Supergroup Chern-Simons theory, q-series, and resurgence

200X.XXXXX w/ Francesca Ferrari

Pavel Putrov

ICTP, Trieste

July 27, 2020

Plan/Summary

- Proposal for the value of the index $\hat{Z}^{\mathfrak{sl}(m|n)}[M^3] := \mathrm{Tr}(-1)^F q^{L_0}$ of a system of M5-branes related by dualities to $SU(M|N)$ CS on M^3 (L_0 generates rotations of both $\mathbb{C}$'s):

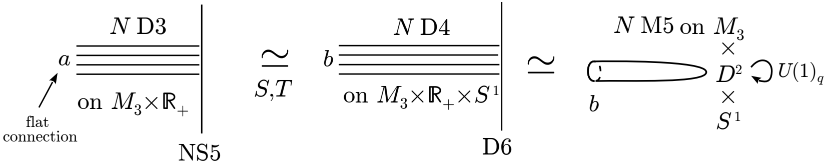
M-theory	T^*M^3	$\times$	$\mathbb{C}$	$\times$	$\mathbb{C}$	$\times$	S^1_{time}
N M5's	M^3	$\times$	$\mathbb{C}$	$\times$	$\{0\}$	$\times$	S^1_{time}
M M5's	M^3	$\times$	$\{0\}$	$\times$	$\mathbb{C}$	$\times$	S^1_{time}

  for a class of closed oriented 3-manifolds M^3.
- Examples and properties of these q-series (resurgence).
- Relation to the quantum invariants of [Costatino–Geer–Patureau-Mirand '14] from non-semisimple category of representations of $\mathcal{U}_q^H(\mathfrak{sl}(M|N))$ (with modified trace) when q is a root of unity ($\mathcal{U}_q^H(\mathfrak{sl}(2|1))$ case worked out in [Ha '18]).

Remark: The case of $SU(N)$ and $U(1|1)$ was considered in [Gukov-PP-Vafa '16, Gukov-Mariño-PP '16, Gukov-Pei-PP-Vafa '17, Gukov-Manolescu '19, . . .]

Brane realization of $SU(N)$ CS

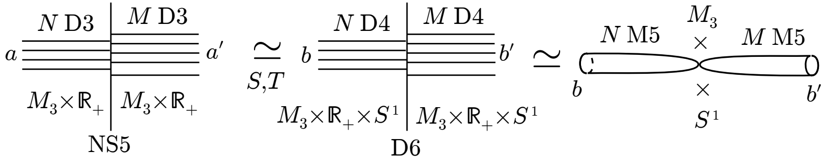
[Witten,. . .]



Coupling constant of the $SU(N)$ SYM living on D3 branes is the analytically continued CS coupling: $-k = \tau$. The fugacity $q = e^{\frac{2\pi i}{k}}$.

Brane realization of $SU(M|N)$ CS

[Vafa,Gaiotto-Witten,Witten-Mikhaylov,...]



Remark: $SU(N|M)$ Chern-Simons also appears in a relation to ABJM theory on S^3 [Drukker-Trancanelli '09, Mariño-Putrov '09]

$U(1|1)$ Chern-Simons theory

$U(1|1)$ CS = Reidemeister-Turaev torsion = SW invariants

[Rozansky-Saleur,Meng-Taubes,Mikhaylov,. . .]

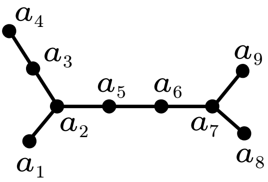
categorification: Monopole/Heegaard Floer homology

Group	Knots		3-manifolds		4-manifolds
	invariant	categorification	invariant	categorification	invariant
$U(1\|1)$	Alexander polynomial	knot Floer homology	3d SW invariants	Monopole Floer homology	4d SW invariants
$SU(2)$	Jones polynomial	Khovanov homology	WRT invariant	?	???

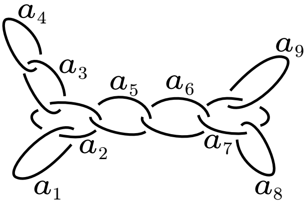
$$U(1|1) \subset SU(2|1) \supset SU(2)$$

Plumbed 3-manifolds

plumbing graph Γ



$M_3(\Gamma)$ = Dehn surgery on



1) cut out tubular neighborhoods of the link components

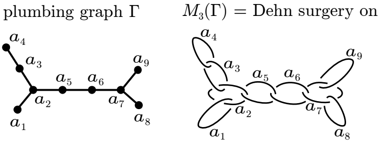
2) glue back in solid tori using $T^{a_i}S \in SL(2,\mathbb{Z})$ transformations of the boundari tori.

L - number of vertices (link components)

$$L \times L \text{ linking matrix } M_{i_1 i_2} = \begin{cases} 1, & i_1, i_2 \text{ connected,} \\ a_i, & i_1 = i_2 = i, \\ 0, & \text{otherwise.} \end{cases}$$

$$H_1(M_3) \cong \mathbb{Z}^L / M\mathbb{Z}^L$$

$SU(2)$ homological blocks for plumbed 3-manifolds



Homological blocks ("Zed-hats") for $SU(2)$ [Gukov-Pei-PP-Vafa '17] (higher rank generalization [Park '19]) :

$$\hat{Z}_b^{\mathfrak{sl}(2)}(q) = q^{-\frac{3L+\sum_i M_{ii}}{4}} \times$$

$$\text{v.p.} \int_{|z_i|=1} \prod_{i \,\in\, \text{Vertices}} \frac{dz_i}{2\pi i z_i} \, (z_i - 1/z_i)^{2-\deg(i)} \cdot \Theta_b^{-2M}(z;q) \quad \in q^{\Delta_b}\mathbb{Z}[[q]]$$

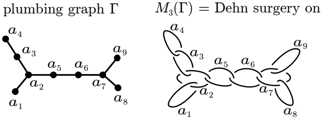
$$\Theta_b^{-2M}(z;q) := \sum_{\ell \in 2M\mathbb{Z}^L + b} q^{-\frac{\ell^T M^{-1} \ell}{4}} \prod_{i=1}^{L} z_i^{\ell_i}$$

M is "weakly negative definite", $\hat{Z}_b^{\mathfrak{sl}(2)} \neq 0$ for

$$b \in Spin^c(M^3) \cong \{b \in \mathbb{Z}^L/2M\mathbb{Z}^L \,|\, b_i = \deg(i) \mod 2\}$$

[Gukov-Manolescu '19]

$SU(2|1)$ homological blocks for plumbed 3-manifolds



Assume the plumbing graph is a tree and $M^3 = M^3(\Gamma)$ is a rational homology sphere ($b_1 = 0$)

$$\hat{Z}^{\mathfrak{sl}(2|1)}_{b,c} = \int_\Gamma \prod_{i\in\mathrm{Vert}} \frac{dy_i}{2\pi i y_i}\frac{dz_i}{2\pi i z_i}\left(\frac{y_i - z_i}{(1-y_i)(1-z_i)}\right)^{2-\mathrm{deg}(i)} \times$$

$$\sum_{\substack{n=b \mod M\mathbb{Z}^L \\ m=c \mod M\mathbb{Z}^L}} q^{m^T M^{-1} n} \prod_j z_j^{m_j} y_j^{n_j}$$

$$b, c \in \mathbb{Z}^L / M\mathbb{Z}^L \cong H_1(M^3, \mathbb{Z})$$

The integral means taking the constant term after expanding the rational function in y and z in the first line in the chamber corresponding to the choice of the contour Γ.

Existence/uniqueness of Γ so that the result is a well defined element of $q^{\Delta_{bc}}\mathbb{Z}[[q]]$? ($\Delta_{bc} = b^T M^{-1} c = \ell k(b,c) \mod 1$)

$SU(2|1)$ homological blocks for plumbed 3-manifolds (some technical details)

$$\hat{Z}^{\mathfrak{sl}(2|1)}_{b,c} = \int\limits_{\Gamma} \prod_{i\in \mathsf{Vert}} \frac{dy_i}{2\pi i y_i}\frac{dz_i}{2\pi i z_i}\left(\frac{y_i - z_i}{(1-y_i)(1-z_i)}\right)^{2-\mathsf{deg}(i)} \times$$

$$\sum_{\substack{n=b \mod M\mathbb{Z}^L \\ m=c \mod M\mathbb{Z}^L}} q^{m^T M^{-1} n} \prod_j z_j^{m_j} y_j^{n_j}$$

1) The contour Γ such that the result is a well defined element of $q^{\Delta_{bc}}\mathbb{Z}[[q]]$ exist iff $-M^{-1}$ is "weakly copositive".

Namely, there exists a vector

$$\alpha_i = \pm 1, i \in \mathrm{Vert}|_{\deg\neq 2}$$

such that the submatrix

$$X_{ij} := -M^{-1}_{ij}\alpha_i\alpha_j,\ i,j \in \mathsf{Vert}|_{\deg>2},$$

is copositive ($v_i \geq 0, \forall i$ *implies* $\sum_{i,j} v_i v_j X_{ij} \geq 0$) *and*

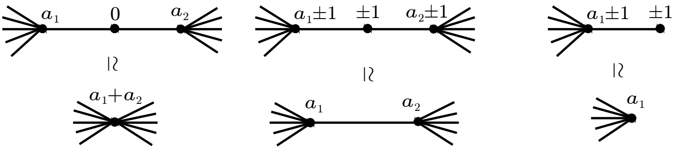
$$\alpha_i\alpha_j M^{-1}_{ij} \leq 0, \qquad \forall i \in \mathsf{Vert}|_{\deg=1},\ j \in \mathsf{Vert}|_{\deg\neq 2},\ j \neq i.$$

2) If such Γ exist there are only two choices (for a generic plumbing), giving the same result.

The choice of the contour is specified by the vector α above.

Invariance under Kirby moves

3d Kirby moves $\Gamma \sim \Gamma'$ moves s.t. $M_3(\Gamma) \cong M_3(\Gamma')$:



The "weak copositivity" condition on M and the q-series $\hat{Z}^{\mathfrak{sl}(2|1)}_{b,c}$ are invariant.

Examples: 3-sphere

- $M^3 = S^3$: $H_1(S^3, \mathbb{Z}) = 0$,

$$\hat{Z}^{\mathfrak{sl}(2|1)} = -1 + 2\sum_{n\geq 1} \frac{q^n}{1-q^n} = -1 + 2\sum_{m} d(m)q^m =$$
$$= -1 + 2(q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 4q^6 + 2q^7 + \ldots)$$

where $d(m)$ is the number of divisors of m (Lambert series).

Examples: Lens spaces

- $M^3 = L(p,1) \cong S^3/\mathbb{Z}_p$: $H_1(S^3,\mathbb{Z}) \cong \mathbb{Z}_p$,

$$\hat{Z}^{\mathfrak{sl}(2|1)}_{bc} = \mathrm{const}_{bc} + 2\sum_m d(m;p,b,c)q^{m/p}$$

where $d(m;p,b,c)$ is the number of positive integer pairs (r,s) satisfying

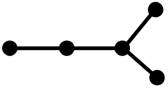
$$\begin{cases} r = b \mod p \\ s = c \mod p \\ rs = m \end{cases}$$

Remarks:

1) $\hat{Z}^{\mathfrak{sl}(n)}[L(p,1)]$ are polynomials in $q^{\frac{1}{p}}$

2) $d(m;p,b,c)$ coincides with Euler characteristic of the moduli space of m $SO(3)$ instantons on $L(p,1)\times\mathbb{R}$ propagating between flat connections labelled by $b\pm c$ [Austin '90].

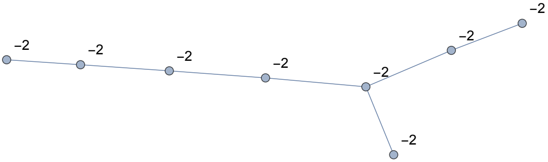
Examples: Seifert 3-manifolds w/ 3 exceptional fiber



For Seifet manifolds with 3 exceptional fibers $\hat{Z}^{\mathfrak{sl}(n)}_{bc}$ are linear combintations of the q-series of the form

$$F(q;\alpha,\beta,\gamma;A,B) := \sum_{m\geq 0} \frac{q^{\alpha m^2+\beta m+\gamma}}{1-q^{Am+B}}.$$

For example for Poincare homology sphere



$$\hat{Z}^{\mathfrak{sl}(2|1)} = -1 + 2q^2 + 2q^3 + 4q^4 + 4q^5 + 6q^6 + 4q^7 + \dots$$

Resurgence

Assume $H_1(M^3, \mathbb{Z}) = 0$ for technical simplicity. Consider asymptotic expansion in $\hbar := -\log q = -2\pi i\tau \to 0$

$$\hat{Z} = \text{singular} + \sum_{n\geq 0} c_n \hbar^n$$

and define the Borel transform

$$B(\xi) := \sum_{n\geq 0} \frac{c_n}{\Gamma(n+1)}\, \xi^n$$

If $\hat{Z}$ originates from a "path-integral of analytically continued CS theory" (for ordinary compact Lie groups studied by [Witten,Kontsevich,Costin-Garoufalidis,Gukov-Mariño-PP,. . .]) we expect the series for $B(\xi)$ to have finite radius of convergence, to be analytically continuable beyond that, with possible singularities at $\xi = 4\pi^2 S$ where $S \mod 1$ is the CS invariant of a flat connection (for the complexified even subgroup).

Is it the case for $\hat{Z}^{\mathfrak{sl}(2|1)}$?

Resurgence for S^3

For S^3 the asymptotic expansion is explicitly known (e.g. [Bettin-Conrey '13]):

$$\hat{Z}^{\mathfrak{sl}(2|1)} = -1 + 2\sum_{n\geq 1} \frac{q^n}{1-q^n} \sim$$

$$\frac{-1+2\pi i/\hbar}{2} - 2\frac{\log(i\hbar)-\gamma}{\hbar} - 2\sum_{n\geq 1} \frac{B_{2n}^2}{(2n)!\,(2n)}\,\hbar^{2n-1}$$

The Borel transform $B(\xi)$ has singularities at $\xi = 4\pi^2 S$, $\forall S \in \mathbb{Z}$ (lifts of CS functional of trivial flat connection).

Remark: $\hat{Z}^{\mathfrak{sl}(n)}[S^3]$ is an entire function in $\hbar$.

Resurgence for Poincare homology sphere

For Seifet manifolds with 3 exceptional fibers it is sufficient to analyze asymptotic expansion of

$$F(q;\alpha,\beta,\gamma;A,B)=\sum_{m\geq 0}\frac{q^{\alpha m^2+\beta m+\gamma}}{1-q^{Am+B}}.$$

for general parameters α,β,γ,A,B. This can be done using Euler-Maclaurin summation formula. Its Borel tranform has singularities at $\xi=4\pi^2 S$, $S=\frac{K^2}{4\alpha}$ and $S=-\frac{(2K\alpha/A)^2}{4\alpha}$, $K\in\mathbb{Z}$.

Example: $M^3=$Poincare homology sphere. Many “fake” singularities cancel out between contributions from different F's. The singularities with $S=0,\frac{1}{120},\frac{49}{120} \mod 1$ do not. They correspond to CS functional of $SU(2)\subset SU(2|1)$ flat connections.

Quantum invariants associated to $\mathcal{U}_q^H(\mathfrak{sl}(2|1))$

Using the standard Witten-Reshetikhin-Turaev approach to topological invariants for quantum $\mathfrak{sl}(2|1)$ leads to trivial invariants as quantum dimensions vanish.

Instead one should use notion of modified trace [Geer–Patureau-Mirand–Turaev '09, Geer–Kujawa–Patureau-Mirand '11] and construction of [Costantino–Geer–Patureau-Mirand '14] applied to the non-semisiple category of representations (satisfying certain conditions) of $\mathcal{U}_q^H(\mathfrak{sl}(2|1))$ at $q = e^{\frac{4\pi i}{k}}$, odd k [Ha '18]. The result is an invariant

$$N_k(M^3, \omega)$$

of closed oriented 3-manifolds M^3 colored by

$$\omega \in H^1(M^3, \mathbb{C}/\mathbb{Z} \times \mathbb{C}/\mathbb{Z}) \setminus \left(H^1(M^3, \frac{1}{2}\mathbb{Z}/\mathbb{Z} \times \mathbb{C}/\mathbb{Z}) \cup \ldots \right)$$

obtained via surgery on a framed link (For each link components one sums over representations with weights in $\mathbb{C} \times \mathbb{C}$ equal to $\omega([\text{meridian}])$ modulo $\mathbb{Z} \times \mathbb{Z}$)

Relation between the q-series and "non-semi-simple quantum invariants"

Conjecture: for a rational homology sphere M^3

$$N_k(M^3,\omega) = \frac{\mathcal{T}(2\omega_1)}{\ell|H_1(M^3,\mathbb{Z})|}\times$$
$$\times \sum_{\substack{\beta,\gamma\in H_1(M^3,\mathbb{Z})\\ b,c\in H^1(M^3,\mathbb{C}/\mathbb{Z})}} e^{2\pi i k\cdot \ell k(\beta,\gamma)+4\pi i(b-\omega_2)(\gamma)+2\pi i(c-(\omega_1+\omega_2))(\beta)}\cdot \hat{Z}^{\mathfrak{sl}(2|1)}_{b,c}\big|_{q\to e^{\frac{4\pi i}{k}}}$$

where $\omega \in H^1(M^3, \mathbb{C}/\mathbb{Z}\times\mathbb{C}/\mathbb{Z})\setminus\ldots$ (as before) and $\mathcal{T}$ is the Reidemeister torsion.

Remark: For a rational homology sphere $H^1(M^3,\mathbb{C}/\mathbb{Z})\cong H^1(M^3,\mathbb{Q}/\mathbb{Z})$ and the linking pairing

$$\ell k : H_1(M^3,\mathbb{Z})\otimes H_1(M^3,\mathbb{Z})\to \mathbb{Q}/\mathbb{Z}$$

provides an isomorphism $H_1(M^3,\mathbb{Z})\cong H^1(M^3,\mathbb{C}/\mathbb{Z})$.

Some open questions and future directions

- In simple examples all coefficients are positive (possibly except the constant term). Is it true in general? Explanation?
- Categorification of the q-series similar to $U(1|1)$ case? Might be easier to understand than for $SU(2)$.
- Relation between $\hat{Z}^{\mathfrak{psl}(n|n)}[M^3]$ and $PSU(N)$ instanton counting on $M^3 \times \mathbb{R}$?
- Manifolds with (torus) boundaries.