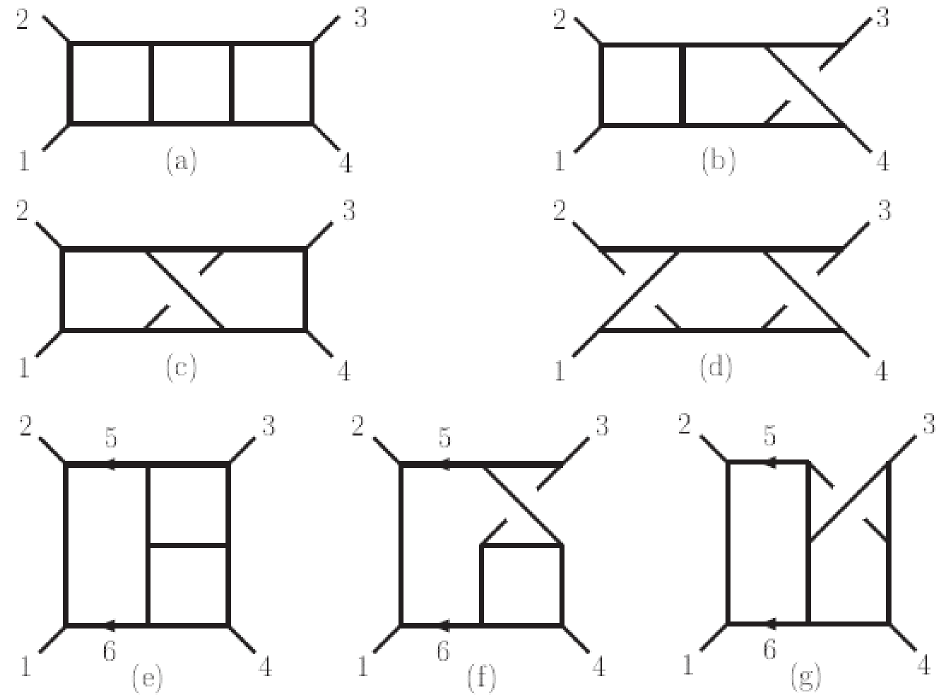
Corners (cusps) are located at x_i^μ – same dual momentum variables appear at weak coupling (in planar theory)

N=4 numerators at 3 loops

Omit overall $st\,A_4^{\text{tree}}$

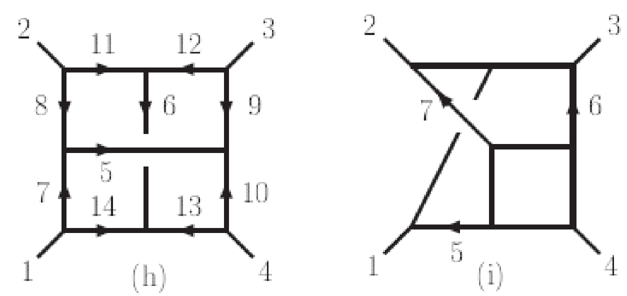
$s_{iM} = (k_i + \ell_M)^2$
$\tau_{iM} = 2k_i \cdot \ell_M$

s_{12}^2



$s_{12}\, s_{46}$

$$s_{12}(\tau_{26} + \tau_{36}) + s_{14}(\tau_{15} + \tau_{25}) + s_{12}s_{14}$$



$$- s_{12}\, s_{45} - s_{14}\, s_{46} - \frac{1}{3}(s_{12} - s_{14})\,\ell_7^2$$

manifestly quadratic in loop momentum ℓ_M

Dual conformal invariance at 4 loops

- Simple graphical rules:

4 (net) lines into inner x_i

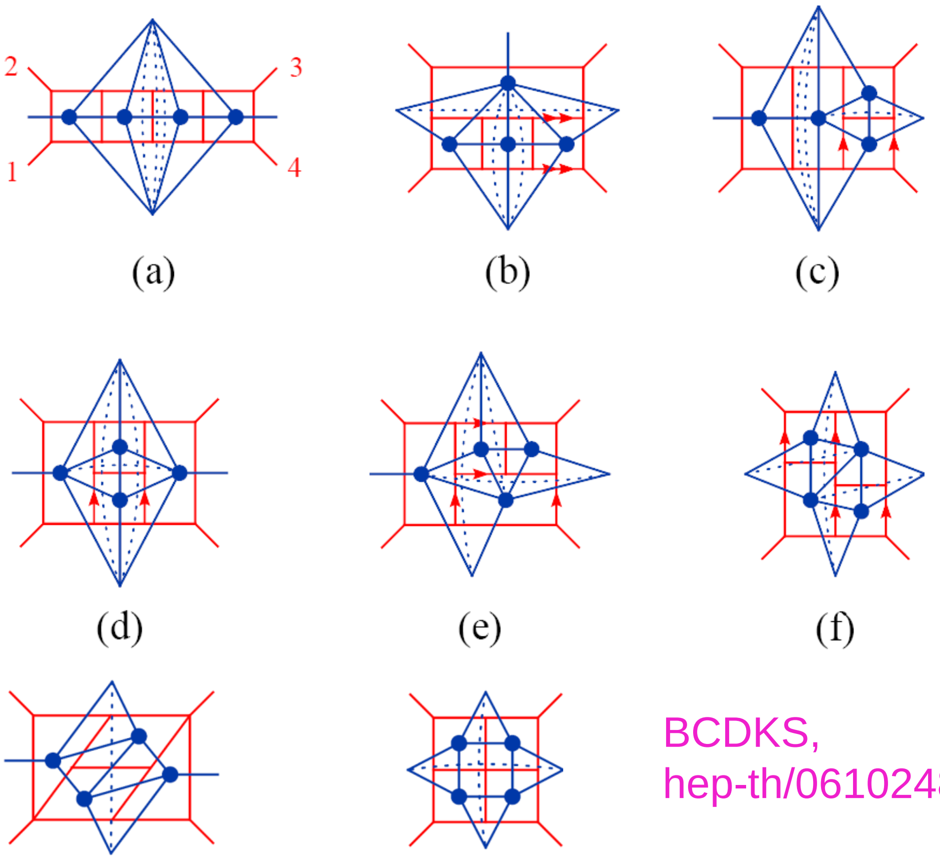
1 (net) line into outer x_i

- Dotted lines are for numerator factors

4 loop planar integrals all of this form

also true at 5 loops

BCJK, 0705.1864



(a) (b) (c)

(d) (e) (f)

(d_2) (f_2)

BCDKS, hep-th/0610248