NLO tensor reductions, where we are

Jochem Fleischer

Univ. Bielefeld
and DESY, Zeuthen

in collaboration with Tord Riemann and Valery Yundin

Outline

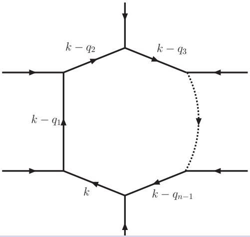
Introduction

n-point tensor integrals of rank R: (n,R)-integrals

$$I_n^{\mu_1\cdots\mu_R} = \int \frac{d^dk}{i\pi^{d/2}} \frac{\prod_{r=1}^{R} k^{\mu_r}}{\prod_{j=1}^{n} c_j^{\nu_j}}, \tag{1}$$

$d = 4 - 2\epsilon$ and denominators c_j have *indices* ν_j and *chords* q_j

$$c_j = (k - q_j)^2 - m_j^2 + i\varepsilon \tag{2}$$



tensor integrals due to, e.g.:

- fermion propagators
- three-gauge boson couplings

Scalar Integrals

- package for all $n \leq 4$ scalar integrals: QCDloop [Ellis:2007 [1]]
- Recent results for scalar integrals (complex masses):
- package for all $n \leq 4$ scalar integrals:ONELOop [A.van Hameren:2010 [2]]
- Analytic results
- Scalar one-loop 4-point integrals: 1-loop 4pt. [A.Denner and S.Dittmaier:2010 [3]]

Status of opensource packages - possibly not complete

- package FF [vanOldenborgh:1990 [4]] ,
- package LoopTools/FF [Hahn:1998,2006 [5]] – covers also 5-point functions, rank $R \leq 4$ $1/\epsilon^2$ not covered, and we observed sometimes problems in certain configurations with light-like external particles
- package Golem95 [Binoth:2008 [6]] for $n \leq 6$, massive propagators in the test phase
- Mathematica package hexagon.m [Diakonidis:2008 [7, 8]] for $n \leq 6$, $rankR \leq 4$
- package for all $n \leq 4$ scalar integrals: QCDloop [Ellis:2007 [1]]
- OPP methods: CutTools and Samurai

Other contributions

Crucial contributions [of course, list is incomplete . . .] ⇒

- [Campbell:1996 [9]]
- [Denner:2002,2005 [10, 11]]
- [Binoth:1999,2005 [12, 13]]
- [Bern:1993 [14]]
- [Ossola:2006 [15]]
- [The SM and NLO Multileg Working Group: Summary report.[16]]

In the following, I will describe recent developments based on

- [Davydychev:1991,Tarasov:1996,Fleischer:1999,Diakonidis:2008,2009 [17, 18, 19, 8, 20, 21]] present work: JF and T.Riemann, arXiv:1009.4436

In view of the importance of stable numerics for tensor reductions, it would be welcome to have one or more complete opensource programs for this task. To our knowledge, none is presently available. It is our aim to provide one.

Tensors expressed in terms of integrals in higher dimension

Following [Davydychev:1991 [17]] ,also [J.F. et al.:2000 [19]] express tensors by means of scalar integrals in higher dimensions($n_{ij} = \nu_{ij} = 1 + \delta_{ij}, n_{ijk} = \nu_{ij}\nu_{ijk}, \nu_{ijk} = 1 + \delta_{ik} + \delta_{jk}$ etc.):

$$I_n^{\mu} = \int^d k^{\mu} \prod_{r=1}^{n} c_r^{-1} = -\sum_{i=1}^{n} q_i^{\mu} \, I_{n,i}^{[d+]} \tag{3}$$

$$I_n^{\mu\,\nu} = \int^d k^{\mu} \, k^{\nu} \prod_{r=1}^{n} c_r^{-1} = \sum_{i,j=1}^{n} q_i^{\mu} \, q_j^{\nu} \, n_{ij} \, I_{n,ij}^{[d+]^2} - \frac{1}{2} g^{\mu\nu} \, I_n^{[d+]} \tag{4}$$

$$I_n^{\mu\,\nu\,\lambda} = \int^d k^{\mu} \, k^{\nu} \, k^{\lambda} \prod_{r=1}^{n} c_r^{-1} = -\sum_{i,j,k=1}^{n} q_i^{\mu} \, q_j^{\nu} \, q_k^{\lambda} \, n_{ijk} \, I_{n,ijk}^{[d+]^3} + \frac{1}{2} \sum_{i=1}^{n} g^{[\mu\nu} q_i^{\lambda]} I_{n,i}^{[d+]^2} \tag{5}$$

$$I_n^{\mu\,\nu\,\lambda\,\rho} = \int^d k^{\mu} \, k^{\nu} \, k^{\lambda} \, k^{\rho} \prod_{r=1}^{n} c_r^{-1} = \sum_{i,j,k,l=1}^{n} q_i^{\mu} \, q_j^{\nu} \, q_k^{\lambda} \, q_l^{\rho} \, n_{ijkl} \, I_{n,ijkl}^{[d+]^4}$$

$$-\frac{1}{2} \sum_{i,j=1}^{n} g^{[\mu\nu} q_i^{\lambda} q_j^{\rho]} \, n_{ij} I_{n,ij}^{[d+]^3} + \frac{1}{4} g^{[\mu\nu} g^{\lambda\rho]} I_n^{[d+]^2} \tag{6}$$

Notations: Integrals

$$I_{p,\,i\,j\,k\cdots}^{[d+]^l,stu\cdots} = \int^{[d+]^l} \prod_{r=1}^{n} \frac{1}{c_r^{1+\delta_{ri}+\delta_{rj}+\delta_{rk}+\cdots-\delta_{rs}-\delta_{rt}-\delta_{ru}-\cdots}}, \int^d \equiv \int \frac{d^d k}{\pi^{d/2}},$$

where $[d+]^l = 4 + 2l - 2\varepsilon$

$$I_{n-1,ab}^{\{\mu_1,\cdots\},s}$$

is obtained from

$$I_n^{\{\mu_1,\cdots\}}$$

by

- shrinking line s
- raising the powers of inverse propagators a, b.

Notations: modified Cayley determinant [Melrose:1965]

Modified Cayley determinant $()_N$ of a diagram with N internal lines and chords q_j:

$$()_N \equiv \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & Y_{11} & Y_{12} & \ldots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \ldots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \ldots & Y_{NN} \end{vmatrix}, \tag{7}$$

with matrix elements

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i,j = 1 \ldots N) \tag{8}$$

Gram determinant G_n: $G_n = |2q_iq_j|, i,j = 1, \ldots n$

For a choice $q_n = 0$, both determinants are related: $()_N = -G_{N-1}$

$\Rightarrow$ The determinant $()_N$ does not depend on the masses.

Notations: signed minors [Melrose:1965]

We also need signed minors of $()_N$, constructed by deleting m rows and m columns from $()_N$, and multiplying with a sign factor:

$$\left(\begin{array}{cccc} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{array}\right)_N \equiv$$

$$\equiv (-1)^{\sum_l (j_l+k_l)} \mathrm{sgn}_{\{j\}} \, \mathrm{sgn}_{\{k\}} \left| \begin{array}{c} \text{rows } j_1 \cdots j_m \text{ deleted} \\ \text{columns } k_1 \cdots k_m \text{ deleted} \end{array} \right| \quad (9)$$

where $\mathrm{sgn}_{\{j\}}$ and $\mathrm{sgn}_{\{k\}}$ are the signs of permutations that sort the deleted rows $j_1 \cdots j_m$ and columns $k_1 \cdots k_m$ into ascending order.

Following [Tarasov:1996,Fleischer:1999 [18, 19]]:
apply recurrence relations relating scalar integrals of different dimensions in order to get rid of the high dimensions.

$$\nu_j \mathbf{j}^+ I_n^{(d+2)} = \frac{1}{()_n} \left[-\binom{j}{0}_5 + \sum_{k=1}^{n} \binom{j}{k}_n \mathbf{k}^- \right] I_n^d \qquad (10)$$

$$(d - \sum_{i=1}^{n} \nu_i + 1) I_n^{(d+2)} = \frac{1}{()_n} \left[\binom{0}{0}_n - \sum_{k=1}^{n} \binom{0}{k}_n \mathbf{k}^- \right] I_n^d, \qquad (11)$$

where the operators $\mathbf{i}^\pm, \mathbf{j}^\pm, \mathbf{k}^\pm$ act by shifting the indices ν_i, ν_j, ν_k by ± 1.

Alternative: Recursions for pentagons

Express any $(5, R)$ pentagon by a $(5, R-1)$ pentagon plus $(4, R-1)$ boxes [Fleischer et al., Diakonidis:2010 [22]]

$$I_5^{\mu_1\cdots\mu_{R-1}\mu} = I_5^{\mu_1\cdots\mu_{R-1}} Q_0^{\mu} - \sum_{s=1}^{5} I_4^{\mu_1\cdots\mu_{R-1},s} Q_s^{\mu} \qquad (12)$$

auxiliary vectors with inverse Gram determinants

$$Q_s^{\mu} = \sum_{i=1}^{5} q_i^{\mu} \frac{\binom{s}{i}_5}{()_5}, \quad s = 0, \ldots, 5 \qquad (13)$$

For e.g. $R = 3$, again $[1/()_5]^3$ will occur.

Algebraic simplifications, 1^{st} step

With the identity

$$\binom{0}{0}_5 \binom{s}{i}_5 = \binom{0s}{0i}_5 ()_5 + \binom{0}{i}_5 \binom{s}{0}_5 \tag{14}$$

we eliminate the inverse Gram determinant from Q_s^μ:

$$\binom{0}{0}_5 I_5^{\mu_1 \cdots \mu_{R-1} \mu} = \left[\binom{0}{0}_5 I_5^{\mu_1 \cdots \mu_{R-1}} - \sum_{s=1}^{5} \binom{s}{0}_5 I_4^{\mu_1 \cdots \mu_{R-1}, s} \right] Q_0^\mu - \sum_{s=1}^{5} I_4^{\mu_1 \cdots \mu_{R-1}, s} \overline{Q}_s^\mu \tag{15}$$

The auxiliary vectors are

$$Q_0^\mu = \sum_{i=1}^{5} q_i^\mu \frac{\binom{0}{i}_5}{()_5} \text{ and } \overline{Q}_s^\mu = \sum_{i=1}^{5} q_i^\mu \binom{0s}{0i}_5 \tag{16}$$

Start recursion analytically

Have to show for the product $T^{\mu_1\cdots\mu_{R-1}} \times Q_0^\mu$ that the Gram determinant cancels.

$$T^{\mu_1\cdots\mu_{R-1}} = \left[\binom{0}{0}_5 I_5^{\mu_1\cdots\mu_{R-1}} - \sum_{s=1}^{5} \binom{s}{0}_5 I_4^{\mu_1\cdots\mu_{R-1},s} \right] \qquad (17)$$

Vector is known:

$$I_5^\mu = \sum_{i=1}^{4} q_i^\mu E_i, \qquad (18)$$

$$E_i \equiv -I_{5,i}^{[d+]} = (d-4)\frac{\binom{0}{i}_5}{\binom{0}{0}_5} I_5^{[d+]} - \frac{1}{\binom{0}{0}_5}\sum_{s=1}^{5}\binom{0i}{0s}_5 I_4^s, \qquad (19)$$

Examples for the use of signed minors

$$\binom{s}{0}_5 \binom{0s}{is}_5 = \binom{s}{i}_5 \binom{0s}{0s}_5 - \binom{s}{s}_5 \binom{0s}{0i}_5 \tag{20}$$

and

$$\binom{s}{0}_5 \binom{ts}{is}_5 = \binom{s}{i}_5 \binom{ts}{0s}_5 - \binom{s}{s}_5 \binom{ts}{0i}_5 . \tag{21}$$

Here the $\binom{s}{s}_5$ term cancels and the remaining factor $\binom{ts}{0i}_5$ is antisymmetric in s, t, yielding a vanishing contribution after summation over s, t. Result:

$$T^{\mu,s} = \sum_{i=1}^{4} q_i^{\mu} T_i^s, \qquad T_i^s = -\binom{s}{i}_5 I_4^{[d+],s}, \tag{22}$$

Higher order tensors I

It is crucial to get the $T^s_{ij\ldots}$,i.e.

$$T^{\mu_1\mu_2\cdots} = \sum_{i,j\cdots=1}^{4} q_i^{\mu_1} q_j^{\mu_2} \cdots T^s_{ij\ldots}, \tag{23}$$

We get for the higher tensors and find a pattern !

$$T^s_{ij} = \binom{s}{i}_5 I^{[d+]^2,s}_{4,j} + \binom{s}{j}_5 I^{[d+]^2,s}_{4,i} \tag{24}$$

With

$$T_5^{\mu\,\nu\,\lambda,s} = \sum_{i,j,k=1}^{4} q_i^{\mu}\, q_j^{\nu}\, q_k^{\lambda}\, T^s_{ijk} + \sum_{i=1}^{4} g^{[\mu\nu} q_i^{\lambda]}\, T^s_{00i}, \tag{25}$$

Higher order tensors II

$$T^s_{00i} = \frac{1}{2}\binom{s}{i}_5 I_4^{[d+]^2,s} \tag{26}$$

$$T^s_{ijk} = -\left\{\binom{s}{i}_5 \nu_{jk} I_{4,jk}^{[d+]^3,s} + \binom{s}{j}_5 \nu_{ik} I_{4,ik}^{[d+]^3,s} + \binom{s}{k}_5 \nu_{ij} I_{4,ij}^{[d+]^3,s}\right\}. \tag{27}$$

$$T_5^{\mu\,\nu\,\lambda\rho} = \sum_{i,j,k,l=1}^{4} q_i^\mu q_j^\nu q_k^\lambda q_l^\rho T_{ijkl} + \sum_{i,j=1}^{4} g^{[\mu\nu} q_i^\lambda q_j^{\rho]} T_{00ij} + g^{[\mu\nu} g^{\lambda\rho]} T_{0000}, \tag{28}$$

$$T^s_{0000} = 0. \tag{29}$$

$$T^s_{00ij} = -\frac{1}{2}\left\{\binom{s}{i}_5 I_{4,j}^{[d+]^3,s} + \binom{s}{j}_5 I_{4,i}^{[d+]^3,s}\right\} \tag{30}$$

$$T^s_{ijkl} = \binom{s}{i}_5 n_{jkl} I_{4,jkl}^{[d+]^4,s} + \binom{s}{j}_5 n_{ikl} I_{4,ikl}^{[d+]^4,s} + \binom{s}{k}_5 n_{ijl} I_{4,ijl}^{[d+]^4,s} + \binom{s}{l}_5 n_{ijk} I_{4,ijk}^{[d+]^4,s} \tag{31}$$

Algebraic simplifications, 2nd step

The metric tensor

$$\binom{s}{i}_5 \frac{\binom{0}{j}_5}{()_5} = -\binom{0i}{sj}_5 + \binom{s}{0}_5 \frac{\binom{i}{j}_5}{()_5}, \qquad g^{\mu\nu} = 2\sum_{i,j=1}^{4} \frac{\binom{i}{j}_5}{()_5} q_i^\mu q_j^\nu \quad (32)$$

$$I_5^{\mu\,\nu\,\lambda} = \sum_{i,j,k=1}^{4} q_i^\mu q_j^\nu q_k^\lambda E_{ijk} + \sum_{k=1}^{4} g^{[\mu\nu} q_k^{\lambda]} E_{00k} \quad (33)$$

$$E_{ijk} = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \left\{ \left[\binom{0j}{sk}_5 I_{4,i}^{[d+]^2,s} + (i \leftrightarrow j) \right] + \binom{0s}{0k}_5 \nu_{ij} I_{4,ij}^{[d+]^2,s} \right\} \quad (34)$$

$$E_{00j} = \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \left[\frac{1}{2} \binom{0s}{0j}_5 I_4^{[d+],s} - \frac{d-1}{3} \binom{s}{j}_5 I_4^{[d+]^2,s} \right] \quad (35)$$

These presentations are evidently free of inverse G-dets.

Isolation of inverse sub-Gram det^s $()_4$ I

For scalar higher dimensional integrals

$$I_4^s, I_4^{[d+],s}, I_4^{[d+]^2,s}, \ldots \tag{36}$$

application of dimension-shifting recurrence relations produces powers of $1/\binom{s}{s}_5$ (see (11)):

Unwanted sub-Gram-determinants $\binom{s}{s}_5$.

We can, however, express in general:

$$I_4^{d,s} = \sum_{t=1}^{5} \frac{\binom{ts}{0s}_5}{\binom{0s}{0s}_5} I_3^{d,st}, \quad \text{for} \binom{s}{s}_5 = 0 \tag{37}$$

for any $d = [d+]^l$. Add corrections for small $\binom{s}{s}_5$!?

Isolation of inverse sub-Gram det^s $()_4$ II

Try to have inverse Gram determinants only in boxes $I_4^{[d+]^l}$, and hold the I_3, I_2, I_1 free of them. Introduce:

$$Z_4^{d,s} = \sum_{t=1}^{5} \frac{\binom{ts}{0s}_5}{\binom{0s}{0s}_5} I_3^{d,st}, \quad Z_{4,i}^{(d),s} = \sum_{t=1, t\neq i}^{5} \frac{\binom{ts}{0s}_5}{\binom{0s}{0s}_5} I_{3,i}^{(d),st} + \frac{\binom{is}{0s}_5}{\binom{0s}{0s}_5} I_4^{(d),s} \quad (38)$$

Special relation needed:

$$\binom{0s}{is}_5 \binom{ts}{0s}_5 - \binom{0s}{0s}_5 \binom{ts}{is}_5 = -\binom{s}{s}_5 \binom{0st}{0si}_5. \quad (39)$$

$$I_{4,i}^{[d+],s} = -\frac{\binom{0s}{is}_5}{\binom{0s}{0s}_5} \frac{\left[I_4^s - Z_4^s \right]}{\binom{s}{s}_5} + \frac{1}{\binom{0s}{0s}_5} \sum_{t=1}^{5} \binom{0st}{0si}_5 I_3^{st} \quad (40)$$

Reduction of higher tensors I

$$\nu_{ij} I_{4,ij}^{[d+]^2,s} = -\frac{\binom{0s}{js}_5}{\binom{s}{s}_5}\left[I_{4,i}^{[d+],s} - Z_{4,i}^{[d+],s}\right] + \frac{1}{\binom{0s}{0s}_5}\left[\binom{0si}{0sj}_5 I_4^{[d+],s} + \sum_{t=1,t\neq i}^{5}\binom{0st}{0sj}_5 I_{3,i}^{[d+],st}\right], \quad (41)$$

$$\begin{aligned}\nu_{ij}\nu_{ijk} I_{4,ijk}^{[d+]^3,s} = \quad & -\frac{\binom{0s}{ks}_5}{\binom{s}{s}_5}\nu_{ij}\left[I_{4,ij}^{[d+]^2,s} - Z_{4,ij}^{[d+]^2,s}\right] \\ & + \frac{1}{\binom{0s}{0s}_5}\left[\binom{0si}{0sk}_5 I_{4,j}^{[d+]^2,s} + \binom{0sj}{0sk}_5 I_{4,i}^{[d+]^2,s} + \sum_{t=1,t\neq i,j}^{5}\binom{0st}{0sk}_5 \nu_{ij} I_{3,ij}^{[d+]^2,st}\right] \quad (42)\end{aligned}$$

and

$$\begin{aligned}n_{ijkl} I_{4,ijkl}^{[d+]^4,s} \quad = \quad & -\frac{\binom{0s}{ls}_5}{\binom{s}{s}_5}\nu_{ij}\nu_{ijk}\left[I_{4,ijk}^{[d+]^3,s} - Z_{4,ijk}^{[d+]^3,s}\right] + \frac{1}{\binom{0s}{0s}_5}\left[\binom{0sk}{0sl}_5 \nu_{ij} I_{4,ij}^{[d+]^3,s} + \right. \\ & \left.\binom{0sj}{0sl}_5 \nu_{ik} I_{4,ik}^{[d+]^3,s} + \binom{0si}{0sl}_5 \nu_{jk} I_{4,jk}^{[d+]^3,s} + \sum_{t=1,t\neq i,j,k}^{5}\binom{0st}{0sl}_5 \nu_{ij}\nu_{ijk} I_{3,ijk}^{[d+]^3,st}\right]. \quad (43)\end{aligned}$$

Reduction of higher tensors II

For the tensor coefficient of rank 2 we obtain

$$\frac{\binom{0}{0}}{()}\left[I_{4,i}^{[d+]}-Z_{4,i}^{[d+]}\right]=$$
$$-(d-2)\left[\frac{\binom{0}{i}}{\binom{0}{0}}(d-1)I_4^{[d+]^2}-\frac{1}{\binom{0}{0}}\sum_{t=1}^{4}\binom{0t}{0i}I_3^{[d+],t}\right] \quad (44)$$

and from (41)

$$\nu_{ij}I_{4,ij}^{[d+]^2}=\frac{\binom{0}{i}}{\binom{0}{0}}\frac{\binom{0}{j}}{\binom{0}{0}}(d-2)(d-1)I_4^{[d+]^2}+\frac{\binom{0i}{0j}}{\binom{0}{0}}I_4^{[d+]}$$
$$-\frac{\binom{0}{j}}{\binom{0}{0}}\frac{d-2}{\binom{0}{0}}\sum_{t=1}^{4}\binom{0t}{0i}I_3^{[d+],t}+\frac{1}{\binom{0}{0}}\sum_{t=1}^{4}\binom{0t}{0j}I_{3,i}^{[d+],t} \quad (45)$$

Reduction of higher tensors III

Dropping the index s : $\binom{s}{s}_5 \to ()_4$ and $I_5^{[d+],s} \to I_4^{[d+]}$

A lengthy calculation yields

$$\frac{\binom{0}{0}}{()}\nu_{ij}\left[I_{4,ij}^{[d+]^2} - Z_{4,ij}^{[d+]^2}\right] = \frac{\binom{0}{i}}{\binom{0}{0}}\frac{\binom{0}{j}}{\binom{0}{0}}(d-1)d(d+1)I_4^{[d+]^3} + (d-1)\frac{1}{\binom{0}{0}}\binom{0i}{0j}I_4^{[d+]^2} - \frac{(d-1)d}{\binom{0}{0}}\frac{\binom{0}{j}}{\binom{0}{0}}\sum_{t=1}^{4}\binom{0t}{0i}I_3^{[d+]^2,t} + \frac{d-1}{\binom{0}{0}}\sum_{t=1}^{4}\binom{0t}{0j}I_{3,i}^{[d+]^2,t}. \qquad (46)$$

Inserting indexed Integrals $I_{4,j}^{[d+]^2}$ we finally have

$$\nu_{ij}\nu_{ijk}I_{4,ijk}^{[d+]^3} = -\frac{\binom{0}{i}}{\binom{0}{0}}\frac{\binom{0}{j}}{\binom{0}{0}}\frac{\binom{0}{k}}{\binom{0}{0}}(d-1)d(d+1)I_4^{[d+]^3} - \frac{\binom{0i}{0j}\binom{0}{k} + \binom{0i}{0k}\binom{0}{j} + \binom{0j}{0k}\binom{0}{i}}{\binom{0}{0}^2}(d-1)I_4^{[d+]^2}$$
$$+\frac{\binom{0}{j}}{\binom{0}{0}}\frac{\binom{0}{k}}{\binom{0}{0}}\frac{(d-1)d}{\binom{0}{0}}\sum_{t=1}^{4}\binom{0t}{0i}I_3^{[d+]^2,t} - \frac{\binom{0}{k}}{\binom{0}{0}}\frac{d-1}{\binom{0}{0}}\sum_{t=1}^{4}\binom{0t}{0j}I_{3,i}^{[d+]^2,t}$$
$$+\sum_{t=1}^{4}\frac{\binom{0i}{0k}\binom{0t}{0j} + \binom{0j}{0k}\binom{0t}{0i}}{\binom{0}{0}^2}I_3^{[d+],t} + \frac{1}{\binom{0}{0}}\sum_{t=1,t\neq i,j}^{4}\binom{0t}{0k}\nu_{ij}I_{3,ij}^{[d+]^2,t} \qquad (47)$$

Reduction of higher tensors IV

We find again a pattern !:

How can we get 47?

Replace in 45 $d \to d + 2$ and multiply with $-(d-1)\frac{\binom{0}{k}}{\binom{0}{0}}$

Factors $(d+i)$ increase by steps of 1

Then add the second part of (42).

Applying this pattern we get the next
higher tensor coefficient:

$$
\begin{aligned}
\nu_{ij}\nu_{ijk}\nu_{ijkl} I_{4,ijkl}^{[d+]^4} &= \frac{\binom{0}{i}}{\binom{0}{0}}\frac{\binom{0}{j}}{\binom{0}{0}}\frac{\binom{0}{k}}{\binom{0}{0}}\frac{\binom{0}{l}}{\binom{0}{0}} d(d+1)(d+2)(d+3) I_4^{[d+]^4} \\
&+ \frac{\binom{0i}{0j}\binom{0}{k}\binom{0}{l} + \binom{0i}{0k}\binom{0}{j}\binom{0}{l} + \binom{0j}{0k}\binom{0}{i}\binom{0}{l} + \binom{0i}{0l}\binom{0}{j}\binom{0}{k} + \binom{0j}{0l}\binom{0}{i}\binom{0}{k} + \binom{0k}{0l}\binom{0}{i}\binom{0}{j}}{\binom{0}{0}^3} d(d+1) I_4^{[d+]^3} \\
&+ \frac{\binom{0i}{0l}\binom{0j}{0k} + \binom{0j}{0l}\binom{0i}{0k} + \binom{0k}{0l}\binom{0i}{0j}}{\binom{0}{0}^2} I_4^{[d+]^2} \\
&- \frac{\binom{0}{j}}{\binom{0}{0}}\frac{\binom{0}{k}}{\binom{0}{0}}\frac{\binom{0}{l}}{\binom{0}{0}} \frac{d(d+1)(d+2)}{\binom{0}{0}} \sum_{t=1}^{4} \binom{0t}{0i} I_3^{[d+]^3,t} + \frac{\binom{0}{k}}{\binom{0}{0}}\frac{\binom{0}{l}}{\binom{0}{0}} \frac{d(d+1)}{\binom{0}{0}} \sum_{t=1}^{4} \binom{0t}{0j} I_{3,i}^{[d+]^3,t} \\
&- \frac{d}{\binom{0}{0}^3} \sum_{t=1}^{4} \left[\binom{0i}{0k}\binom{0t}{0j} + \binom{0j}{0k}\binom{0t}{0i} \right] \binom{0}{l} I_3^{[d+]^2,t} \\
&- \frac{d}{\binom{0}{0}^3} \sum_{t=1}^{4} \left[\binom{0j}{0l}\binom{0t}{0i}\binom{0}{k} + \binom{0i}{0l}\binom{0t}{0j}\binom{0}{k} + \binom{0k}{0l}\binom{0t}{0i}\binom{0}{j} \right] I_3^{[d+]^2,t} \\
&+ \frac{1}{\binom{0}{0}^2} \sum_{t=1}^{4} \left[\binom{0j}{0l}\binom{0t}{0k} I_{3,i}^{[d+]^2,t} + \binom{0i}{0l}\binom{0t}{0k} I_{3,j}^{[d+]^2,t} + \binom{0k}{0l}\binom{0t}{0j} I_{3,i}^{[d+]^2,t} \right] \\
&- \frac{\binom{0}{l}}{\binom{0}{0}} \frac{d}{\binom{0}{0}} \sum_{t=1}^{4} \binom{0t}{0k} \nu_{ij} I_{3,ij}^{[d+]^3,t} + \frac{1}{\binom{0}{0}} \sum_{t=1,t\neq i,j}^{4} \binom{0t}{0l} \nu_{ij}\nu_{ijk} I_{3,ijk}^{[d+]^3,t} .
\end{aligned} \tag{48}
$$

Corrections for small Gram determinants I

$$Z_4^{[d+]^l} = \frac{1}{\binom{0}{0}} \sum_{t=1}^{4} \binom{t}{0} I_3^{[d+]^l,t}. \tag{49}$$

Recursion and 1 Iteration:

$$I_4^{[d+]^l} = Z_4^{[d+]^l} + \frac{()}{\binom{0}{0}} \left[(2l+1) - 2\varepsilon\right] I_4^{[d+]^{(l+1)}} \tag{50}$$

$$= Z_4^{[d+]^l} + \frac{()}{\binom{0}{0}} \left[(2l+1) - 2\varepsilon\right] \left\{ Z_4^{[d+]^{l+1}} + \frac{()}{\binom{0}{0}} \left[(2l+3) - 2\varepsilon\right] I_4^{[d+]^{(l+2)}} \right\} \tag{51}$$

$I_4^{[d+]^l} = F_4^{[d+]^l} + D_4^{[d+]^l} \frac{1}{\varepsilon}$ and $Z4d^l$= finite part of $Z_4^{[d+]^l}$

Corrections for small Gram determinants II

$$F_4^{[d+]^l} = Z4d^l + \frac{()}{\binom{0}{0}} \left[(2l+1) F_4^{[d+]^{(l+1)}} - 2\ D_4^{[d+]^{(l+1)}} \right]. \quad (52)$$

$$\delta Z4d_i^l = \frac{()}{\binom{0}{0}} \left[(2l+1) Z4d_i^{(l+1)} - 2\ D_4^{[d+]^{(l+1)}} \right] \quad i = 0, 1, 2 \cdots (53)$$

$$Z4d_i^l = Z4d^l + \delta Z4d_{(i-1)}^l, \quad i = 1, 2, \cdots \quad . \quad (54)$$

Start: $F_4^{[d+]^{(l+1)}} = Z4d_0^{(l+1)} = Z4d^l_{|()_4=0} \quad l = 1, \ldots, l_{max} + 1.$

Corrections for small Gram determinants III

Technical detail: iteration also for the divergent part

$$DivZ4d_0^l = \frac{1}{\binom{0}{0}} \sum_{t=1}^{4} \binom{t}{0} D_3^{[d+]^l}(t); \quad l = 1, \ldots, l_{max} + 1 \quad (55)$$

$$\delta DivZ4d_0^l = \frac{()}{\binom{0}{0}} (2l + 1) DivZ4d_0^{l+1}; \quad l = 3, \ldots, l_{max} \quad (56)$$

and

$$\begin{aligned} DivZ4d_{(i+1)}^{(l-i)} &= DivZ4d_{()}^{(l-i)} + \delta DivZ4d_i^l \\ \delta DivZ4d_{(i+1)}^l &= \frac{()}{\binom{0}{0}} 2(l-i) DivZ4d_{(i+1)}^{(l-i)}; \end{aligned} \quad (57)$$

Corrections for small Gram determinants IV

"Solving" (53) and (54) :

$$Z4d_i^L = \sum_{j=0}^{i-1} c(j) r^j Z4d^{(L+j)} - 2 \sum_{j=0}^{i-1} c(j) r^{j+1} D_4^{[d+]^{(L+j+1)}} + c(i) r^i Z4d_0^{(L+i)}, \qquad (58)$$

where $r = \frac{()}{\binom{0}{0}}$ and $c(j) = 2^j \frac{\Gamma(L+j+\frac{1}{2})}{\Gamma(L+\frac{1}{2})}$.

$i \to \infty$ and assuming convergence:

$$I_4^{[d+]^L} = \sum_{j=0}^{\infty} c(j) r^j Z4d^{(L+j)} - 2 \sum_{j=0}^{\infty} c(j) r^{j+1} D_4^{[d+]^{(L+j+1)}}. \qquad (59)$$

Padé approximants

Padé approximants in terms of the ε-algorithm:
1^{st} column is zero, 2^{nd} : sequence $S^i = Z4d_i^L$

$$\varepsilon_{-1}^{(i)} = 0, \tag{60}$$

$$\varepsilon_{0}^{(i)} = Z4d_i^L, i = 0, \cdots, l_{\max} - L, \tag{61}$$

$$\varepsilon_{k+1}^{(i)} = \varepsilon_{k-1}^{(i+1)} + \frac{1}{\varepsilon_k^{(i+1)} - \varepsilon_k^{(i)}} \tag{62}$$

ε-table and the Padé table are related:

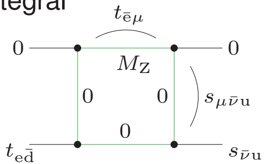
$$\varepsilon_{2k}^{(i)} = [k + i/k], \tag{63}$$

$$Z4d^L_{l_{\max}-L} = \varepsilon_{2k}^{(0)} \equiv [k/k]_{Z4d^L}, \quad k = l_{\max} - L \tag{64}$$

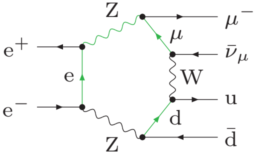
An example: D_{111}, following A. Denner

A.Denner, plenary talk DESY Theory Workshop 2009, p.69 (backup transparency)

box integral



appears, e.g., in subgraph of diagram



Gram det.: $\Delta^{(N)} \to 0$ if $t_{e\bar{d}} \to t_{crit} \equiv \frac{s_{\mu\bar{\nu}u}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}u} - s_{\bar{\nu}u}}$

x		$\Re e\ D_{1111}$	$\Im m\ D_{1111}$
0.	[exp 0,0]	2.05969289730 E-10	1.55594910118 E-10
10^{-8}	[exp x,2]	2.05969289342 E-10	1.55594909187 E-10
	[exp 0,2]	2.05969289349 E-10	1.55594909187 E-10
10^{-4}	[exp x,5]	2.05965609497 E-10	1.55585605343 E-10
	[exp 0,5]	2.05965609495 E-10	1.55585605343 E-10
0.001	[exp 0,6]	2.05932484380 E-10	1.55501912433 E-10
	[exp x,6]	2.05932484381 E-10	1.55501912433 E-10
	$I_{4,2222}^{[d+]^4}$	2.02292295240 E-10	1.54974785467 E-10
	D_{1111}	2.01707671668 E-10	1.62587142251 E-10
0.005	[exp 0,6]	2.05786054801 E-10	1.55131031024 E-10
	[pade 0,3]	2.05785198947 E-10	1.55131031003 E-10
	[exp x,6]	2.05786364440 E-10	1.55131031024 E-10
	[pade x,3]	2.05785199805 E-10	1.55131030706 E-10
	$I_{4,2222}^{[d+]^4}$	2.05778894114 E-10	1.55135794453 E-10
	D_{1111}	2.05779811490 E-10	1.55136343923 E-10
0.01	[exp 0,6]	2.05703298143 E-10	1.54669910676 E-10
	[pade 0,3]	2.05600940065 E-10	1.54669907784 E-10
	[exp 0,10]	2.05600964693 E-10	1.54669910676 E-10
	[pade 0,5]	2.05600955381 E-10	1. 54669910676E-10
	[exp x,10]	2.05600963675 E-10	1.54669910676 E-10
	[pade x,5]	2.05600955381 E-10	1.54669910676 E-10
	$I_{4,2222}^{[d+]^4}$	2.05600013702 E-10	1.54670651917 E-10
	D_{1111}	2.05600239280 E-10	1.54670771210 E-10

Table: Numerical values for the tensor coefficient D_{1111}. Values marked by D_{1111} are evaluated with LoopTools, the $I_{4,2222}^{[d+]^4}$ corresponds to (48) The labels [exp 0,2n] and [pade 0,n] denote iteration $2n$ and Pade approximant $[n, n]$ when the small Gram determinant expansion starts at $x = 0$, and [exp x,2n] and [pade x,n] are the corresponding numbers for an expansion starting at x.

x	$\Re e\ D_{1111}$	$\Im m\ D_{1111}$
0.01 [exp 0,6]	2.05703298143 E-10	1.54669910676 E-10
[pade 0,3]	2.05600940065 E-10	1.54669907784 E-10
[exp 0,10]	2.05600964693 E-10	1.54669910676 E-10
[pade 0,5]	2.05600955381 E-10	1. 54669910676E-10
[exp x,10]	2.05600963675 E-10	1.54669910676 E-10
[pade x,5]	2.05600955381 E-10	1.54669910676 E-10
$I^{[d+]^4}_{4,2222}$	2.05600013702 E-10	1.54670651917 E-10
D_{1111}	2.05600239280 E-10	1.54670771210 E-10
0.05 [exp 0,6]	4.83822963052 E-09	1.51077429118 E-10
[pade 0,3]	2.01518061131 E-10	1.50591643209 E-10
[exp 0,20]	2.04218962072 E-10	1.51077424143 E-10
[pade 0,10]	2.04122727654 E-10	1.51077424149 E-10
[exp x,20]	2.04190274030 E-10	1.51077424143 E-10
[pade x,10]	2.04122727971 E-10	1.51077423985 E-10
$I^{[d+]^4}_{4,2222}$	2.04122726387 E-10	1.51077422901 E-10
D_{1111}	2.04122726601 E-10	1.51077423320 E-10
0.1 [exp 0,26]	2.20215264409 E-08	1.46815247004 E-10
[pade 0,13]	2.01749674352 E-10	1.46681287362 E-10
[exp x,26]	2.08190721550 E-08	1.46815247004 E-10
[pade x,13]	2.03995221326 E-10	1.46785977364 E-10
$I^{[d+]^4}_{4,2222}$	2.02269485177 E-10	1.46815247061 E-10
D_{1111}	2.02269485217 E-10	1.46815247051 E-10
1. $I^{[d+]^4}_{4,2222}$	1.72115440143 E-10	9.74550747662 E-11
D_{1111}	1.72115440148 E-10	9.74550747662 E-11

Table: Numerical values for the tensor coefficient D_{1111}. Values marked by D_{1111} are evaluated with LoopTools, the $I^{[d+]^4}_{4,2222}$ corresponds to (48) The labels [exp 0,2n] and [pade 0,n] denote iteration $2n$ and Pade approximant $[n, n]$ when the small Gram determinant expansion starts at $x = 0$, and [exp x,2n] and [pade x,n] are the corresponding numbers for an expansion starting at x.

x		$\Re e\ D_{111}$	$\Im m\ D_{111}$
0	[exp 0,0]	–3.15407250453 E-10	–3.31837792634 E-10
10^{-8}	[exp x,1]	–3.15407250057 E-10	–3.31837790700 E-10
	[exp 0,1]	–3.15407250057 E-10	–3.31837790700 E-10
10^{-4}	[exp x,4]	–3.15403282194 E-10	–3.31818461838 E-10
	[exp 0,4]	–3.15403282194 E-10	–3.31818461838 E-10
0.001	[exp x,6]	–3.15367545429 E-10	–3.31644587150 E-10
	[exp 0,6]	–3.15367545429 E-10	–3.31644587150 E-10
	$I^{[d+]^3}_{4,222}$	–3.15372092999 E-10	–3.31645245644 E-10
	D_{111}	–3.15372823537 E-10	–3.31635736868 E-10
0.005	[exp x,6]	–3.15208222856 E-10	–3.30874035862 E-10
	[pade x,3]	–3.15208230282 E-10	–3.30874035931 E-10
	[exp 0,6]	–3.15208224867 E-10	–3.30874035862 E-10
	[pade 0,3]	–3.15208230411 E-10	–3.30874035867 E-10
	$I^{[d+]^3}_{4,222}$	–3.15208269791 E-10	–3.30874006110 E-10
	D_{111}	–3.15208264077 E-10	–3.30874002667 E-10
0.01	[exp 0,6]	–3.15006665284 E-10	–3.29915926110 E-10
	[pade 0,3]	–3.15007977830 E-10	–3.29915888075 E-10
	[exp 0,10]	–3.15007991203 E-10	–3.29915926110 E-10
	[pade 0,5]	–3.15007991324 E-10	–3.29915926110 E-10
	[exp x,10]	–3.15007991217 E-10	–3.29915926110 E-10
	[pade x,5]	–3.15007991324 E-10	–3.29915936110 E-10
	$I^{[d+]^3}_{4,222}$	–3.15008000292 E-10	–3.29915916848 E-10
	D_{111}	–3.15008000292 E-10	–3.29915915368 E-10

Table: Numerical values for the tensor coefficient D_{111}. Values marked by D_{111} are evaluated with LoopTools, the $I^{[d+]^3}_{4,222}$, defined in (47). The labels [exp 0,2n] and [pade 0,n] denote iteration $2n$ and Pade approximant $[n, n]$ when the small Gram determinant expansion starts at $x = 0$, and [exp x,2n] and [pade x,n] are the corresponding numbers for an expansion starting at x.

x	$\Re e\ D_{111}$	$\Im m\ D_{111}$
0.01 [exp 0,6]	–3.15006665284 E-10	–3.29915926110 E-10
[pade 0,3]	–3.15007977830 E-10	–3.29915888075 E-10
[exp 0,10]	–3.15007991203 E-10	–3.29915926110 E-10
[pade 0,5]	–3.15007991324 E-10	–3.29915926110 E-10
[exp x,10]	–3.15007991217 E-10	–3.29915926110 E-10
[pade x,5]	–3.15007991324 E-10	–3.29915936110 E-10
$I_{4,222}^{[d+]^3}$	–3.15008000292 E-10	–3.29915916848 E-10
D_{111}	–3.15008000292 E-10	–3.29915915368 E-10
0.05 [exp 0,6]	–1.34278470211 E-11	–3.22448580722 E-10
[pade 0,3]	–3.13432516570 E-10	–3.22580791799 E-10
[exp 0,20]	–3.13359445767 E-10	–3.22448581032 E-10
[pade 0,10]	–3.13365675001 E-10	–3.22448581024 E-10
[exp x,20]	–3.13361302214 E-10	–3.22448581032 E-10
[pade x,10]	–3.13365674956 E-10	–3.22448581051 E-10
$I_{4,222}^{[d+]^3}$	–3.13365675084 E-10	–3.22448581110 E-10
D_{111}	–3.13365675070 E-10	–3.22448581084 E-10
0.1 [exp 0,26]	–2.49466252165 E-09	–3.13582331984 E-10
[pade 0,13]	–3.11144777695 E-10	–3.13599283949 E-10
[exp x,26]	–2.34010823441 E-09	–3.135823319836 E-10
[pade x,13]	–3.10806582023 E-10	–3.135870111996 E-10
$I_{4,222}^{[d+]^3}$	–3.11226750699 E-10	–3.13582331977 E-10
D_{111}	–3.11226750695 E-10	–3.13582331978 E-10
1. $I_{4,222}^{[d+]^3}$	–2.70193791372 E-10	–2.10251973821 E-10
D_{111}	–2.70193791373 E-10	–2.10251973821 E-10

Table: Numerical values for the tensor coefficient D_{111}. Values marked by D_{111} are evaluated with LoopTools, the $I_{4,222}^{[d+]^3}$, defined in (47). The labels [exp 0,2n] and [pade 0,n] denote iteration $2n$ and Pade approximant $[n, n]$ when the small Gram determinant expansion starts at $x = 0$, and [exp x,2n] and [pade x,n] are the corresponding numbers for an expansion starting at x.

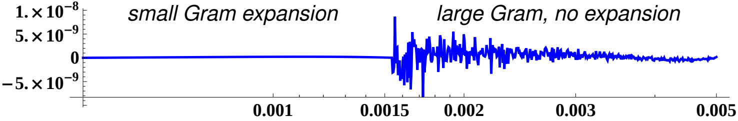
PJFry — numerical package

Numerical implementation of described algorithms:
C++ package **PJFry** by V. Yundin [in preparation]

- Reduction of **5-point** 1-loop tensor integrals up to **rank 5**
- No limitations on internal/external masses combinations
- Small Gram determinants treatment by expansion
- Interfaces for C, C++, FORTRAN and MATHEMATICA

Example:

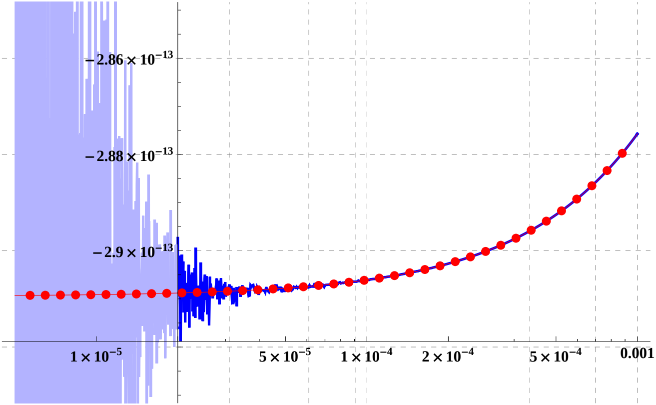
Relative accuracy of E_{3333} coef. around small Gram4 region



PJFry — small Gram region example

Example: E_{3333} coefficient in small Gram region ($x = 0$)

Comparison of Regular and Expansion formulae:



$x{=}0$: $E_{3333}(0, 0, -6{\times}10^4, 0, 0, 10^4, -3.5{\times}10^4, 2{\times}10^4, -4{\times}10^4, 1.5{\times}10^4, 0, 6550, 0, 0, 8315)$

Summary

- Kompakt expression for the tensor components
- No limitation on masses
- Works for vanishing sub-Gram determinants
- No limitations for scalar diagrams containing $\frac{1}{\varepsilon^2}$ terms
- Find patterns how to prodeed to higher tensors
- Analytic simplification of original diagrams (not shown)

References I

- R. K. Ellis and G. Zanderighi, *Scalar one-loop integrals for QCD*, *JHEP* **02** (2008) 002, [0712.1851].
- A. van Hameren, *OneLOop: for the evaluation of one-loop scalar functions*, 1007.4716.
- A. Denner and S. Dittmaier, *Scalar one-loop 4-point integrals*, 1005.2076.
- G. J. van Oldenborgh, *FF: A Package to evaluate one loop Feynman diagrams*, *Comput. Phys. Commun.* **66** (1991) 1–15.
- T. Hahn and M. Perez-Victoria, *Automatized one-loop calculations in four and d dimensions*, *Comput. Phys. Commun.* **118** (1999) 153, [hep-ph/9807565].
- T. Binoth, J. P. Guillet, G. Heinrich, E. Pilon, and T. Reiter, *Golem95: a numerical program to calculate one-loop tensor integrals with up to six external legs*, *Comput. Phys. Commun.* **180** (2009) 2317–2330, [0810.0992].
- T. Diakonidis, J. Fleischer, J. Gluza, K. Kajda, T. Riemann, and J. Tausk, *On the tensor reduction of one-loop pentagons and hexagons*, *Nucl. Phys. Proc. Suppl.* **183** (2008) 109–115, [0807.2984].
- T. Diakonidis *et al.*, *A complete reduction of one-loop tensor 5- and 6-point integrals*, *Phys. Rev.* **D80** (2009) 036003, [0812.2134].
- J. Campbell, E. W. N. Glover, and D. Miller, *One-loop tensor integrals in dimensional regularisation*, *Nucl. Phys.* **B498** (1997) 397–442, [hep-ph/9612413].

References II

A. Denner and S. Dittmaier, *Reduction of one-loop tensor 5-point integrals*, *Nucl. Phys.* **B658** (2003) 175–202, [hep-ph/0212259].

A. Denner and S. Dittmaier, *Reduction schemes for one-loop tensor integrals*, *Nucl. Phys.* **B734** (2006) 62–115, [hep-ph/0509141].

T. Binoth, J. P. Guillet, and G. Heinrich, *Reduction formalism for dimensionally regulated one-loop N-point integrals*, *Nucl. Phys.* **B572** (2000) 361–386, [hep-ph/9911342].

T. Binoth, J. Guillet, G. Heinrich, E. Pilon, and C. Schubert, *An algebraic / numerical formalism for one-loop multi-leg amplitudes*, *JHEP* **10** (2005) 015, [hep-ph/0504267].

Z. Bern, L. J. Dixon, and D. A. Kosower, *Dimensionally regulated pentagon integrals*, *Nucl. Phys.* **B412** (1994) 751–816, [hep-ph/9306240].

G. Ossola, C. Papadopoulos, and R. Pittau, *Reducing full one-loop amplitudes to scalar integrals at the integrand level*, *Nucl. Phys.* **B763** (2007) 147–169, [hep-ph/0609007].

SM and NLO Multileg Working Group Collaboration, J. R. Andersen *et al.*, *The SM and NLO multileg working group: Summary report*, 1003.1241.

A. I. Davydychev, *A Simple formula for reducing Feynman diagrams to scalar integrals*, *Phys. Lett.* **B263** (1991) 107–111.

O. V. Tarasov, *Connection between Feynman integrals having different values of the space-time dimension*, *Phys. Rev.* **D54** (1996) 6479–6490, [hep-th/9606018].

References III

- J. Fleischer, F. Jegerlehner, and O. Tarasov, *Algebraic reduction of one-loop Feynman graph amplitudes*, *Nucl. Phys.* **B566** (2000) 423–440, [hep-ph/9907327].
- T. Diakonidis, J. Fleischer, T. Riemann, and J. B. Tausk, *A recursive reduction of tensor Feynman integrals*, *Phys. Lett.* **B683** (2010) 69–74, [arXiv:0907.2115].
- J. Fleischer and T. Riemann, *Some variations of the reduction of one-loop Feynman tensor integrals*, 2010. Subm. to Proc. ACAT 2010, Feb. 2010, Jaipur, India.
- T. Diakonidis, J. Fleischer, T. Riemann, and B. Tausk, *A recursive approach to the reduction of tensor Feynman integrals*, *PoS* **RADCOR2009** (2009) 033, [1002.0529].