# Scalar one-loop 4-point functions with complex masses

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- Motivation
- Complex-mass scheme
- Results for scalar 4-point function

# Motivation

- All NLO calculations require scalar one-loop integrals as input ($A_0$, $B_0$, $C_0$, $D_0$)
- calculations with unstable particles require complex internal masses
  $\Rightarrow$ need one-loop integrals with complex masses
- for electroweak corrections most general case needed
- existing results cover all cases for $A_0$, $B_0$, $C_0$
  for $D_0$ only special cases but no general result for arbitrary complex internal masses was available

# Complex-mass scheme

# Relevance of unstable particles

Almost all interesting elementary particles are unstable:

- known: leptons $\mu, \tau$, heavy quarks $\mathrm{b}, \mathrm{t}$, massive gauge bosons $\mathrm{W,Z}$
- Higgs bosons $\mathrm{H_{SM}}$, $\{\mathrm{h}, \mathrm{H}, \mathrm{A}, \mathrm{H}\}_{\mathrm{MSSM}}$
- new particles, e.g. in SUSY: $\tilde{l}, \tilde{q}, \tilde{g}, \tilde{\chi}$

lifetimes $\tau$ too short for detection (e.g. $\tau_{\mathrm{Z,W}} \sim 10^{-25}\,\mathrm{s}$)

$\hookrightarrow$ experiments detect only decay products
unstable particles appear as resonances in certain distributions

interesting reactions at the LHC involving unstable particles:

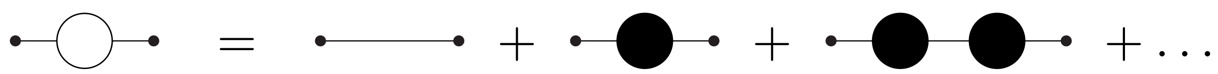
$\mathrm{pp} \to \mathrm{W/Z}(+\mathrm{jets}) \to 2l(+\mathrm{jets})$, $\mathrm{pp} \to \mathrm{H} + 2q \to \mathrm{ZZ} + 2q \to 4l + 2\mathrm{jets}$,
$\mathrm{pp} \to \mathrm{t\bar{t}} \to \mathrm{b\bar{b}WW} \to 2l + 2\mathrm{jets} + \not{E}$, ...

Need consistent treatment of unstable particles
in perturbative evaluation of gauge theories

# Mass and width of unstable particles

Description of resonance requires resummation of propagator corrections

**Dyson series and propagator poles** (scalar example)



$$G^{\phi\phi}(p) \;=\; \frac{\mathrm{i}}{p^2-m^2} + \frac{\mathrm{i}}{p^2-m^2}\,\mathrm{i}\Sigma(p^2)\,\frac{\mathrm{i}}{p^2-m^2} + \ldots \;=\; \frac{\mathrm{i}}{p^2-m^2+\Sigma(p^2)}$$

$\Sigma(p^2) =$ renormalized self-energy, $m =$ renormalized mass

**stable particle:** $\mathrm{Im}\{\Sigma(p^2)\} \;=\; 0$ at $p^2 \sim m^2$

$\hookrightarrow$ propagator pole for real value of $p^2$,
renormalization condition for physical mass $m$: $\Sigma(m^2) = 0$
physical mass = pole of propagator

**unstable particle:** $\mathrm{Im}\{\Sigma(p^2)\} \;\neq\; 0$ at $p^2 \sim m^2$

$\hookrightarrow$ propagator pole shifted into complex $p^2$ plane,
define mass $M$ and width $\Gamma$ via complex pole position $\mu^2 = M^2 - \mathrm{i}M\Gamma$
$\mu^2 - m^2 + \Sigma(\mu^2) = 0$ $\Rightarrow$ gauge-independent $\mu^2$

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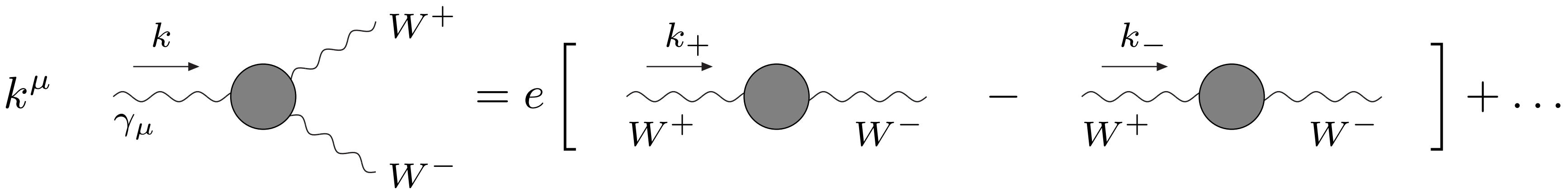
# Unstable particles and gauge invariance

resonances require (partial) Dyson summation of resonant propagators

$\hookrightarrow$ perturbative orders mixed

$\hookrightarrow$ gauge invariance in general violated !

example: electromagnetic Ward identity for photon–W-boson vertex




valid for lowest order or complete higher orders

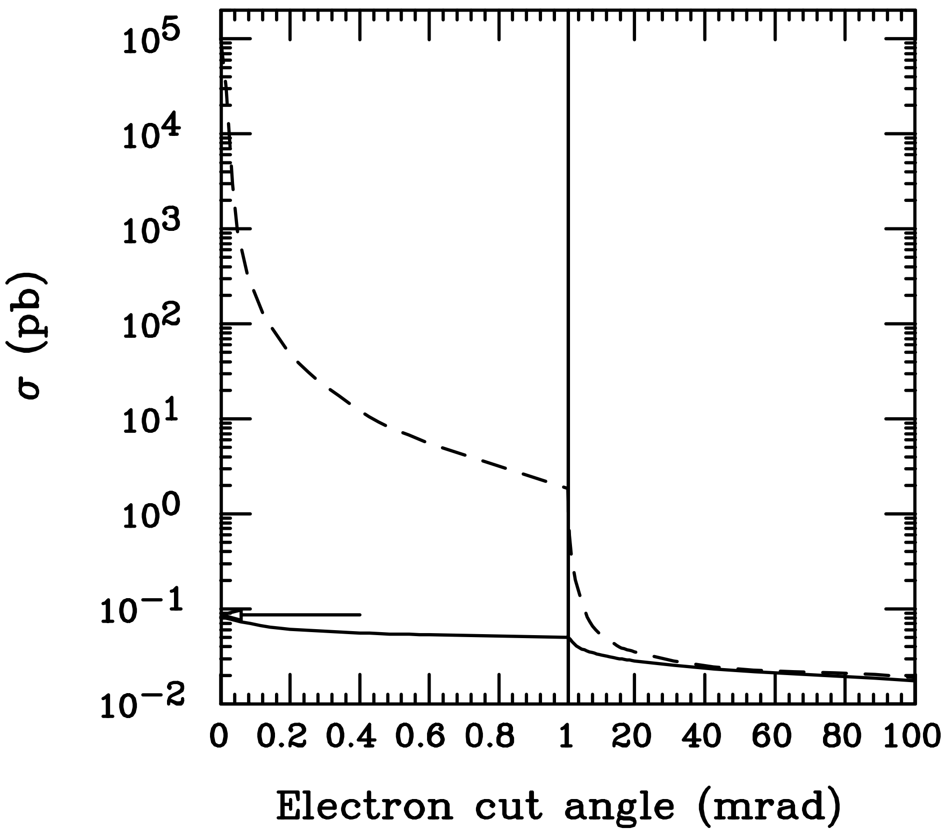
violated for Dyson-resummed propagators

$$k_+^2 - k_-^2 \neq \left[k_+^2 - M_\mathrm{W}^2 + \Sigma(k_+^2)\right] - \left[k_-^2 - M_\mathrm{W}^2 + \Sigma(k_-^2)\right] = k_+^2 - k_-^2 + \Sigma(k_+^2) - \Sigma(k_-^2)$$

unless $\Sigma(k_+^2) = \Sigma(k_-^2)$ or if also vertex is changed

note: gauge-invariance-violating terms are formally of higher order,
but can be dramatically enhanced

# An example: $e^-e^+ \rightarrow e^-\bar{\nu}_e u\bar{d}$




Kurihara, Perret-Gallix, Shimizu '95

$\sqrt{s} = 180\,\text{GeV}$

cross section as a function of the cut on the electron scattering angle

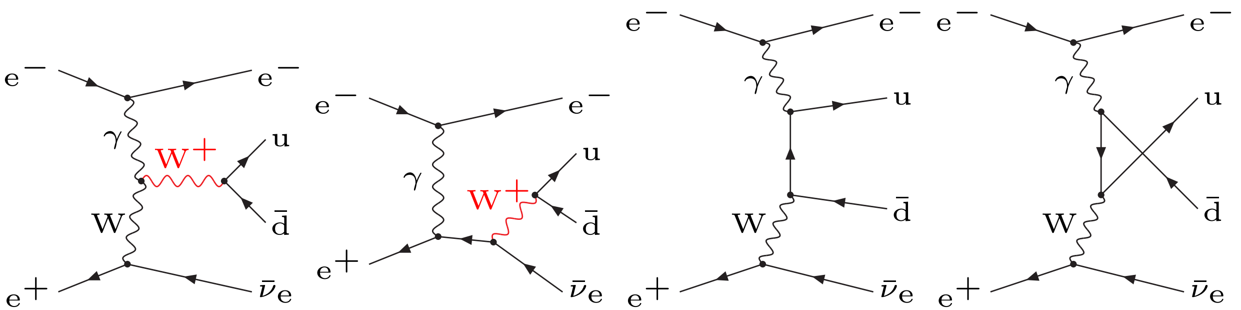
solid: gauge-invariant
(fudge factor) scheme

dashed: constant width
only in resonant propagator

$\hookrightarrow$ crude U(1) gauge-invariance violation

dominant diagrams:

nearly real photon !



# Finite-width schemes

- Naive propagator substitutions in full amplitudes:

$$\frac{1}{k^2-m^2} \to \frac{1}{k^2-m^2+\mathrm{i}m\Gamma(k^2)} \qquad \text{for resonant or all propagators}$$

  - constant width $\Gamma(k^2) = \text{const.}$ $\to$ U(1) respected (if all propagators dressed), SU(2) “mildly” violated
  - running width $\Gamma(k^2) \propto \theta(k^2) \times k^2$ $\to$ U(1) and SU(2) violated

- fudge factor approaches: multiply full amplitudes without widths with factors $\dfrac{p^2-m^2}{p^2-m^2+\mathrm{i}m\Gamma}$ for each potentially resonant propagator

  $\hookrightarrow$ procedure preserves gauge invariance, introduces spurious factors of $\mathcal{O}(\Gamma/m)$, mistreats non-resonant terms

- pole scheme: consistent expansion about resonance Stuart ’91; Aeppli et al. ’93, ’94; ...
  gauge invariant, applicable to higher orders (but cumbersome)
  not reliable in off-shell tails, near thresholds (presence of small scales)

- effective field theory approach: Beneke et al. ’04, ’07, Hoang, Reisser ’04
  close to pole expansion, can be combined with threshold expansion

# Consistent implementation of unstable particles

- Gauge independence
  use pole mass and width from complex pole (gauge independent)
  instead of full or more complete Dyson summation $\Rightarrow$ no running width
- Ward identities
  - QCD and QED
    Ward identities depend only on masses and strong/elmg. coupling constants
    $\Rightarrow$ Ward identities hold after replacing real by complex masses
    use complex pole masses in *all* propagators $\Rightarrow$ fixed width scheme
    QCD and QED corrections: complex masses are independent parameters
    $\Rightarrow$ no further complications
  - Electroweak theory
    couplings are related to masses via weak mixing angle $c_{\mathrm{w}} = M_{\mathrm{W}}/M_{\mathrm{Z}}$
    Ward identities require use of complex masses also in couplings
    $\Rightarrow$ complex weak mixing angle $\Rightarrow$ complex-mass scheme
    Electroweak corrections
    avoid double counting (electroweak corrections in widths)
    mass renormalization must be done for complex pole

# The complex-mass scheme

Denner, Dittmaier, Roth, Wackeroth '99, Denner, Dittmaier, Roth, Wieders '05

Basic idea: (renormalized mass)$^2$ = location of propagator pole in complex $p^2$ plane

$\hookrightarrow$ consistent use of complex masses everywhere !

replace $M_{\mathrm{W}}^2 \to \mu_{\mathrm{W}}^2 = M_{\mathrm{W}}^2 - \mathrm{i} M_{\mathrm{W}} \Gamma_{\mathrm{W}}, \qquad M_{\mathrm{Z}}^2 \to \mu_{\mathrm{Z}}^2 = M_{\mathrm{Z}}^2 - \mathrm{i} M_{\mathrm{Z}} \Gamma_{\mathrm{Z}}$

and define (complex) weak mixing angle via $\cos^2\theta_{\mathrm{w}} \equiv c_{\mathrm{w}}^2 = 1 - s_{\mathrm{w}}^2 = \frac{\mu_{\mathrm{W}}^2}{\mu_{\mathrm{Z}}^2}$

$\hookrightarrow$ complex mass renormalization: $\underbrace{M_{\mathrm{W},0}^2}_{\text{bare mass}} = \mu_{\mathrm{W}}^2 + \underbrace{\delta\mu_{\mathrm{W}}^2}_{\text{ren. constant}}, \quad \ldots$

virtues

- gauge invariance (Ward identities), gauge-parameter independence hold exactly
- perturbative calculations as usual (with complex masses and counterterms)
- no double counting (bare Lagrangian unchanged !)

drawbacks

- loop integrals with complex masses (available)
- spurious terms of $\mathcal{O}(\Gamma/M)$, of higher order [$\mathcal{O}(\alpha^2)$ in $\mathcal{O}(\alpha)$ calculation]

# Complex renormalization: W-boson as example

Direct generalization of on-shell renormalization scheme

Aoki et al. ’81; Denner ’93; Denner, Dittmaier, Weiglein ’94

$\Rightarrow$ complex field renormalization $\underbrace{W_0^\pm}_{\text{bare field}} = \left(1 + \frac{1}{2}\underbrace{\delta\mathcal{Z}_W}_{\text{ren. constant}}\right)W^\pm,$ etc.

- complex $\delta\mathcal{Z}_W$ applies to both $W^+$ and $W^-$ $\Rightarrow (W^+)^\dagger \neq W^-$
- $\delta\mathcal{Z}_W$ drops out in $S$-matrix elements without external W-bosons

on-shell renormalization conditions for W-boson self-energy

$$\hat\Sigma_{\mathrm{T}}^W(\mu_{\mathrm{W}}^2) = 0, \qquad \hat\Sigma_{\mathrm{T}}'^W(\mu_{\mathrm{W}}^2) = 0$$

$\hookrightarrow$ $\mu_{\mathrm{W}}^2$ is location of propagator pole, and “residue = 1”

solutions of renormalization conditions

$$\delta\mu_{\mathrm{W}}^2 = \Sigma_{\mathrm{T}}^W(\mu_{\mathrm{W}}^2), \qquad \delta\mathcal{Z}_W = -\Sigma_{\mathrm{T}}'^W(\mu_{\mathrm{W}}^2)$$

require self-energy for complex squared momenta $(p^2 = \mu_{\mathrm{W}}^2)$

$\hookrightarrow$ analytic continuation of the 2-point functions to unphysical Riemann sheet

# Expansion of counterterms about real momenta

Way around: appropriate expansions about real arguments

$$\Sigma^W_\mathrm{T}(\mu^2_\mathrm{W}) = \Sigma^W_\mathrm{T}(M^2_\mathrm{W}) + (\mu^2_\mathrm{W} - M^2_\mathrm{W})\Sigma'^W_\mathrm{T}(M^2_\mathrm{W}) + \underbrace{\mathcal{O}(\alpha^3)}_{\text{beyond } \mathcal{O}(\alpha) \text{ and UV-finite}}$$

modified counterterms

$$\delta\mu^2_\mathrm{W} = \Sigma^W_\mathrm{T}(M^2_\mathrm{W}) + (\mu^2_\mathrm{W} - M^2_\mathrm{W})\Sigma'^W_\mathrm{T}(M^2_\mathrm{W}), \qquad \delta\mathcal{Z}_W = -\Sigma'^W_\mathrm{T}(M^2_\mathrm{W})$$

$\Rightarrow$ renormalized self-energy

$$\hat{\Sigma}^W_\mathrm{T}(k^2) = \Sigma^W_\mathrm{T}(k^2) - \delta M^2_\mathrm{W} + (k^2 - M^2_\mathrm{W})\delta Z_W$$

with

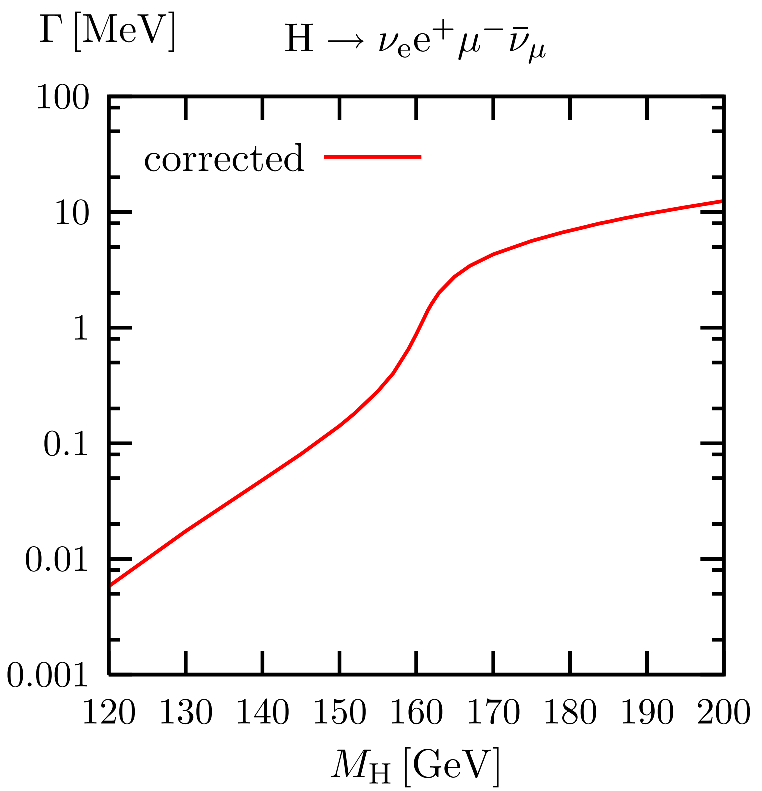
$$\delta M^2_\mathrm{W} = \Sigma^W_\mathrm{T}(M^2_\mathrm{W}), \qquad \delta Z_W = -\Sigma'^W_\mathrm{T}(M^2_\mathrm{W})$$

exactly the form of the renormalized self-energies in usual on-shell scheme

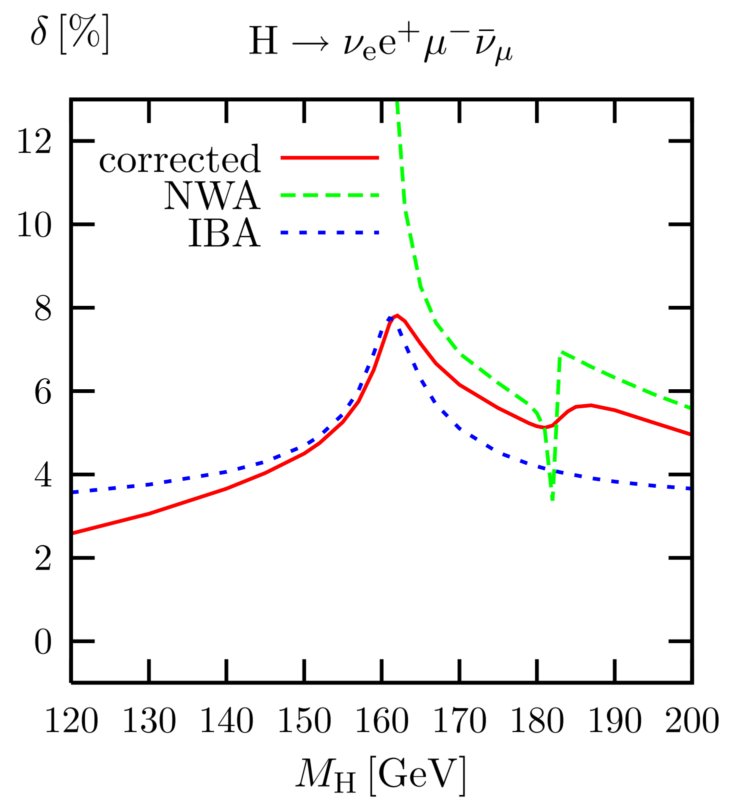
but
- no real parts are taken from $\Sigma^W$
- $\Sigma^W$ depends on complex masses and complex mixing angle

# Example: Partial width for $\mathrm{H} \to \mathrm{WW} \to \nu_\mathrm{e}\mathrm{e}^+\mu^-\bar{\nu}_\mu$

Partial width:




relative corrections ($G_\mu$-scheme):




$$\delta = \frac{\Gamma}{\Gamma_\mathrm{Born}} - 1$$

real $M_\mathrm{W}$, $M_\mathrm{Z}$

complex $M_\mathrm{W}$, $M_\mathrm{Z}$

↑ Coulomb singularity for $M_\mathrm{H} \sim 2M_\mathrm{W}$

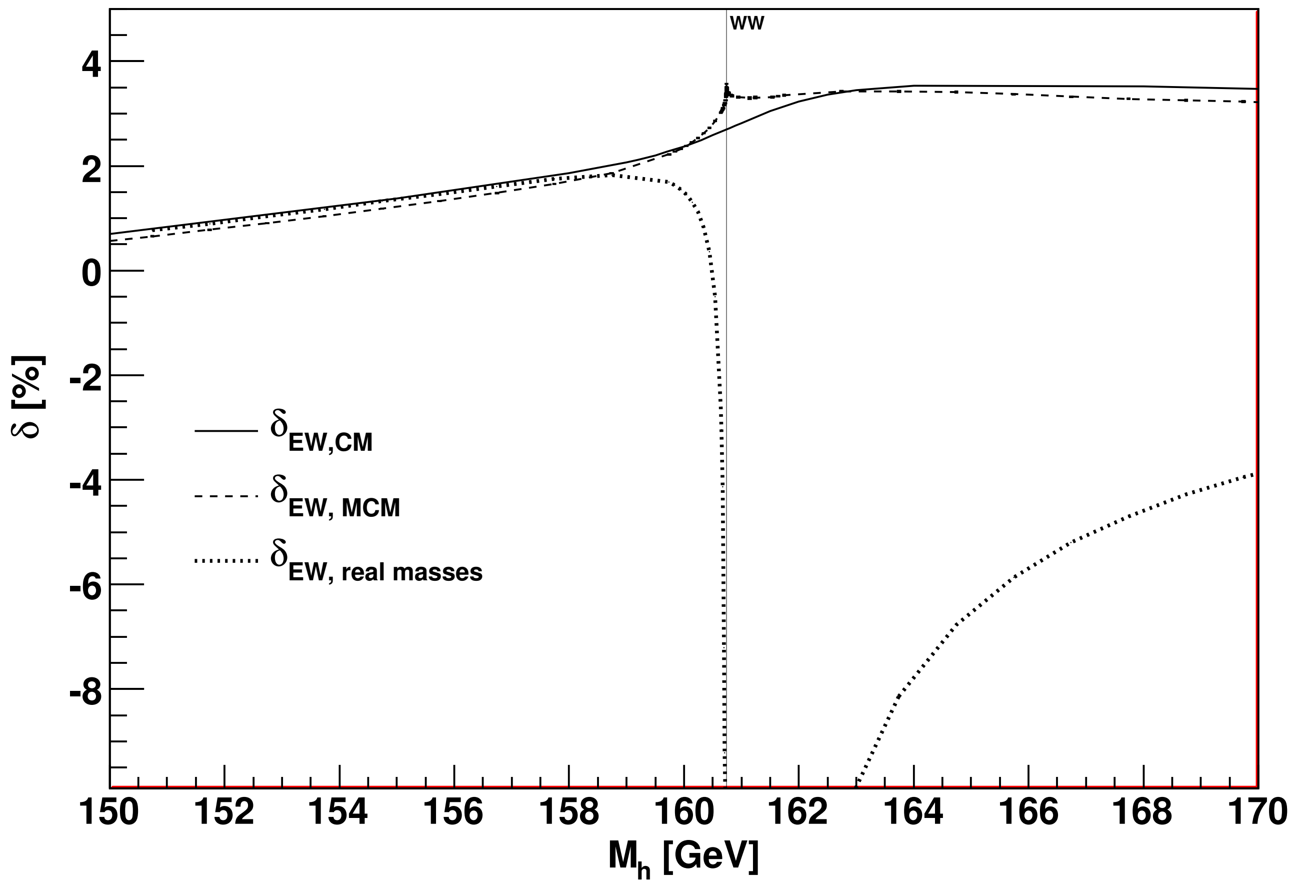
↑ threshold effect in loops for $M_\mathrm{H} \sim 2M_\mathrm{Z}$

NWA = narrow-width approximation

IBA = improved Born approximation

Bredenstein, Denner, Dittmaier, Weber '06, PROPHECY4F

# Example: electroweak two-loop corrections to $H \to \gamma\gamma$




Actis, Passarino, Sturm, Uccirati '08

CM = complex-mass scheme

MCM = "minimal" complex-mass scheme

$\Rightarrow$ consistent (gauge-invariant) use of complex masses mandatory!

# Scalar 4-point function

# Existing results and tools for scalar one-loop integrals

## Documented results

- 't Hooft, Veltman '79: regular 1-, 2-, 3-point integrals with complex masses,
  4-point integrals with real masses
- Beenakker, Denner '90: IR singular 4-point integrals with mass reg.
- Denner, Nierste, Scharf '91: compact result for regular 4-point integral
- Dao, Le '09: regular 4-point integral with complex masses
- many special cases spread over the literature Bern et al. '93, Duplancic, Nizic '01,'02, ...

## Tools for evaluation of scalar one-loop integrals

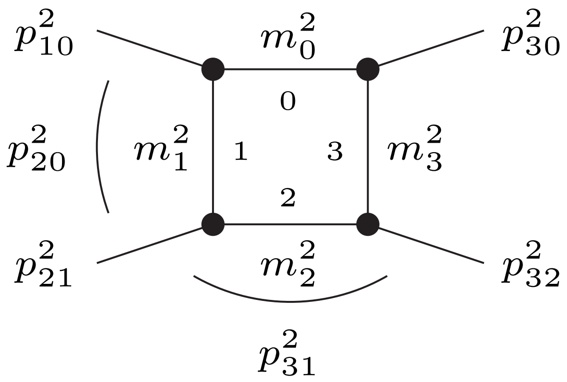
- van Oldenborgh '90, FF: regular integrals
- R.K. Ellis, Zanderighi '07, QCDloop: IR-singular integrals for QCD in dim. reg.
- van Hameren '10 Oneloop regular and IR singular integrals in dim. reg.

Fully general results for complex masses were not available

New: Denner, Dittmaier '10: all regular and singular 4-point integrals with complex masses in dimensional and mass regularization

# Regular 4-point integral

Feynman-parameter representation: ($D = 4$ for regular case)




$$\equiv \quad D_0(p_1, p_2, p_3, m_0^2, m_1^2, m_2^2, m_3^2)$$

$$\equiv \quad D_0(p_{10}^2, p_{21}^2, p_{32}^2, p_{30}^2, p_{20}^2, p_{31}^2, m_0^2, m_1^2, m_2^2, m_3^2)$$

$$= \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int \mathrm{d}^D q \frac{1}{(q^2 - m_0^2)[(q+p_1)^2 - m_1^2][(q+p_2)^2 - m_2^2][(q+p_3)^2 - m_3^2]}$$

$$= \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \int_0^{1-x_1-x_2} \mathrm{d}x_3 \frac{1}{[P(1-x_1-x_2-x_3, x_1, x_2, x_3)]^{2+\epsilon}},$$

with

$$P(x_0, x_1, x_2, x_3) = \sum_{i=0}^{3} m_i^2 x_i^2 + \sum_{\substack{i,j=0 \\ i<j}}^{3} Y_{ij} x_i x_j, \qquad Y_{ij} = Y_{ji} = m_i^2 + m_j^2 - p_{ij}^2$$

difficulties:

- linearization of quadratic form in integration variables
- analytic continuation of integrated result to complex masses

# Solution following Denner, Nierste, Scharf ’91

**step 1:** start from real masses and perform projective transformation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{1+x+(1-1/r_{13})y+(1-1/r_{20})z} \begin{pmatrix} y \\ z \\ x-y/r_{13} \end{pmatrix}$$

$\Rightarrow P$ becomes linear in both $y$ and $z$.
Assuming $r_{20}, r_{13} > 0$, $D_0$ becomes

$$D_0 = \int_0^\infty \mathrm{d}x \int_0^{x r_{13}} \mathrm{d}y \int_0^{r_{20}} \mathrm{d}z \, \frac{1}{[P(1-z/r_{20}, y, z, x-y/r_{13}) - \mathrm{i}\varepsilon]^2}$$

where $r_{ij} = r_{ij,1}$ fulfills $m_i^2 + Y_{ij}x + m_j^2 x^2 = m_i^2\,(1+r_{ij,1}x)\,(1+r_{ij,2}x)$

**step 2:** integration over $z$ and $y$ via partial fractioning

**step 3:** analytic continuation to arbitrary complex $r_{13}$

**step 4:** integration over $x \Rightarrow D_0$ in terms of 16 dilogarithms,
but result requires $r_{20} > 0$, i.e. kinematics/masses restricted

# Generalization with propagator identity

For $p_{20}^2 = 0$, we have $r_{20} = 1 > 0$ and result holds.

Use propagator identity of 't Hooft/Veltman '79

$$\frac{1}{(q+p_1)^2 - m_1^2}\frac{1}{(q+p_3)^2 - m_3^2} = \frac{1}{(q+p)^2 - M^2}\left[\frac{1-\gamma}{(q+p_1)^2 - m_1^2} + \frac{\gamma}{(q+p_3)^2 - m_3^2}\right]$$

where $\gamma$ is arbitrary and $p = \gamma p_1 + (1-\gamma)p_3$,
$M^2 = p^2 + \gamma(m_1^2 - p_1^2) + (1-\gamma)(m_3^2 - p_3^2)$

$\Rightarrow$ relation for $D_0$ functions:

$$\begin{aligned} &D_0(p_{10}^2, p_{21}^2, p_{32}^2, p_{30}^2, p_{20}^2, p_{31}^2, m_0^2, m_1^2, m_2^2, m_3^2) \\ &\quad = \gamma D_0(p_{30}^2, \gamma^2 p_{31}^2, (p_2 - p)^2, p_{20}^2, p^2, p_{32}^2, m_0^2, m_3^2, M^2, m_2^2) \\ &\qquad + (1-\gamma) D_0(p_{10}^2, (1-\gamma)^2 p_{31}^2, (p_2 - p)^2, p_{20}^2, p^2, p_{21}^2, m_0^2, m_1^2, M^2, m_2^2) \end{aligned}$$

Fix $\gamma$ so that $0 = p^2 = \gamma^2 p_{31}^2 + \gamma(p_{10}^2 - p_{31}^2 - p_{30}^2) + p_{30}^2$, with $0 \leq \gamma \leq 1$
$\Rightarrow \mathrm{Im}\, M^2 \leq 0$ and $r_{20} = 1$ for both $D_0$ functions on r.h.s.

$\Rightarrow$ general $D_0$ function with complex masses expressed in terms of 32 dilogarithms in all physically relevant regions of phase space

# Alternative solutions for general case

- ’t Hooft, Veltman ’79 provided sketch for result on terms of 108 dilogarithms implemented into Fortran code by Dao, Le ’09
- Denner, Dittmaier ’10 contains additional result in terms of 72 dilogarithms based on modified approach of ’t Hooft, Veltman ’76

# Singular 4-point integrals

Complete list of singular 4-point integrals

- all soft- and/or collinear singular cases
- in mass, dimensional and mixed regularization
- with complex masses

derivation

- calculate in suitable (simple) regularization
- translate to other regularizations by adding and subtracting appropriate 3-point functions

checks

- integrals that do not need dimensional regularization: directly extracted from general result as limit of small masses
- integrals that need dimensional regularization: compared with results in collection of Ellis, Zanderighi '07 or special cases derived therefrom

# General structure of IR singularities at one loop

## Singular parts entirely contained in 3-point functions

explicit construction of singular part    Beenakker et al. ’02, Dittmaier ’03

$$T^{(N)}_{\mu_1\ldots\mu_P}(p_0,\ldots,p_{N-1},m_0,\ldots,m_{N-1})\Big|_{\mathrm{sing}}$$

$$= \sum_{n=0}^{N-1} \sum_{\substack{k=0\\k\neq n,n+1}}^{N-1} \underbrace{A_{nk}} \underbrace{C_{\mu_1\ldots\mu_P}(p_n,p_{n+1},,p_k,m_n,m_{n+1},m_k)}_{\text{all singular 3-point subintegrals}}$$

$A_{nk}$: simple algebraic functions of momenta and masses

Application: translation of singular integrals between regularization schemes

$$\left[T^{(N),\mathrm{regA}}_{\mu_1\ldots\mu_P} - T^{(N),\mathrm{regB}}_{\mu_1\ldots\mu_P}\right]\Bigg|_{\mathrm{nonsing}} = (\mathrm{IR\ finite}) = \text{(reg.-scheme independent)}$$

$\Rightarrow$ transition from scheme A to scheme B:

$$T^{(N),\mathrm{regB}}_{\mu_1\ldots\mu_P} = T^{(N),\mathrm{regA}}_{\mu_1\ldots\mu_P} \underbrace{-T^{(N),\mathrm{regA}}_{\mu_1\ldots\mu_P}\Big|_{\mathrm{sing}} + T^{(N),\mathrm{regB}}_{\mu_1\ldots\mu_P}\Big|_{\mathrm{sing}}}_{\text{singular 3-point integrals}}$$

# Conclusion

- consistent treatment of unstable particles needed for NLO calculations
- complex-mass scheme: gauge-invariant, straightforward to use
- scalar one-loop integrals with complex masses required
- explicit expression exist for all basic scalar one-loop integrals