Continuum Quantum Field Theory for Fractons

String Math 2020

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Seiberg-SHS, arXiv: 2003.10466, 2004.00015, 2004.06115

Common Lore

- Starting at short distances with some lattice model with local interactions, there is an effective quantum field theory (QFT) description at low energy/long distances.

- A more limited version: Every gapped phase of matter can be described by a topological quantum field theory (TQFT) at low energy.

- TQFT has been studied extensively by both physicists and mathematicians. It captures the universal behaviors of these gapped topological phases.

Challenge to the Common Lore

- Fractons [...Haah 2011...] : a large class of lattice spin models exhibiting peculiar features.

- Do **not** seem to admit a standard continuum field theory limit.

- In particular, the gapped fracton phases do **not** seem to admit a TQFT description at low energy.

Introduction

- We will extend the framework of QFT to incorporate these systems.

- This talk will center around the X-cube model [Vijay, Haah, Fu 2016], one of the most celebrated gapped fracton models, and its related systems.

- We will restrict ourselves to (3+1)-dimensional flat spacetime.

- Our QFTs are not topological, but they have quasi-topological defects and operators.

What is the X-cube model?

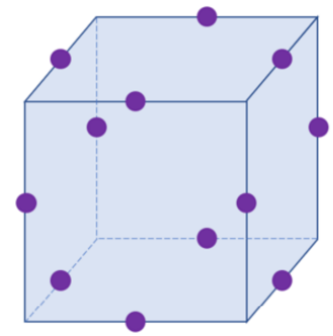
X-cube Model [Vijay, Haah, Fu 2016]

Figures taken from
Pretko-Chen-You 2020

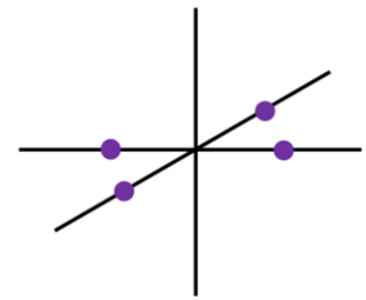
- 3d cubic lattice
- 2-dimensional Hilbert space at each link
- Pauli operators at each link ℓ

$$\sigma_\ell^1 \sigma_\ell^3 = -\sigma_\ell^3 \sigma_\ell^1$$
$$\sigma_\ell^1 \sigma_\ell^1 = \sigma_\ell^3 \sigma_\ell^3 = 1$$

- All the terms in the Hamiltonian commute with each others



$$A_c = \prod_{\ell \in c} \sigma_\ell^1$$



$$B_v^z = \prod_{\substack{\ell \ni v \\ \ell \in (xy)}} \sigma_\ell^3$$

$$H = -\sum_c A_c - \sum_v (B_v^x + B_v^y + B_v^z)$$

Peculiarities of the X-cube Model

- Number of ground states grows exponentially in the linear size L^x, L^y, L^z of the system (assuming periodic boundary condition).

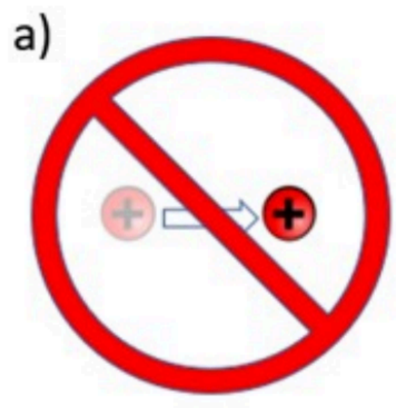
  $$\text{ground state degeneracy} = 2^{2L^x+2L^y+2L^z-3}$$

  It becomes infinite in the continuum limit $L^i \to \infty$.

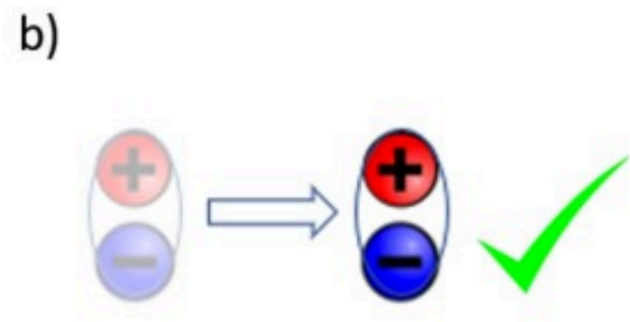
- The ground state degeneracy is robust: small deformations by local operators cannot lift the degeneracy in perturbation theory.

- Excitations have restricted mobility.

Restricted Mobility



a single fracton
cannot move



but two of them
can move together

Seems impossible to be described by QFT???

Figures taken from
Pretko-Chen-You 2020

As a warm up, consider a more familiar example:

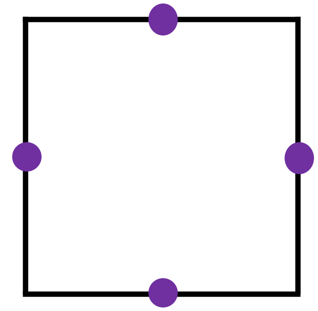
Toric Code

Toric Code [Kitaev 1997]

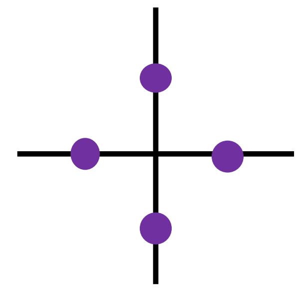
- 2d square lattice
- 2-dimensional Hilbert space at each link
- Pauli operators at each link ℓ

$$\sigma_\ell^1 \sigma_\ell^3 = -\sigma_\ell^3 \sigma_\ell^1$$
$$\sigma_\ell^1 \sigma_\ell^1 = \sigma_\ell^3 \sigma_\ell^3 = 1$$

- All the terms in the Hamiltonian commute with each others

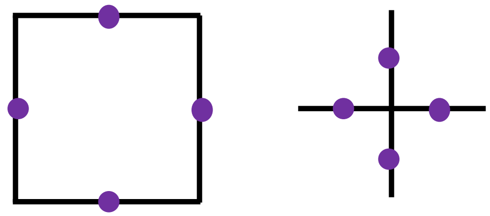
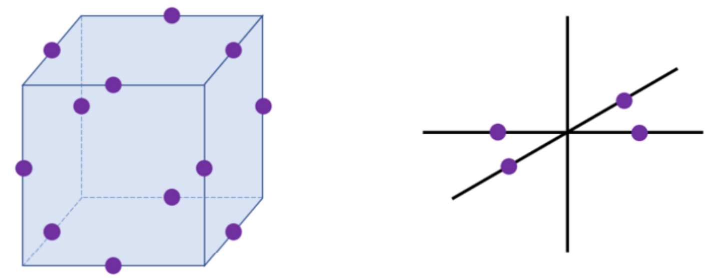


$$A_p = \prod_{\ell \in p} \sigma_\ell^1$$



$$B_v = \prod_{\ell \ni v} \sigma_\ell^3$$

$$H = -\sum_p A_p - \sum_v B_v$$

Lattice model	(2+1)d Toric code	(3+1)d X-cube model
Lattice interaction		
Ground states	2^2	$2^{2L^x+2L^y+2L^z-3}$
Excitations	Anyons	Fractons, Lineons
Low energy QFT	$\mathbb{Z}_2$ gauge theory $\frac{2}{2\pi}\,\hat{a} \wedge da$	???

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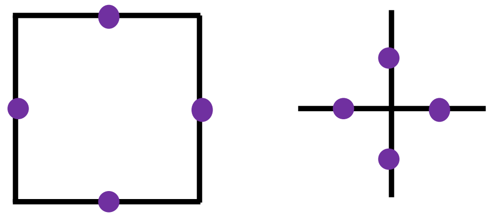
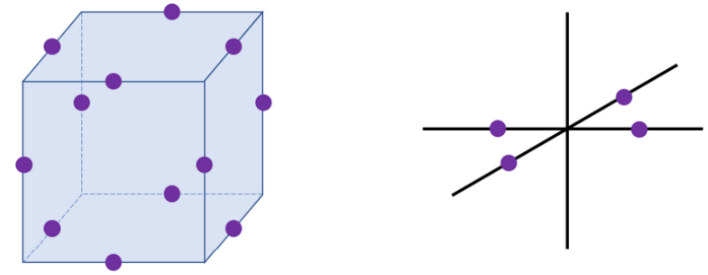
Lattice interaction		
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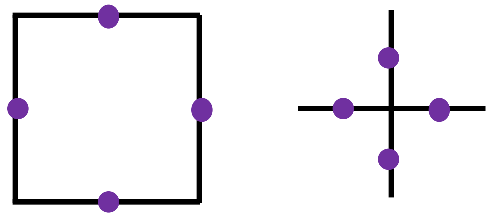
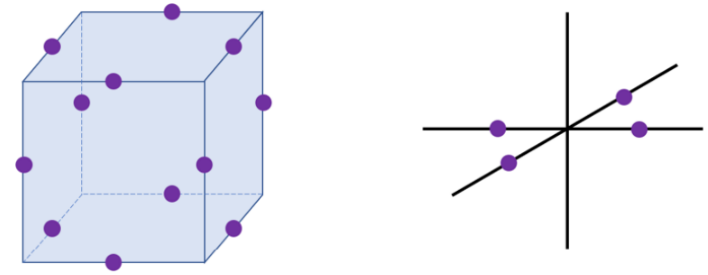
Lattice interaction		
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(2+1)d $\mathbb{Z}_2$ Gauge Theory

- Higgs the ordinary $U(1)$ gauge theory a_μ to $\mathbb{Z}_2$:

(dimensionful couplings suppressed)

$$\mathcal{L}_{Higgs} = \left(\partial_\mu \varphi - \mathbf{2} a_\mu\right)^2 - f_{\mu\nu} f^{\mu\nu}$$

$$a_\mu \sim a_\mu + \partial_\mu \alpha$$

$$\varphi \sim \varphi + \mathbf{2}\alpha$$

$\varphi \sim \varphi + 2\pi$ is a charge-2, real Stueckelberg scalar field. $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the field strength.

- At low energy, this Higgs model can be dualized to a Chern-Simons theory [Maldacena-Moore-Seiberg, Banks-Seiberg, Kapustin-Seiberg]

$$\mathcal{L} = \frac{\mathbf{2}}{2\pi} \epsilon^{\mu\nu\rho} \hat{a}_\mu \partial_\nu a_\rho$$

Duality: $\hat{a}_\mu \leftrightarrow \varphi$

$\mathbb{Z}_2$ Gauge Theory

- The $\mathbb{Z}_2$ gauge theory can be derived by the following steps:

1. Gauge theory a_μ with ordinary $U(1)$ gauge symmetry

2. Higgs the gauge group to $\mathbb{Z}_2$

Strategy for the X-cube Model

- We will follow the same path as in the toric code

1. Gauge theory A with **dipole** $U(1)$ gauge symmetry

2. Higgs the gauge group to $\mathbb{Z}_2$

QFT for Fractons

- New elements in our nonstandard QFT for fractons
  - Spacetime symmetry: non-relativistic, no rotation symmetry
  - Exotic global and gauge symmetries
  - Discontinuous field configuration

Spacetime Symmetry

- Spacetime symmetry in 3+1d:
  - 90 degree spatial rotations = cubic group = S_4
  - Continuous translations in space and time

- In relativistic QFT, fields are in the irreps of the Lorentz group, e.g. scalar, vector, n-form…

- Our fields are in the irreps of the cubic group.

Space Symmetry: Cubic Group S_4

(ignoring reflections)

Its representations are

- **1** – the trivial representation (a scalar)
- **3** – the vector representation V^i: V^x, V^y, V^z

The $SO(3)$ traceless symmetric tensor is decomposed as $\mathbf{3}' \oplus \mathbf{2}$

- **3′** – $T^{(ij)}$ with $i \neq j$: T^{xy}, T^{yz}, T^{zx}
- **2** – D^{ii} with vanishing trace: D^{xx}, D^{yy}, $D^{zz} = -D^{xx} - D^{yy}$. We will also label them as $D^{[ij]k} = \epsilon^{ijk} D^{kk}$
- **1′** – $P^{(xyz)}$. It arises in the three index symmetric tensor of $SO(3)$

Strategy

- We will follow the same path as in toric code

1. Gauge theory A with **dipole** $U(1)$ gauge symmetry

2. Higgs the gauge group to $\mathbb{Z}_2$

Dipole Gauge Symmetry

- Gauge fields and gauge transformations [Xu-Wu, Slagle-Kim, Bulmash-Barkeshli, Ma-Hermele-Chen, You-Burnell-Hughes…]

S_4 irrep

$$A_0 \sim A_0 + \partial_0 \alpha \quad \mathbf{1}$$

$$A_{ij} \sim A_{ij} + \partial_i \partial_j \alpha \quad \mathbf{3'}$$

$i, j, k = x, y, z$
$(i \neq j \neq k)$

- Gauge-invariant electric and magnetic fields

$$E_{ij} = \partial_0 A_{ij} - \partial_i \partial_j A_0 \quad \mathbf{3'}$$

$$B_{[ij]k} = \partial_i A_{jk} - \partial_j A_{ik} \quad \mathbf{2}$$

$U(1)$ Tensor Gauge Theory

$$A_0 \sim A_0 + \partial_0 \alpha \qquad A_{ij} \sim A_{ij} + \partial_i \partial_j \alpha$$

- $U(1)$ tensor gauge theory

$$\mathcal{L}_{U(1)} = \frac{1}{2g_e^2} E_{ij} E^{ij} - \frac{1}{2g_m^2} B_{[ij]k} B^{[ij]k}$$

- Gauss law

$$\partial_x \partial_y E^{xy} + \partial_y \partial_z E^{yz} + \partial_z \partial_x E^{zx} = 0$$

- This is a gapless fracton model.

Strategy

- We will follow the same path as in toric code

1. Gauge theory A with **dipole** $U(1)$ gauge symmetry
2. Higgs the gauge group to $\mathbb{Z}_2$

Higgsing the Dipole Gauge Symmetry

[Seiberg-SHS 2020]

(dimensionful couplings suppressed)

- Higgs the $U(1)$ tensor gauge theory A to $\mathbb{Z}_2$:

$$\mathcal{L}_{Higgs} = (\partial_0\phi - \mathbf{2}A_0)^2 + \left(\partial_{ij}\phi - \mathbf{2}A_{ij}\right)^2 + \left(E_{ij}^2 - B_{[ij]k}^2\right)$$

$$A_0 \sim A_0 + \partial_0\alpha, \qquad A_{ij} \sim A_{ij} + \partial_i\partial_j\alpha$$

$$\phi \sim \phi + \mathbf{2}\alpha$$

$\phi \sim \phi + 2\pi$ is a charge-2, real Stueckelberg scalar field.

- At low energy, this Higgs model can be dualized to a BF-type Lagrangian

$$\mathcal{L} = \frac{\mathbf{2}}{2\pi}(\hat{A}_0 B + \hat{A}E)$$

Duality: $\left(\hat{A}_0^{[ij]k}, \hat{A}^{ij}\right) \leftrightarrow \phi$

$\mathbb{Z}_2$ Tensor Gauge Theory

[Slagle-Kim 2017, Seiberg-SHS 2020]

$$\mathcal{L} = \frac{2}{2\pi}\left(\hat{A}_0 B + \hat{A}E\right) = \frac{2}{2\pi}(A_0\hat{B} + A\hat{E})$$

(indices suppressed)

- Two sets of gauge fields (A_0, A_{ij}) and $(\hat{A}_0^{[ij]k}, \hat{A}^{ij})$

  S_4 irrep: (**1** , **3'**) (**2** , **3'**)

- Gauge symmetry:

$$A_0 \to A_0 + \partial_0\alpha, \qquad A \to A + \partial\partial\alpha$$

$$\hat{A}_0 \to \hat{A}_0 + \partial_0\hat{\alpha}, \qquad \hat{A} \to \hat{A} + \partial\hat{\alpha}$$

$i, j, k = x, y, z$
$i \neq j \neq k$

Higgsing and Dualizing

(2+1)d $\mathbb{Z}_2$ gauge theory	(3+1)d $\mathbb{Z}_2$ tensor gauge theory [Seiberg-SHS 2020]
$\mathcal{L} = \left(\partial_\mu \varphi - 2a_\mu\right)^2 - f_{\mu\nu} f^{\mu\nu}$	$\mathcal{L} = (\partial_0 \phi - 2A_0)^2 + \left(\partial_i \partial_j \phi - 2A_{ij}\right)^2 + (E_{ij}^2 - B_{[ij]k}^2)$
Higgsing RG ↓ Dualizing: $\hat{a}_\mu \leftrightarrow \varphi$	Higgsing RG ↓ Dualizing: $\left(\hat{A}_0^{[ij]k}, \hat{A}^{ij}\right) \leftrightarrow \phi$
$\mathcal{L} = \frac{2}{2\pi} \epsilon^{\mu\nu\rho} \hat{a}_\mu \partial_\nu a_\rho$	$\mathcal{L} = \frac{2}{2\pi} \left(\hat{A}_0 B + \hat{A} E\right)$

$\mathbb{Z}_N$ Tensor Gauge Theory

- It is straightforward to generalize to $\mathbb{Z}_N$ tensor gauge theory (indices suppressed):

$$\mathcal{L} = \frac{N}{2\pi}\left(\hat{A}_0 B + \hat{A}E\right) = \frac{N}{2\pi}(A_0\hat{B} + A\hat{E})$$

- The level $N \in \mathbb{Z}$ is quantized by a discontinuous large gauge transformation α on the Euclidean four-torus with lengths $\ell^\tau, \ell^x, \ell^y, \ell^z$:

$$\alpha(\tau, x, y, z) = 2\pi[\frac{x}{\ell^x}\Theta(y - y_0) + \frac{y}{\ell^y}\Theta(x - x_0) - \frac{xy}{\ell^x\ell^y}]$$

Does this do the job?

Let's reproduce all the peculiarities of fractons.

Peculiarities of the X-cube Model

- Number of ground states grows exponentially in the linear size L^x, L^y, L^z of the system (assuming periodic boundary condition).

  $$\text{ground state degeneracy} = 2^{2L^x+2L^y+2L^z-3}$$

  It becomes infinite in the continuum limit $L^i \to \infty$.

- The ground state degeneracy is robust: small deformations by local operators cannot lift the degeneracy in perturbation theory.

- Excitations have restricted mobility.

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  ↑ will return to this in the end

- The ground state degeneracy is robust: small deformations by local operators cannot lift the degeneracy in perturbation theory.

- Excitations have restricted mobility.

Robustness

- Small perturbations at short distances become local operators in the long-distance QFT. If they are relevant, they destabilize the system.

$$\mathcal{L} = \frac{2}{2\pi}(\hat{A}_0 B + \hat{A} E)$$

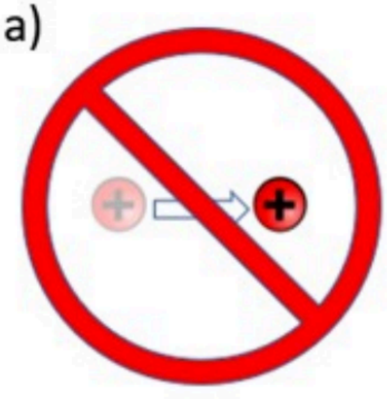
- Equation of motion sets all the local operators (electric and magnetic fields) to **zero**:

$$E = B = \hat{E} = \hat{B} = 0$$

- Since there is no local operator in the $\mathbb{Z}_2$ tensor gauge theory, it is robust.
- Similar to Chern-Simons theory.

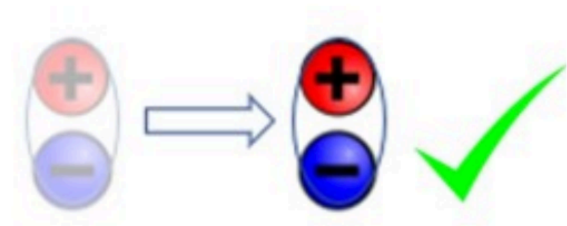
Restricted Mobility

a)



a single fracton
cannot move

b)



but two of them
can move together

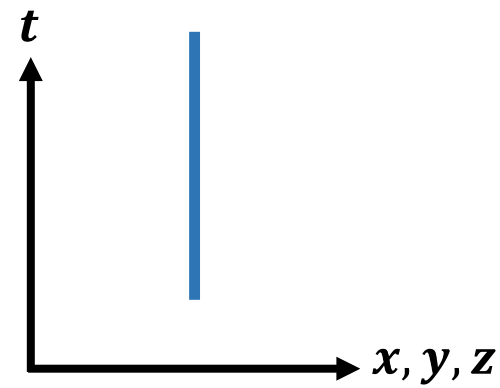
How do we explain this from quantum field theory?

Fractons as Defects of A

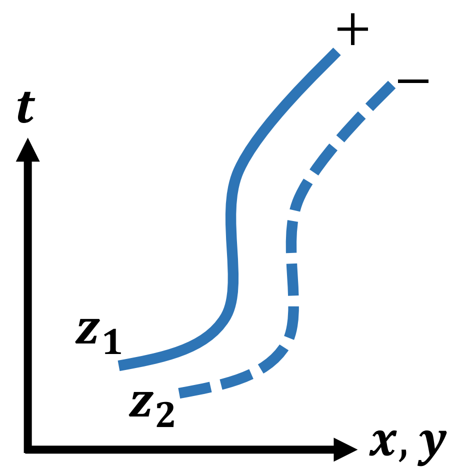
$$A_0 \sim A_0 + \partial_0 \alpha \qquad A_{ij} \sim A_{ij} + \partial_i \partial_j \alpha$$

- A single fracton

$$\exp[i \int_{-\infty}^{\infty} dt\, A_0]$$



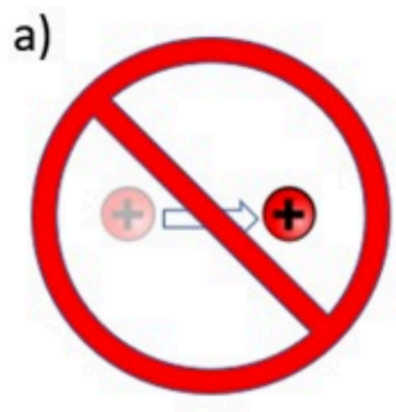
- It can **NOT** move because there is no connection A_i.
- A pair of fractons with opposite charges can move



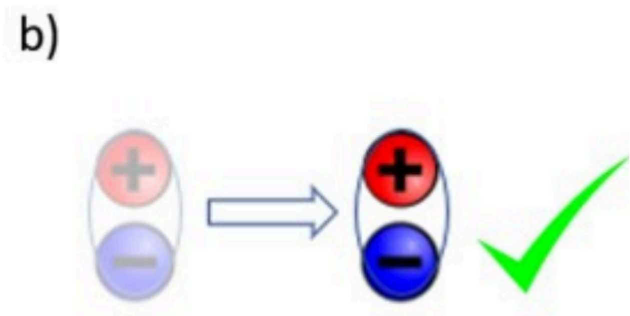
Wilson strip

$$\exp\left[i \int_{z_1}^{z_2} dz \int_{C \in (t,x,y)} (\partial_z A_0\, dt + A_{zx}\, dx + A_{zy}\, dy) \right]$$

where C is a spacetime curve in t, x, y. It is a planon.



a single fracton
cannot move



but two of them
can move together

The restricted mobility of fractons is explained by the gauge invariance of the defects

Lineons as Defects of $\hat{A}$

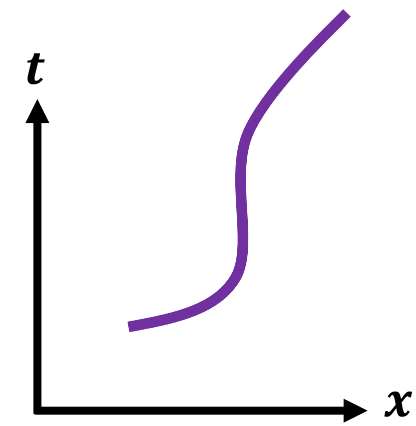
$$\hat{A}_0^{k(ij)} \sim \hat{A}_0^{k(ij)} + \partial_0 \hat{\alpha}^{k(ij)} \quad \hat{A}^{ij} \sim \hat{A}^{ij} + \partial_k \hat{\alpha}^{k(ij)}$$

- Lineons come with three species associated with x, y, z
- A single lineon of species x can move along the x direction, but not y, z

Wilson line

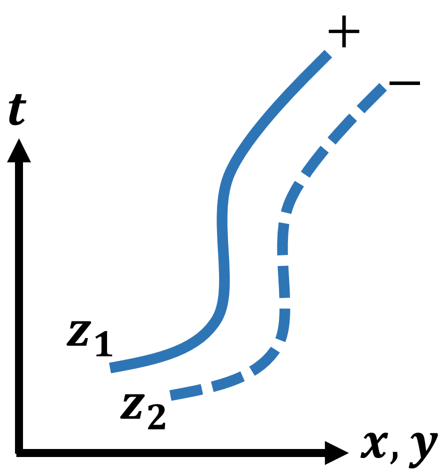
$$\exp\left[i \int_{\hat{C} \in (t,x)} (\hat{A}_0^{x(yz)} dt + \hat{A}^{yz} dx) \right]$$

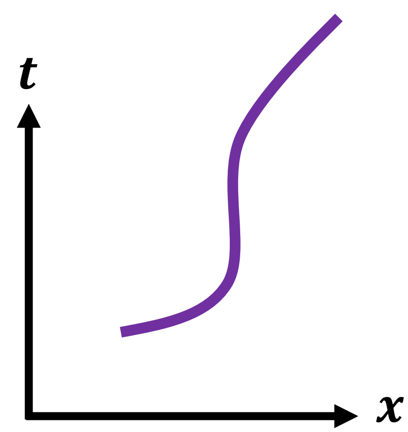
where $\hat{C}$ is a spacetime curve in t, x.



Cast of Characters in X-cube Model

- Fractons and lineons are represented by Wilson defects in the continuum $\mathbb{Z}_2$ tensor gauge theory.
- Similar to the anyons of the ordinary $\mathbb{Z}_2$ gauge theory.

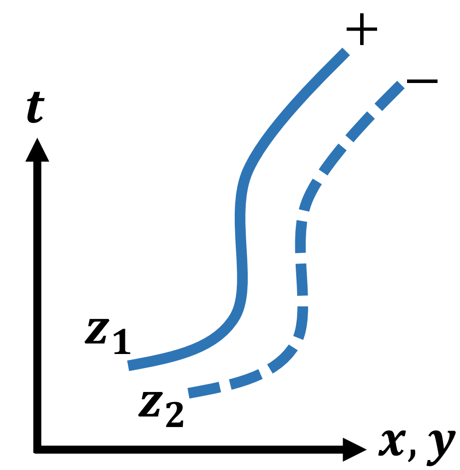




Quasi-Topological Defects

$$\exp\left[i\int_{z_1}^{z_2} dz \int_{C\in(t,x,y)} (\partial_z A_0\, dt + A_{zx}\, dx + A_{zy}\, dy)\right]$$

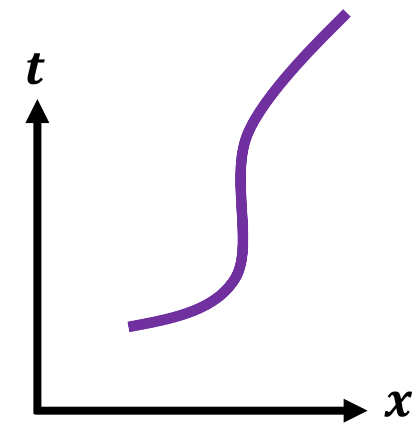
- The Wilson strip (planon) is not completely topological.
- But since the electric and magnetic fields $E = B = 0$ vanish, the Wilson strip is topological in C.
- It's topological in t, x, y, but not in z.
- At a fixed time it is a strip operator generating $\mathbb{Z}_2$ dipole global symmetry. In quantum information theory, this is known as a logical operator.



Quasi-Topological Defects

$$\exp\left[i \int_{\hat{C}\in(t,x)} (\hat{A}_0^{x(yz)} dt + \hat{A}^{yz} dx)\right]$$

- Similarly the hatted Wilson line (lineon) is <u>not</u> completely topological.
- But since the electric and magnetic fields $\hat{E} = \hat{B} = 0$ vanish, the hatted Wilson line is topological in $\hat{C}$.
- It's topological in t, x, but not in y, z.
- At a fixed time it is a line operator generating a $\mathbb{Z}_2$ tensor global symmetry. In quantum information theory, this is known as a logical operator.



Ground State Degeneracy = $2^{2L^x+2L^y+2L^z-3}$

- The ground state degeneracy is infinite in the continuum limit.
- We will regularize the theory on a lattice.
- Global symmetry operators (logical operators)

$\mathbb{Z}_2$ tensor global symmetry (line) $\mathbb{Z}_2$ dipole global symmetry (strip)

nontrivial algebra

- The symmetry operators form $2L^x + 2L^y + 2L^z - 3$ pairs of Heisenberg algebra (clock and shift)

$$AB = -BA\,, \qquad A^2 = B^2 = 1$$

- The $2^{2L^x+2L^y+2L^z-3}$ ground states are in the minimal representation of this algebra.

Fracton Peculiarities	**QFT Explanations**
Ground State Degeneracy $= 2^{2L^x+2L^y+2L^z-3}$	Exotic Global Symmetries and Their Algebras
Robustness	Absence of Local Operators
Restricted Mobility	Gauge Invariance of the Wilson Defects

Conclusion

- Extending the framework of QFT to incorporate fractons.
- $\mathbb{Z}_2$ tensor gauge theory: the low-energy limit of the X-cube model.

$$\mathcal{L} = \frac{2}{2\pi}\left(\hat{A}_0 B + \hat{A} E\right)$$

- It captures all the universal peculiarities of the X-cube model.
- Its defects and operators are topological in some directions but not in others.

Outlook on Fractons

- QFT for other fracton models
  - e.g. checkerboard model [Gorantla-Lam-Seiberg-SHS, to appear]
- Place the theory on more general manifolds
- Braiding between fractons and lineons
- Is there a mathematical formalism similar to the unitary modular tensor category classifying gapped fracton phases?

Many more to explore!

Thank you!

And stay healthy!