Geometric characterisation of topological string partition functions

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Topological string partition functions

Consider A/B model topological string on Calabi-Yau manifold X/Y.

World-sheet definition of Z_{top} yields asymptotic (?) series

$$\log Z_{\text{top}} \sim \sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}_g \qquad (1)$$

Question: Existence of summations?

Do there exist functions Z_{top} having (1) as asymptotic expansion?

(a) Functions on which space?

(b) Functions, sections of a line bundle, or what?

Z_{top} could be locally defined functions on $\mathcal{M}_{\text{Käh}}(X)$ or $\mathcal{M}_{\text{cplx}}(Y)$.

$$Z_{\text{top}} = Z_{\text{top}}(t), \quad t = (t_1, \dots, t_d): \text{ coordinates on } \mathcal{M}_{\text{Käh}}(X).$$

Dream: There exists a natural geometric structure on $\mathcal{M}_{\text{cplx}}(Y)$ allowing us to represent Z_{top} as "local section".

Our playground: Local Calabi-Yau manifolds Y_Σ of class Σ:

$uv - f_\Sigma(x,y) = 0$ s.t. $\Sigma = \{(x,y) \in T^*C; f_\Sigma(x,y) = 0\} \subset T^*C$ smooth,

$f_\Sigma(x,y) = y^2 - q(x)$, $q(x)(dx)^2$: quadratic differential on cplx. surface C.

Moduli space $\mathcal{B} \equiv \mathcal{M}_{\text{cplx}}(Y)$: Space of pairs (C,q), C: Riemann surface, q: quadratic differential.

Special geometry: Coordinates $a^r = \int_{\alpha^r} \sqrt{q}$, $\check{a}^r = \int_{\check{\alpha}_r} \sqrt{q} = \frac{\partial}{\partial a^r}\mathcal{F}(a)$, where $\{(\alpha^r, \check{\alpha}_r); r = 1,\dots,d\}$ is a canonical basis for $H_1(\Sigma,\mathbb{Z})$.

Integrable structure: (Donagi-Witten, Freed) $\exists$ canonical torus fibration

$$\pi : \mathcal{M}_{\text{int}}(Y) \to \mathcal{B}, \qquad \Theta_b := \pi^{-1}(b) = \mathbb{C}^d/\mathbb{Z}^d + \tau(b)\cdot\mathbb{Z}^d,$$

$\tau(b)_{rs} = \frac{\partial}{\partial a_\imath^r}\frac{\partial}{\partial a_\imath^s}\mathcal{F}(a_\imath)$, coordinates $(\theta_\imath^r, \check{\theta}_\imath^r)$, $r = 1,\dots,d$, on torus fibers.

(a) $\mathcal{M}_{\text{int}}(Y)$ moduli space of pairs $(\Sigma,\mathcal{D})$, $\mathcal{D}$: divisor on Σ (Abel-Jacobi)
(b) $\mathcal{M}_{\text{int}}(Y) \simeq \mathcal{M}_{\text{Hit}}(Y)$, moduli space of Higgs pairs $(\mathcal{E},\varphi)$ (Hitchin)
(c) $\mathcal{M}_{\text{int}}(Y) \simeq$ intermediate Jacobian fibration (Diaconescu-Donagi-Pantev)

Our starting point

Some of Y_Σ: limits of toric CY $\Rightarrow$ compute Z_{top} with topological vertex[1].

Comparison with instanton counting[2] and AGT-correspondence
$\Rightarrow$ $Z_{\text{top}} \sim$ conformal block of Virasoro VOA at $c = 1$.

String dualities relate[3] $Z_{\text{top}}(t;\hbar) \overset{\text{[MNOP]}}{\sim} Z_{\text{D0-D2-D6}}(t;\hbar)$ to free fermions

$$\sum_{p \in H^2(Y,\mathbb{Z})} e^{p\xi} Z_{\text{top}}(t + \hbar p;\hbar) \overset{\text{[MNOP]}}{\sim} Z_{\text{D0-D2-D4-D6}}(\xi, t;\hbar) \overset{\text{[DHSV]}}{=} Z_{\text{ff}}(\xi, t;\hbar),$$

which can be inverted to get Z_{top}. Recent progress[4] on the relations

Free fermion CFT $\leftrightarrow$ Tau-functions $\leftrightarrow$ Virasoro VOA,

and relation to exact WKB/abelianisation allow us to interpret the results for Z_{top} in geometric terms, leading to the picture outlined below.

[1] Aganagic, Klemm, Marino, Vafa

[2] Moore-Nekrasov-Shatashvili; Losev-Nekrasov-Shatashvili; Nekrasov

[3] Dijkgraaf-Hollands-Sulkowski-Vafa [DHSV] using Maulik-Nekrasov-Okounkov-Pandharipande [MNOP]

[4] Gamayun-Iorgov-Lisovyy; Iorgov-Lisovyy-J.T.

Our proposal in a nutshell: (compare with Alexandrov, Persson, Pioline – later!)

Main geometric players:

- Moduli space $\mathcal{B} \equiv \mathcal{M}_{\rm cplx}(Y)$ of complex structures,
- torus fibration $\mathcal{M}_{\rm int}(Y)$ over $\mathcal{B}$ canonically associated to the special geometry on $\mathcal{B}$ ($\sim$ intermediate Jacobian fibration).

There then exist

(A) a canonical one-parameter ($\hbar$) family of deformations of the complex structures on $\mathcal{M}_{\rm int}(Y)$, defined by an atlas of Darboux coordinates $x_\imath = (\mathrm{x}_\imath, \check{\mathrm{x}}^\imath)$ on $\mathcal{Z} := \mathcal{M}_{\rm int}(Y) \times \mathbb{C}^*$,

(B) a canonical pair $(\mathcal{L}_\Theta, \nabla_\Theta)$ consisting of

- $\mathcal{L}_\Theta$: line bundle on $\mathcal{Z}$, transition functions: Difference generating functions of changes of coordinates $x_\imath$,
- ∇_Θ: connection on $\mathcal{L}_\Theta$, flat sections: Tau-functions $\mathcal{T}_\imath(\mathrm{x}_\imath, \check{\mathrm{x}}^\imath)$,

defining the topological string partition functions via

$$\boxed{\mathcal{T}_\imath(\mathrm{x}_\imath, \check{\mathrm{x}}^\imath) = \sum_{\mathrm{n}\in\mathbb{Z}^d} e^{2\pi \mathrm{i}\,(n,\check{\mathrm{x}}^\imath)} Z^\imath_{\rm top}(\mathrm{x}_\imath - \mathrm{n}).}$$

(A) The BPS Riemann-Hilbert problem (Gaiotto-Moore-Neitzke; Bridgeland)

Define $\hbar$-deformed complex structures by atlas of coordinates on $\mathcal{Z} \simeq \mathcal{M}_{\mathrm{cplx}}(Y) \times \mathbb{C}^\times$ with charts $\{\mathcal{U}_\imath; \imath \in \mathbb{I}\}$, Darboux coordinates

$$x_\imath = (\mathrm{x}_\imath, \check{\mathrm{x}}^\imath) = x_\imath(\hbar), \qquad \Omega = \sum_{r=1}^{d} dx_\imath^r \wedge d\check{x}_r^\imath, \quad \text{such that}$$

- changes of coordinates across $\{\hbar \in \mathbb{C}^\times; a_\gamma/\hbar \in \mathrm{i}\mathbb{R}_-\}$ represented as

$$X_{\gamma'}^\jmath = X_{\gamma'}^\imath (1 - X_\gamma)^{\langle \gamma', \gamma \rangle \Omega(\gamma)}, \qquad X_\gamma^\jmath = e^{2\pi \mathrm{i} \langle \gamma, x_\imath \rangle} = e^{2\pi \mathrm{i} (p_r^\imath x_\imath^r - q_\imath^r \check{x}_r^\imath)}, \quad \text{if } \gamma = (q_\imath^1, \ldots, q_\imath^d; p_1^\imath, \ldots, p_d^\imath),$$

  determined by data $\Omega(\gamma)$ satisfying Kontsevich-Soibelman-WCF.
- asymptotic behaviour

$$\mathrm{x}_\imath^r \sim \frac{1}{\hbar} a_\imath^r + \vartheta_\imath^r + \mathcal{O}(\hbar), \qquad \check{\mathrm{x}}_\imath^r \sim \frac{1}{\hbar} \check{a}_r^\imath + \check{\vartheta}_r^\imath + \mathcal{O}(\hbar),$$

  with $(a_\imath^r, \check{a}_r^\imath)$ coordinates on $\mathcal{B}$, $\theta_r^\imath := \vartheta_r^\imath - \tau \cdot \check{\vartheta}_\imath^r$ coordinates on Θ_b.

Solving the BPS-RH problem

$1^{\rm st}$ Solution: NLIE (Gaiotto-Moore-Neitzke (GMN); Gaiotto)

$$X_\gamma(\hbar) = X_\gamma^{\rm sf}(\hbar)\exp\left[-\frac{1}{4\pi \rm i}\sum_{\gamma'}\langle\gamma,\gamma'\rangle\Omega(\gamma')\int_{l_{\gamma'}}\frac{d\hbar'}{\hbar'}\frac{\hbar'+\hbar}{\hbar'-\hbar}\log(1-X_{\gamma'}(\hbar'))\right]$$

with $\log X_\gamma^{\rm sf}(\hbar) = \frac{1}{\hbar}a_\gamma + \vartheta_\gamma$. (Gaiotto: Conformal limit of GMN-NLIE)

$2^{\rm nd}$ Solution: Quantum curves

Quantum curves: Opers, certain pairs $(\mathcal{E},\nabla_\hbar) = (\text{bundle},\text{connection})$ $\leftrightsquigarrow$

$$\text{differential operators}\quad \hbar^2\partial_x^2 - q_\hbar(x).$$

Coordinates $X_\gamma^\imath(\hbar)$, $\check{X}_\imath^\gamma(\hbar)$ for space of monodromy data defined by Borel summation of exact WKB solution $\rightsquigarrow$ charts $\mathcal{U}_\imath$ labelled by spectral networks (Gaiotto-Moore-Neitzke; Hollands-Neitzke).

2^{nd} Solution: Quantum curves

Equation $y^2 = q(x)$ defining Σ admits canonical quantisation $y \to \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x}$,

$$\rightsquigarrow \quad \text{oper} \quad \hbar^2 \frac{\partial^2}{\partial x^2} - q(x) \quad \leftrightsquigarrow \quad \nabla_\hbar = \hbar \frac{\partial}{\partial x} - \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}.$$

Observation: There is an essentially canonical generalisation $\hbar$-deforming pairs $(\Sigma, \mathcal{D})$, representable by opers with apparent singularities

$$\hbar^2 \partial_x^2 - q_\hbar(x), \quad q_\hbar(x) = \frac{3\hbar^2}{4(x-u_r)^2} + \mathcal{O}((x-u_r)^{-1}), \ \ r = 1, \dots, d.$$

Conjecture

Solution of BPS-RH-problem given by composition of holonomy map with rational coordinates for space of monodromy data,

$$\mathcal{M}_{\text{char}}(Y) : \quad \begin{array}{l} \text{Algebraic variety having coordinate ring} \\ \text{generated by trace functions } \mathrm{tr}(\rho(\gamma)) \end{array}$$

having Borel summable $\hbar$-expansion.

Expansion in $\hbar$ - exact WKB: Solutions to $\left(\hbar^2 \frac{\partial^2}{\partial x^2} - q_\hbar(x)\right)\chi(x) = 0$,

$$\chi_\pm^{(b)}(x) = \frac{1}{\sqrt{S_{\rm odd}(x)}} \exp\left[\pm \int^x dx' \, S_{\rm odd}(x')\right],$$

with $S_{\rm odd} = \frac{1}{2}(S^{(+)} - S^{(-)})$, $S^{(\pm)}(x)$ being formal series solutions to

$$q_\hbar = \lambda^2(S^2 + S'), \qquad S(x) = \sum_{k=-1}^{\infty} \hbar^k S_k(x), \qquad S_{-1}^{(\pm)} = \pm\sqrt{q_0}. \qquad (2)$$

It is believed[5] that series (2) is **Borel-summable** away from **Stokes-lines**,

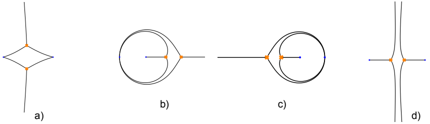
$$\mathrm{Im}(w(x)) = \mathrm{const.}, \qquad w(x) = e^{-\mathrm{i}\arg(\lambda)} \int^x dx' \sqrt{q(x')}$$

Voros symbols $V_\beta := \int_\beta dx \, S_{\rm odd}(x)$ can be Borel-summable, then representing ingredients of the solution to the BPS-RH-problem.

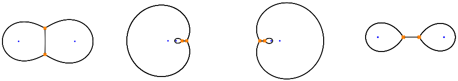
[5] Probably proven by Koike-Schäfke (unpublished), and by Nikolaev (to appear).

Borel summability depends on the topology of Stokes graph formed by Stokes lines (determined by $q_0 \sim$ point on $\mathcal{B}$). Two "extreme" cases:

FG Stokes graph $\leftsquigarrow\!\!\rightsquigarrow$ triangulation of C



FN Stokes graph $\leftsquigarrow\!\!\rightsquigarrow$ pants decomposition



In between there exist several hybrid types of graphs.

Case FG: D. Allegretti has proven conjecture of T. Bridgeland: Voros symbols $\sim$ Fock-Goncharov (FG) type coordinates solve BPS-RH problem.

Important: Extension to case FN needed for topological string applications: Case FN: Real[6] "skeleton" in $\mathcal{B}$, described by Jenkins-Strebel differentials[7].

[6] Real values of $\hbar$ and special coordinates $a^r_\imath$

[7] Stokes graphs decompose C into ring domains

Second half of our proposal:

There exists a canonical pair $(\mathcal{L}_\Theta, \nabla_\Theta)$ consisting of

- $\mathcal{L}_\Theta$: line bundle on $\mathcal{Z}$, transition functions: Difference generating functions of changes of coordinates $x_\imath$
- ∇_Θ: connection on $\mathcal{L}_\Theta$, flat sections: Tau-functions $\mathcal{T}_\imath(\mathsf{x}_\imath, \check{\mathsf{x}}^\imath)$, determining Z_{top} with the help of

$$\mathcal{T}_\imath(\mathsf{x}_\imath, \check{\mathsf{x}}^\imath) = \sum_{\mathsf{n}\in\mathbb{Z}^d} e^{2\pi \mathrm{i}\,(n,\check{\mathsf{x}}^\imath)} Z^\imath_{\text{top}}(\mathsf{x}_\imath - \mathsf{n}).$$

This means that there are wall-crossing relations

$$\mathcal{T}_\imath(\mathsf{x}_\imath, \check{\mathsf{x}}^\imath) = F_{\imath\jmath}(\mathsf{x}_\imath, \mathsf{x}_\jmath)\mathcal{T}_\jmath(\mathsf{x}_\jmath, \check{\mathsf{x}}^\jmath),$$

on overlaps $\mathcal{U}_\imath \cap \mathcal{U}_\jmath$ of charts, with transition functions $F_{\imath\jmath}(\mathsf{x}_\imath, \mathsf{x}_\jmath)$: difference generating functions, defined by the changes of coordinates $x_\imath = x_\imath(x_\jmath)$.

Difference generating functions:

$$\mathcal{T}(\mathsf{x},\check{\mathsf{x}}) = \sum_{\mathsf{n}\in\mathbb{Z}^d} e^{2\pi\mathrm{i}(\mathsf{n},\check{\mathsf{x}})} Z(\mathsf{x}-\mathsf{n}) \;\Leftrightarrow\; \begin{cases} \mathcal{T}(\mathsf{x},\check{\mathsf{x}}+\delta_r) = \mathcal{T}(\mathsf{x},\check{\mathsf{x}}) \\ \mathcal{T}(\mathsf{x}+\delta_r,\check{\mathsf{x}}) = e^{2\pi\mathrm{i}\,\check{\mathsf{x}}_r}\mathcal{T}(\mathsf{x},\check{\mathsf{x}}) \end{cases} \tag{3}$$

Coordinates considered here are such that $\mathsf{x}_\imath = \mathsf{x}_\imath(\mathsf{x}_\jmath,\check{\mathsf{x}}^\jmath)$ can be solved for $\check{\mathsf{x}}^\jmath$ in $\mathcal{U}_\imath \cap \mathcal{U}_\jmath$, defining $\check{\mathsf{x}}^\jmath(\mathsf{x}_\imath,\mathsf{x}_\jmath)$. Having defined tau-functions $\mathcal{T}_\imath(\mathsf{x}_\imath,\check{\mathsf{x}}^\imath)$ and $\mathcal{T}_\jmath(\mathsf{x}_\jmath,\check{\mathsf{x}}^\jmath)$ on charts $\mathcal{U}_\imath$ and $\mathcal{U}_\jmath$, respectively, there is a relation of the form

$$\mathcal{T}_\imath(\mathsf{x}_\imath,\check{\mathsf{x}}^\imath) = F_{\imath\jmath}(\mathsf{x}_\imath,\mathsf{x}_\jmath)\mathcal{T}_\jmath(\mathsf{x}_\jmath,\check{\mathsf{x}}^\jmath),$$

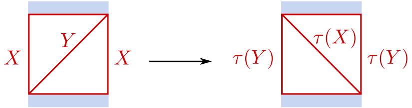
on the overlaps $\mathcal{U}_{\imath\jmath} = \mathcal{U}_\imath \cap \mathcal{U}_\jmath$. To ensure that both $\mathcal{T}_\imath$ and $\mathcal{T}_\jmath$ satisfy the relations (3), $F_{\imath\jmath}(\mathsf{x}_\imath,\mathsf{x}_\jmath)$ must satisfy

$$F_{\imath\jmath}(\mathsf{x}_\imath+\delta_r,\mathsf{x}_\jmath) = e^{+2\pi\mathrm{i}\,\check{\mathsf{x}}^\imath_r} F_{\imath\jmath}(\mathsf{x}_\imath,\mathsf{x}_\jmath), \tag{4a}$$

$$F_{\imath\jmath}(\mathsf{x}_\imath,\mathsf{x}_\jmath+\delta_r) = e^{-2\pi\mathrm{i}\,\check{\mathsf{x}}^\jmath_r} F_{\imath\jmath}(\mathsf{x}_\imath,\mathsf{x}_\jmath). \tag{4b}$$

We will call functions $F_{\imath\jmath}(\mathsf{x}_\imath,\mathsf{x}_\jmath)$ satisfying the relations (4) associated to a change of coordinates $x_\imath = x_\imath(x_\jmath)$ difference generating functions.

Basic example:



$$X' = \tau(X) = Y^{-1}, \qquad (5)$$

$$Y' = \tau(Y) = X(1 + Y^{-1})^{-2}.$$

Introduce logarithmic variables x, y, x', y',

$$X = e^{2\pi\mathrm{i}\,x}, \qquad Y = -e^{2\pi\mathrm{i}\,y}, \qquad X' = -e^{2\pi\mathrm{i}\,x'}, \qquad Y' = e^{2\pi\mathrm{i}\,y'}.$$

The equations (5) can be solved for Y and Y',

$$Y(x, x') = -e^{-2\pi\mathrm{i}\,x'}, \qquad Y'(x, y) = e^{2\pi\mathrm{i}\,x}(1 - e^{2\pi\mathrm{i}\,x'})^{-2}.$$

The difference generating function $\mathcal{J}(x, x')$ associated to (5) satisfies

$$\frac{\mathcal{J}(x+1, x')}{\mathcal{J}(x, y)} = -(Y(x, x'))^{-1}, \qquad \frac{\mathcal{J}(x, x'+1)}{\mathcal{J}(x, y)} = Y'(x, x').$$

A function satisfying these properties is

$$\mathcal{J}(x, x') = e^{2\pi\mathrm{i}xx'}(E(x'))^2, \qquad E(z) = (2\pi)^{-z} e^{-\frac{\pi\mathrm{i}}{2}z^2} \frac{G(1+z)}{G(1-z)},$$

where $G(z)$ is the Barnes G-function satisfying $G(z+1) = \Gamma(z)G(z)$.

Tau-functions as solutions to the secondary RH problem

In arXiv:2004.04585 and work in progress we explain how to define solutions $\mathcal{T}_\imath(\mathrm{x}_\imath, \check{\mathrm{x}}^\imath)$ to the secondary RH problem by combining

free fermion CFT with exact WKB.

Key features:

- Proposal covers real slice in $\mathcal{B}$ represented by Jenkins-Strebel differentials using FN type coordinates,
- agrees with topological vertex calculations on the real slice, whenever available,
- and defines canonical extensions into strong coupling regions[8] (for $C = C_{0,2}$ using important work of Its-Lisovyy-Tykhyy).

Exact WKB for quantum curves fixes normalisation ambiguities $\Rightarrow$ the $\hbar$-deformation is "as canonical as possible".

[8] In the sense of Seiberg-Witten theory

The picture found in the class Σ examples suggests:

The higher genus corrections in the topological string theory on X are encoded in a canonical $\hbar$-deformation of the moduli space $\mathcal{M}_{\text{cplx}}(Y)$ of complex structures on the mirror Y of X.

There are hints that this picture may generalise beyond the class Σ examples:

(A) Relation to geometry of hypermultiplet moduli spaces – see below
(B) Relation to spectrum of BPS-states, geometry of space of stability conditions (T. Bridgeland)
(C) Relations to spectral determinants (Marino et.al.)?

Take-outs: (see below)

1) Relation classical-quantum
2) Relation with Theta-functions on intermediate Jacobian fibration
3) Interplay between 2d-4d wall-crossing and free fermion picture

(A) Relation to geometry to hypermultiplet moduli spaces

A similar characterisation of Z_{top} follows from the proposal of Alexandrov, Persson, and Pioline (APP) for NS5-brane corrections to the geometry of hypermultiplet moduli spaces:

- SUSY $\rightsquigarrow$ describe quantum corrections using twistor space geometry,

  $$\text{locally} \quad \mathcal{Z} \simeq \mathcal{M}_{\text{cplx}}(Y) \times \mathbb{P}^1,$$

  having atlas of Darboux coordinates $x_\imath = (\mathrm{x}_\imath, \check{\mathrm{x}}^\imath)$ on $\mathcal{Z}$.
- Combining mirror symmetry, S-duality, and twistor space geometry $\Rightarrow$ quantum correction from one NS5-brane encoded in locally defined holomorphic functions $H_{\text{NS5}}(\mathrm{x}_\imath, \check{\mathrm{x}}^\imath)$ having representation of the form

  $$H_{\text{NS5}}(\mathrm{x}_\imath, \check{\mathrm{x}}^\imath) = \sum_{\mathrm{n} \in \mathbb{Z}^d} e^{2\pi \mathrm{i}\,(n, \check{\mathrm{x}}^\imath)} K^\imath_{\text{NS5}}(\mathrm{x}_\imath - \mathrm{n}).$$
- Using the DT-GW-relation (MNOP): $K^\imath_{\text{NS5}}(\mathrm{x}_\imath) \sim Z^\imath_{\text{top}}(\mathrm{x}_\imath)$.

This suggests: $\begin{cases} \text{Our results} \rightsquigarrow \text{confirmation of APP-proposal,} \\ \text{APP-framework predicts generalisations of our results.} \end{cases}$

1) Relation classical-quantum: The magic formula

$$\mathcal{T}_\imath(\mathsf{x}_\imath,\check{\mathsf{x}}^\imath)=\sum_{\mathsf{n}\in\mathbb{Z}^d} e^{2\pi\mathrm{i}\,(n,\check{\mathsf{x}}^\imath)} Z^\imath_{\rm top}(\mathsf{x}_\imath-\mathsf{n}) \tag{6}$$

can be interpreted as a relation between an honest quantum deformation of $\mathcal{B}$ and the $\hbar$-deformation of a classical space discussed in this talk.

The main observation in Iorgov-Lisovyy-J.T. was that the transform (6) simultaenously diagonalises all operators in a realisation of the quantised algebra of functions on $\mathcal{M}_{\rm char}(Y)$ generated by Verlinde loop operators.

In work by Alexandrov-Pioline, it was shown that the wall-crossing relations $\mathcal{T}_\imath(\mathsf{x}_\imath,\check{\mathsf{x}}^\imath)=F_{\imath\jmath}(\mathsf{x}_\imath,\mathsf{x}_\jmath)\mathcal{T}_\jmath(\mathsf{x}_\jmath,\check{\mathsf{x}}^\jmath)$, translate into integral transforms

$$Z^\imath_{\rm top}(\mathsf{x}_\imath)=\int d\mathsf{x}_\jmath\;K(\mathsf{x}_\imath,\mathsf{x}_\jmath)Z^\jmath_{\rm top}(\mathsf{x}_\jmath).$$

In view of the relation with Theta-functions (next slide) this is probably best understood in connection with the ideas related to quantisation of the intermediate Jacobian going back to Witten.

2) Relation with Theta-functions on intermediate Jacobian fibration

Let us use the isomonodromic tau-functions to define $\Theta_{\Sigma_\hbar}(\mathrm{a},\theta;z;\hbar)$,

$$\Theta_{\Sigma_\hbar}(\mathrm{a},\theta;z;\hbar) := \mathcal{T}\big(\sigma(\mathrm{a},\theta;\hbar)\,,\,\tau(\mathrm{a},\theta;\hbar)\,;\,z\,;\,\hbar\big), \qquad (7)$$

when $d=1$, $\sigma \equiv x^1_\imath$, $\eta \equiv \check{x}^\imath_1$, $\theta = \theta^\imath_1$.

Claim

The limit

$$\log\Theta_\Sigma(\mathrm{a},\theta;z) := \lim_{\hbar\to 0}\Big[\log\Theta_{\Sigma_\hbar}(\mathrm{a},\theta;z;\hbar) - \log\mathcal{Z}_{\mathrm{top}}(\sigma(\mathrm{a},\theta);z;\hbar)\Big] \qquad (8)$$

exists, with function $\Theta_\Sigma(\mathrm{a},\theta;z)$ defined in (8) being the theta function

$$\Theta_\Sigma(\mathrm{a};\theta;z) = \sum_{n\in\mathbb{Z}} e^{2\pi \mathrm{i} n\theta}\, e^{\pi \mathrm{i} n^2 \tau_\Sigma(\mathrm{a})}, \qquad (9)$$

with $\tau_\Sigma(\mathrm{a})$ related to $\mathcal{F}(\mathrm{a},z)$ by $\tau_\Sigma = \dfrac{1}{2\pi\mathrm{i}}\dfrac{\partial^2\mathcal{F}}{\partial\mathrm{a}^2}$.

3) Interplay between 2d-4d wall-crossing and free fermion picture

Background Y_Σ can be modified to open-closed background by inserting Aganagic-Vafa branes located at points of Σ. Generalisation of the formula

$$\mathcal{T}_\imath(\mathrm{x}_\imath, \check{\mathrm{x}}^\imath) \equiv \langle\, \Omega\,,\, \mathfrak{f}_\Psi\,\rangle = \sum_{\mathrm{n}\in\mathbb{Z}^d} e^{2\pi\mathrm{i}\,(n,\check{\mathrm{x}}^\imath)} Z^\imath_{\rm top}(\mathrm{x}_\imath - \mathrm{n})$$

due to Iorgov-Lisovyy-J.T. will then relate free fermion expectation values

$$\Psi(x,y) = \langle\!\langle\, \bar{\psi}(x)\psi(y)\rangle\!\rangle = \frac{\langle \Omega, \bar{\psi}(x)\psi(y)\mathfrak{f}_\Psi\rangle}{\langle\, \Omega\,,\, \mathfrak{f}_\Psi\,\rangle},$$

to expectation values of degenerate fields of the Virasoro algebra, representing the fermions of Aganagic-Dijkgraaf-Klemm-Marino-Vafa in our context.

Noting that $\Psi(x,y)$ represents the solution to the classical RH-problem associated to the tau-function $\mathcal{T}_\imath = \langle\, \Omega\,,\, \mathfrak{f}_\Psi\rangle$ one sees that:

relation between classical RH-problem to BPS-RH problem:
Example for 4d-2d wall crossing (GMN).

Exact WKB fixes the normalisations for $\Psi(x,y)$, via 4d-2d wall crossing determining the normalisations of $\mathcal{T}_\imath$.