# $\mathcal{W}$-algebras

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# Outline

1. Introduction
2. Triality of $\mathcal{W}$-algebras (with Andrew Linshaw)
3. Translation of $\mathcal{W}$-algebras (with Tomoyuki Arakawa and Boris Feigin)

# Introduction

Historic use of VOAs:

# VOAs in string theory

- The two-dimensional worldsheet theory of a string is described by a two-dimensional conformal field theory (CFT)
- The algebraic notion of the symmetry or chiral algebra of a CFT is that of a vertex operator algebra (VOA)
- Roughly speaking: understanding CFTs is the same as understanding the representation theory of VOAs.

# VOAs for mathematics

- The denominator identity of the monster Lie algebra of physical states of the bosonic string associated to the monster VOA allowed to prove the monster moonshine conjecture (Borcherds)
- Representation categories of strongly regular VOAs are modular tensor categories (Moore-Seiberg, Huang)
- Interesting connections to geometry, topology and representation theory, ...

Modern use of VOAs:

# VOAs for higher dimensional physics

There are many 2-D/3-D and 2-D/4-D correspondences and a common theme is that representation theory data of vertex algebras (VOAs) appears as meaningful invariants of higher dimensional theories.

This gives many more and new connections to geometry, topology and number theory.

The basic VOA is associated to a Lie (super)algebra with invariant bilinear form and called the affine vertex (super)algebra. All known VOAs can be constructed from affine ones by iterating standard constructions as invariant subalgebras, extensions and cohomologies.

# Affine VOA

1. $\mathfrak{g}$ Lie (super)algebra with $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ non-degenerate and $k \in \mathbb{C}$. The universal affine VOA $V^k(\mathfrak{g}, B) = V^k(\mathfrak{g})$ is strongly and freely generated by fields $J^x$ with $x$ in a basis of $\mathfrak{g}$ and operator product

$$J^x(z)J^y(w) \sim \frac{kB(x,y)}{(z-w)^2} + \frac{J^{[x,y]}}{(z-w)}$$

2. $L_k(\mathfrak{g})$ simple quotient of $V^k(\mathfrak{g})$
3. $\mathfrak{g}$ simple Lie algebra, $B$ Killing form normalized as usual (long roots have norm two), then $L_k(\mathfrak{g})$ integrable if $k$ is a positive integer $\to$ WZW theory.

# $\mathcal{W}$-algebra (Kac-Roan-Wakimoto)

$f$ nilpotent in $\mathfrak{g}$, then there exists a ghost superalgebra $C_f$ and a fermionic field $d^{f,k}$, such that $(V^k(\mathfrak{g}) \otimes C_f, d_0^{f,k})$ is a complex and the homology is the $\mathcal{W}$-algebra associated to $\mathfrak{g}, f$ and $k$,

$$H(V^k(\mathfrak{g}) \otimes C_f, d_0^{f,k}) = \mathcal{W}^k(\mathfrak{g}, f).$$

Examples:

1. $f = 0$, then $\mathcal{W}^k(\mathfrak{g}, f) = V^k(\mathfrak{g})$,
2. $f$ principal nilpotent then $\mathcal{W}^k(\mathfrak{g}, f)$ is the "usual" $\mathcal{W}$-algebra of $\mathfrak{g}$.
3. The $\mathcal{W}$-algebra of $\mathfrak{sl}_2$ is the Virasoro algebra.

# Some modules for $\mathfrak{g}$ Lie algebra

1. Let $\psi = k + h^\vee$ ($h^\vee$ the dual Coxeter number of $\mathfrak{g}$)
2. Set $M^\psi(\lambda)$ the Weyl module of $V^k(\mathfrak{g})$ of highest-weight $\lambda \in P^+$, i.e. $M^\psi(\lambda)$ has top level the irreducible highest-weight representation of $\mathfrak{g}$ of highest-weight $\lambda$ (and $M^\psi(\lambda)$ is simple for generic $k$)
3. Set $M_f^\psi(\lambda) = H(M^\psi(\lambda) \otimes C_f, d_0^{f,k})$.
4. The category whose simple objects are the $M_f^\psi(\lambda)$ for $\lambda \in P^+$ is denoted by $KL_f^\psi(\mathfrak{g})$.
5. Conjecturally $KL_f^\psi(\mathfrak{g}) \cong KL_0^\psi(\mathfrak{g})$ as ribbon categories, but the existence of tensor category is not known unless $f = 0$ (Kazhdan-Lusztig) or $\mathfrak{g} = \mathfrak{sl}_2$ (C-Jiang-Hunziker-Ridout-Yang).

## Definition

Let $W \subset V$ be VOAs, then the subalgebra $C = \mathrm{Com}(W, V)$ that commutes with $W$ is called a coset VOA.
The notation

$$C = \frac{V}{W}$$

is used in physics.

## Remark (Feigin, I. B. Frenkel-Garland-Zuckerman)

*If $W = V^k(\mathfrak{g})$ for generic $k$, then $C$ is also realized as a relative semi-infinite Lie algebra cohomology*

$$C = H^0_{rel}(\mathfrak{g}, V \otimes V^{-k-2h^\vee}(\mathfrak{g}))$$

*In physics this is essentially the realization of cosets as gauged WZW-models.*

# Examples

Let $\mathfrak{g} = \mathfrak{sl}_{n+m}$ and $f_{n,m}$ be the nilpotent element corresponding to the partition $(n, 1, \ldots, 1)$ of $n+m$. Let $\psi = k + n + m$. For $n > 0$, we define $\mathcal{W}^\psi(n, m) := \mathcal{W}^k(\mathfrak{sl}_{n+m}, f_{n,m})$ (for $n = 0$ the definition is a bit different) The best known cases are

1. the principal $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_n) \cong \mathcal{W}^\psi(n, 0)$,
2. the subregular $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{\text{subreg}}) \cong \mathcal{W}^\psi(n, 1)$,
3. the affine vertex algebra $V^k(\mathfrak{sl}_{m+1}) \cong \mathcal{W}^k(\mathfrak{sl}_{m+1}, 0) \cong \mathcal{W}^\psi(1, m)$,
4. the minimal $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_{m+2}, f_{\text{min}}) \cong \mathcal{W}^\psi(2, m)$.

# Examples

Let $\mathfrak{g} = \mathfrak{sl}_{n|m}$ and the nilpotent element $f_{n|m}$ corresponding to the super partition $(n|1, \ldots, 1)$ of $n|m$. Let $\psi = k + n - m$. For $n + m \geq 2$ and $n \neq m$, we define $\mathcal{V}^{\psi}(n, m) := \mathcal{W}^k(\mathfrak{sl}_{n|m}, f_{n|m})$ and slightly different for other cases.

The best known cases are

1. the principal $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_n) \cong \mathcal{V}^{\psi}(n, 0)$,
2. the principal $\mathcal{W}$-superalgebra $\mathcal{W}^k(\mathfrak{sl}_{n|1}, f_{n|1}) \cong \mathcal{V}^{\psi}(n, 1)$,
3. the affine vertex superalgebra $V^k(\mathfrak{sl}_{1|m}) \cong \mathcal{W}^k(\mathfrak{sl}_{1|m}, 0) \cong \mathcal{V}^{\psi}(1, m)$,
4. the minimal $\mathcal{W}$-superalgebra $\mathcal{W}^k(\mathfrak{sl}_{2|m}, f_{\min}) \cong \mathcal{V}^{\psi}(2, m)$ for $m \neq 2$.

# Examples of cosets of $\mathcal{W}$-algebras by their affine subVOA

$$\mathcal{C}^{\psi}(n,m) := \begin{cases} \mathrm{Com}\left(V^{\psi-m-1}(\mathfrak{gl}_m), \mathcal{W}^{\psi}(n,m)\right) & \text{for } m \geq 1, \\ \mathcal{W}^{\psi}(n,0) & \text{for } m = 0. \end{cases}$$

$$\mathcal{D}^{\psi}(n,m) := \begin{cases} \mathrm{Com}\left(V^{-\psi-m+1}(\mathfrak{gl}_m), \mathcal{V}^{\psi}(n,m)\right) & \text{for } n \neq m \text{ and } m \geq 1, \\ \mathrm{Com}\left(V^{-\psi-n+1}(\mathfrak{sl}_n), \mathcal{V}^{\psi}(n,n)\right)^{\mathrm{GL}_1} & \text{for } n = m \text{ and } n \geq 2, \\ \mathcal{V}(1,1)^{\mathrm{GL}_1} & \text{for } n = m = 1, \\ \mathcal{V}^{\psi}(n,0) & \text{for } m = 0. \end{cases}$$

# VOAs for higher dimensional physics

Examples:

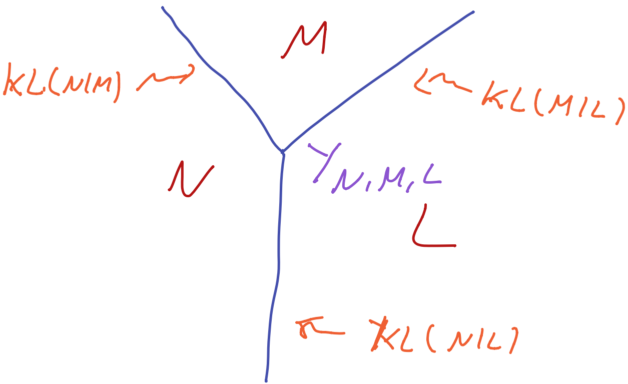
1. AGT-correspondence: $N = 2$ supersymmetric 4-dimensional gauge theory to Toda field theory correspondence
2. Corner VOAs in S-duality in $N = 4$ supersymmetric 4-dimensional GL-twisted gauge theories
3. VOAs and 4-manifolds
4. many more

These three examples predict new properties for $\mathcal{W}$-algebras.

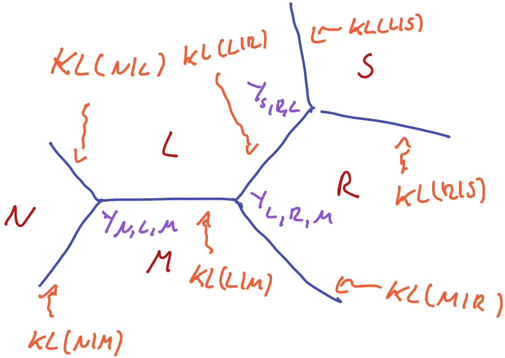
# Webs of $Y$-algebras (Gaiotto, Rapcak, Prochazka)

1. VOA $Y_{N,M,L}[\psi]$ at the junction of interfaces in $N = 4$ supersymmetric 4-dimensional GL-twisted gauge theories with gauge groups $U(N), U(M), U(L)$ and coupling $\psi$.
2. interface line defects are associated with categories $KL(N|M), KL(M|L)$ and $KL(L|N)$ of modules of $Y_{N,M,L}$
3. To a web one associates an extension of tensor products of $Y$-algebras along the common subcategories.

# Y-algebras



# Web


KL(N|L)
KL(L|R)
KL(L|S)
S
$\gamma_{S,R,L}$
L
R
KL(R|S)
N
$\gamma_{N,L,M}$
$\gamma_{L,R,M}$
M
KL(L|M)
KL(M|R)
KL(N|M)

# Some $Y$-algebras

$$
\begin{aligned}
Y_{0,M,N}[\psi] &= \mathcal{C}^{\psi}(N-M, M) \otimes \mathcal{H}, \qquad M \leq N,\\
Y_{0,M,N}[\psi] &= \mathcal{C}^{-\psi+1}(M-N, N) \otimes \mathcal{H}, \qquad M > N,\\
Y_{L,0,N}[\psi] &= \mathcal{D}^{\psi}(N, L) \otimes \mathcal{H},\\
Y_{L,M,0}[\psi] &= \mathcal{D}^{-\psi+1}(M, L) \otimes \mathcal{H}.
\end{aligned}
$$

$\mathcal{H}$ denotes a rank one Heisenberg VOA (a free boson).

# Gluing VOAs

The problem of VOA extension is a categorical problem.
Under certain technical assumptions on the representation categories of two VOAs $V_1$ and $V_2$, the following holds:

**Theorem (C-Kanade-McRae)**

*$V_1 \otimes V_2$ can be extended to a larger simple VOA $V$ for which $V_1$ and $V_2$ form a mutually commuting pair if and only if these two VOAs have subcategories that are braid-reversed equivalent. In formula (and if $\mathcal{C}$ is semisimple)*

$$V = V_1 \boxtimes_{\mathcal{C}} V_2 \cong \bigoplus_{X \in Irr(\mathcal{C})} X \otimes \tau(X)^*.$$

*Here $\tau$ denotes the equivalence.*

Sloppily speaking: two VOAs can glue to a larger VOA if and only if they share a common category.

Progress on many powerful conjectures from gauge theory:

# Triality of Gaiotto-Rapcak

Let $\psi$ be defined by

$$\psi = -\frac{\epsilon_2}{\epsilon_1}, \qquad \epsilon_1 + \epsilon_2 + \epsilon_3 = 0$$

and set

$$Y^{\epsilon_1,\epsilon_2,\epsilon_3}_{N_1,N_2,N_3} := Y_{N_1,N_2,N_3}[\psi].$$

Then with this notation the triality conjecture is

$$Y^{\epsilon_{\sigma(1)},\epsilon_{\sigma(2)},\epsilon_{\sigma(3)}}_{N_{\sigma(1)},N_{\sigma(2)},N_{\sigma(3)}} \cong Y^{\epsilon_1,\epsilon_2,\epsilon_3}_{N_1,N_2,N_3} \qquad \text{for } \sigma \in S_3.$$

**Remark**

*A related triality had been conjectured earlier by Gaberdiel-Gopakumar in the context of higher spin gravity on $AdS_3/CFT_2$ correspondence*

# Triality of $\mathcal{W}$-algebras

**Theorem (C-Linshaw)**

*Let $n \geq m$ be non-negative integers. As one-parameter vertex algebras*

$$\mathcal{D}^{\psi}(n,m) \cong \mathcal{C}^{\psi^{-1}}(n-m,m) \cong \mathcal{D}^{\psi'}(m,n)$$

*with $\psi'$ defined by $\frac{1}{\psi} + \frac{1}{\psi'} = 1$.*

A *one-parameter vertex algebra* is a vertex algebra over some localization of a polynomial ring in one variable, which in our case is the level $k$ or equivalently the critically shifted level $\psi$.

# On the proof

1. We show that $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$ are simple as one-parameter vertex algebras; equivalently, they are simple for generic values of $\psi$.
2. We find minimal strong generating sets for $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$.
3. We show that aside from certain extreme cases $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$ are one-parameter quotients of the universal two-parameter vertex algebra $\mathcal{W}(c, \lambda)$ constructed by Linshaw. Its simple one-parameter quotients are in bijection with a family of curves in the parameter space $\mathbb{C}^2$, and we finish the proof by explicitly describing these curves. The extreme cases are easily verified separately.

# Consequences

## Corollary (C-Linshaw)

*The following conjectures are true.*

1. *The Gaiotto-Rapcak conjecture that* $Y_{L,M,N}[\psi]$ *is a simple quotient of* $\mathcal{H} \otimes \mathcal{W}(c, \lambda)$*, when one of the labels* $L, M, N$ *is zero.*
2. *The Gaiotto-Rapcak triality conjecture for the algebras* $Y_{L,M,N}[\psi]$ *when one of the labels is zero.*
3. *The Prochazka-Rapcak formula for the truncation curve of* $Y_{0,M,N}[\psi]$*.*
4. *The Prochazka-Rapcak conjecture that* $Y_{0,M,N}[\psi]$ *is of type* $\mathcal{W}(1, 2, 3, \dots, (M-1)(N-1)-1)$*.*

Is there a direct connection between the $\mathcal{W}$-superalgebras and not only via their cosets?

# Peter-Weyl for VOAs

Recall that the space of functions on a compact Lie group is

$$\bigoplus_{\lambda \in P^+} \rho_\lambda \otimes \rho_\lambda^*$$

as a module for the Lie algebra of left and right invariant vector fields. VOA-versions of this are key!

# qGL-kernel VOAs

## Conjecture (C-Gaiotto)

*Let* $\frac{1}{\psi} + \frac{1}{\psi'} = 1$ *and* $\psi$ *be generic and* $f, f'$ *be nilpotent, then*

- $A[\mathfrak{sl}_N, \psi] := \bigoplus_{\lambda \in P^+} M^{\psi}(\lambda) \otimes M^{\psi'}(\lambda) \otimes V_{\sqrt{N}\mathbb{Z} + \frac{s(\lambda)}{\sqrt{N}}}$
- $A[\mathfrak{sl}_N, f, \psi] := \bigoplus_{\lambda \in P^+} M_f^{\psi}(\lambda) \otimes M^{\psi'}(\lambda) \otimes V_{\sqrt{N}\mathbb{Z} + \frac{s(\lambda)}{\sqrt{N}}}$
- $A[\mathfrak{sl}_N, f, f'\psi] := \bigoplus_{\lambda \in P^+} M_f^{\psi}(\lambda) \otimes M_{f'}^{\psi'}(\lambda) \otimes V_{\sqrt{N}\mathbb{Z} + \frac{s(\lambda)}{\sqrt{N}}}$

*are simple super VOAs for generic* $\psi$.
*(*$s : P^+ \to P/Q \xrightarrow{\cong} \frac{1}{\sqrt{N}}\mathbb{Z}/\sqrt{N}\mathbb{Z}$ *and* $V_L$ *denotes the lattice VOA of* $L = \sqrt{N}\mathbb{Z}$*.)*

Conjecture and its variants are known to be true in many but not all cases.

## Conjecture (C-Linshaw)

*For generic k and $n \geq m$.,*

1. $Com\left(V^{\psi-m-1}(\mathfrak{sl}_m) \otimes \mathcal{H}, A[\mathfrak{sl}_N, f_{n-m,m}, \psi]\right) \cong \mathcal{V}^{1-\psi'}(m,n).$
2. $H^0_{rel}(\mathfrak{gl}_m, \mathcal{W}^{\psi}(n-m,m) \otimes A[\mathfrak{sl}_m, 1-\psi]) \cong \mathcal{V}^{\psi^{-1}}(n,m).$

## Remark

*By construction*

$Com\left(V^{\psi'-n}(\mathfrak{sl}_n) \otimes \mathcal{H}, A[\mathfrak{sl}_n, f_{n-m,m}, \psi]\right) \cong \mathcal{W}^{\psi}(n-m,m)$

## Theorem (C-Linshaw)

*First part of the Conjecture is true on the level of characters.*

Connection to geometry, that is generalizations of Schiffmann-Vasserot.

Reminder: A famous result of Schiffmann and Vasserot says that the equivariant cohomology of the moduli space of torsion free coherent sheaves on $\mathbb{CP}^2$ equipped with a framing along $\mathbb{CP}^1_\infty \subset \mathbb{CP}^2$ is a $\mathcal{W}(\mathfrak{gl}_n)$-algebra Verma module and its fundamental class is a Whittaker vector. The norm of the Whittaker vector is Nekrasov's instanton partition function, i.e. the regularized integral of the class 1.

**Theorem (Rapcak, Soibelman, Yang, Zhao)**

*The $\mathcal{W}_{N,M,L}$-algebras of Bershtein-Feigin-Merzon act on the cohomology of the moduli space of spiked instantons of Nekrasov.*

**Conjecture (Bershtein-Feigin-Merzon)**

$\mathcal{W}_{N,M,L} \cong Y_{N,M,L}$ *as one-parameter VOAs.*

The $\mathcal{W}_{N,M,L}$-algebras are defined as intersection of kernels of screening operators on free field algebras.

**Theorem**

*The Conjecture is true for all $N$ and*

1. $M = L = 0$ *(Feigin-Frenkel)*
2. $M = 1$, $L = 0$ *(C-Genra-Nakatsuka)*

The Urod and VOAs for 4-manifolds (Boris Feigin and Sergei Gukov)

Idea: The equivariant cohomology of the moduli space of $G$-Instantons of a smooth 4-manifold $M$ is a module for a VOA, $VOA[M, G]$.

- 4-manifolds are labelled by VOAs
- 3-dimensional boundaries are labelled by full tensor categories of VOA-modules
- Gluing 4-manifolds along common boundaries amounts to gluing VOAs along the common category of modules.

# Blowup

- Take a 4-manifold $M$ and remove a small 4-ball $B$, $M \setminus B$.
- Do the same with $\overline{\mathbb{CP}}^2$.
- Get a glued 4-manifold $M \# \overline{\mathbb{CP}}^2$ by identifying the common boundary $S^3$.
- Claim: $VOA[M \# \overline{\mathbb{CP}}^2] = VOA[M] \otimes \mathcal{U}$.
- $\mathcal{U}$ is the Urod.

### Theorem (Arakawa-C-Feigin (close to completion))

*The quantum Hamiltonian reduction functor commutes with tensoring with integrable representations:*

$$H(M \otimes L, d_0^{f,k+n}) \cong H(M, d_0^{f,k}) \otimes L.$$

*Here $M$ is a module of $V^k(\mathfrak{g})$ at level $k$ and $L$ an integrable module at level $n$. If $M$ and $L$ are vertex algebras this is an isomorphism of vertex algebras, otherwise it is an isomorphism of vertex algebra modules.*

This means that the tensor product of a $\mathcal{W}$-algebra module at level $k$ with an integrable representation $L$ at level $n$ carries an action of the $\mathcal{W}$-algebra at level $k+n$, hence any integrable representation serves as a translation functor of $\mathcal{W}$-algebra modules.

### Remark

*The case $\mathfrak{g} = \mathfrak{sl}_2$ is due to Bershtein-Feigin-Litvinov and this proves a prediction of $S$-duality as observed by Gaiotto and myself.*

Let $G$ be the compact Lie group with simply-laced Lie algebra $\mathfrak{g}$, then the Urod of type $G$ is $L_1(\mathfrak{g})$, but with an unusual choice of conformal vector.

## Theorem (Arakawa-C-Linshaw)

*Let $\mathfrak{g}$ be simply-laced and $k$ generic. Let $\psi = k + h^\vee$ and $\psi'$ satisfy*

$$\frac{1}{\psi} + \frac{1}{\psi'} = 1$$

*Let $f$ be principal nilpotent, $Q$ be the root lattice of $\mathfrak{g}$ and $P^+$ its set of dominant weights, then*

$$V^{k-1}(\mathfrak{g}) \otimes L_1(\mathfrak{g}) \cong \bigoplus_{\lambda \in P^+ \cap Q} M^{\psi}(\lambda) \otimes M_f^{\psi'}(\lambda).$$

## Corollary

*With the same set-up and $f'$ another nilpotent element*

$$\mathcal{W}^{k-1}(\mathfrak{g}, f') \otimes L_1(\mathfrak{g}) \cong \bigoplus_{\lambda \in P^+ \cap Q} M_{f'}^{\psi}(\lambda) \otimes M_f^{\psi'}(\lambda).$$

# Outlook

1. Orthosymplectic triality
2. The Urod allows for an algebraic derivation of the famous Nakajima-Yoshioka blow-up equations
3. Many more miraculous properties of the quantum Hamiltonian reduction functor
4. equivalences between tensor categories of different $\mathcal{W}$-algebras (quantum geometric Langlands predictions)
5. $\mathcal{W}$-algebras and toric Calabi-Yau 3-folds
6. Big VOA challenge: representation theory of affine and $\mathcal{W}$-superalgebras