Quantum modularity of 3-manifold invariants

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SISSA, Trieste

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The talk is based on

• **"3d Modularity"**, [arXiv:1809.10148]
with M. C. N. Cheng, S. Chun, S. Gukov, S. M. Harrison

• **"Three-Manifold Quantum Invariants and Mock Theta Functions"**, [arXiv:1912.07997]
with M. C. N. Cheng, G. Sgroi

• [arXiv:200X.XXXXX]
with P. Putrov

• [arXiv:200X.XXXXX]
with M. C. N. Cheng, S. Chun, B. Feigin, S. Gukov, S. M. Harrison

Number Theory

Quantum modularity of
3-manifold invariants

Physics

Topology

Modular forms

Definition

A **modular form** $\varphi(\tau)$ of weight k, multiplier system χ with respect to the group Γ is a holomorphic function $\varphi : \mathcal{H} \to \mathbb{C}$ which satisfies

$$\varphi|_{k,\chi}(\tau) = \varphi(\tau), \qquad \gamma \in \Gamma$$

A **cusp form** is a modular form with Fourier expansion $\varphi(\tau) = \sum_{n>0} c(n)q^n$, $q = e^{2\pi i\tau}$.

The weight-k **slash operator** is defined as

$$\varphi|_{k,\chi}(\tau) = (c\tau + d)^{-k}\chi(\gamma)^{-1}\varphi(\gamma\tau), \qquad \gamma \in \Gamma$$

where the action of Γ on $\mathcal{H}$ is given by fractional linear transformations and the multiplier system is a map $\chi : \Gamma \to S^1$. From now on, $\Gamma = SL(2,\mathbb{Z})$.

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Quantum Modular Forms

[Zagier (2010)]

- QMFs are defined at the boundary of $\mathcal{H}$ $(\mathbb{Q} \cup \{i\infty\})$
- QMFs are neither analytic nor Γ-covariant functions

A **quantum modular form** of weight k and multiplier χ with respect to Γ is a function $Q : \mathbb{Q} \to \mathbb{C}$ such that for every $\gamma \in \Gamma$ the function $p_\gamma(x) : \mathbb{Q} \backslash \{\gamma^{-1}(\infty)\} \to \mathbb{C}$,

$$p_\gamma(x) := Q(x) - Q|_{k,\chi}\gamma(x)\,,$$

has a better analytic behavior than $Q(x)$.
The function $\gamma \mapsto p_\gamma$ is a cocycle on Γ (i.e. $p_{\gamma_1\gamma_2} = p_{\gamma_1}|_{k,\chi}\gamma_2 + p_{\gamma_2}$).

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Strong QMFs

[Zagier (2010)]

A **strong quantum modular form** is a function Q which associates to each element $x \in \mathbb{Q}$ a formal power series over $\mathbb{C}$, so that the identity

$$p_\gamma(x+it) = Q(x+it) - Q|_{k,\chi}\gamma(x+it)\,, \qquad t \to 0^+\,,\ \gamma \in \Gamma$$

holds as an identity between countable collections of formal power series.

The formal function Q might extend to a globally defined function $Q : (\mathbb{C} \setminus \mathbb{R}) \cup \mathbb{Q} \to \mathbb{C}$

Witten-Reshetikhin-Turaev invariant

[Witten (1988)], [Reshetikhin-Turaev (1990)]

Consider the 3d $SU(2)$ Chern-Simons theory, whose partition function is

$$Z_{\mathrm{CS}}(M_3;k) = \int_{\mathcal{A}} \mathcal{D}A\, e^{\frac{i(k-2)}{4\pi}\int_{M_3}\mathrm{Tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)}$$

where $k \in \mathbb{Z}$ denotes the (shifted) Chern-Simons level.

$Z_{\mathrm{CS}}(M_3;k) : \mathbb{Q} \to \mathbb{C}$ was proven to be a topological invariant, known as the Witten-Reshetikhin-Turaev invariant (WRT).

$Z_{\mathrm{CS}}(M_3;k)$ is an example of **strong quantum modular form.**

[Lawrence, Zagier (1999)]

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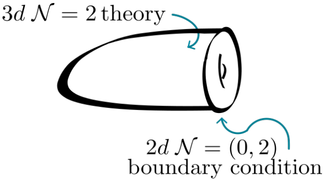
6d $\mathcal{N} = (2, 0)$ SCFT

[Gukov, Putrov, Vafa (2016)], [Gukov, Pei, Putrov, Vafa (2017)]

A_1 $6d$ $\mathcal{N} = (2, 0)$
on $M_3 \times D^2 \times_q S^1$

Supersymmetric three-dimensional gauge theory $T[M_3]$



Topological Quantum Field Theory

$\longrightarrow$ Three-dimensional Chern-Simons theory on M_3

The supersymmetric index of $T[M_3]$ is the **homological block**

$$\widehat{Z}_a(\tau) = Z(D^2 \times_q S^1; \mathcal{B}_a) = \sum_{\substack{i \in \mathbb{Z} + \Delta_a \\ j \in \mathbb{Z}}} (-1)^j q^i \dim \mathcal{H}_a^{i,j}$$

6d $\mathcal{N}=(2,0)$ SCFT

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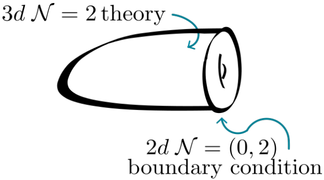

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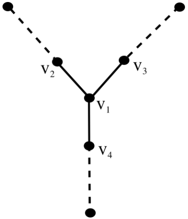
Plumbed 3-manifolds

A plumbed 3-manifold $M_3(\mathcal{G})$ is determined by a weighted simple graph (V, E, α), which represents Dehn surgery on the framed link $\mathcal{L}(\mathcal{G})$.



$M_3(\mathcal{G})$ is equivalently described by an adjacency matrix M, a square matrix of size $|V|$ with entries

$$M_{vv'} = \begin{cases} \alpha(v) & \text{if} \quad v = v' \\ 1 & \text{if} \quad (v, v') \in E \\ 0 & \text{otherwise} \end{cases}$$

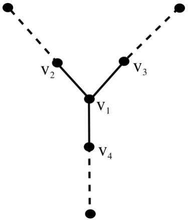
Plumbed 3-manifolds

M induces a non-degenerate symmetric bilinear pairing on $\mathrm{Tor} H_1(M_3, \mathbb{Z})$

$$\left(\mathrm{Tor} H_1(M_3, \mathbb{Z}),\ x \mapsto -(x, M^{-1}x) + \mathbb{Z}\right)$$

where $\mathrm{Tor} H_1(M_3, \mathbb{Z}) = \mathbb{Z}^{|V|}/M\mathbb{Z}^{|V|}$ when $b_1(M_3) = 0$.

M_3 is a weakly negative plumbed manifold if M^{-1} is negative-definite when restricted to the subspace generated by high-valency vertices.



$SU(2)$ Homological Blocks

[Gukov, Pei, Putrov, Vafa (2017)], [Gukov, Manolescu (2019)]

For a weakly negative plumbed three-manifold M_3 and gauge group $SU(2)$ the **homological block** $\widehat{Z}_a(M_3;\tau)$ is

$$\widehat{Z}_a(M_3;\tau) := (-1)^\pi q^{\frac{3\sigma - \sum_{v\in V}\alpha(v)}{4}} \times$$
$$\times \operatorname{vp} \prod_{v\in V} \oint_{|y_v|=1} \frac{dy_v}{2\pi i y_v} \left(y_v - {y_v}^{-1}\right)^{2-\delta_v} \Theta_a^M(\tau, \mathbf{z})$$

where $q = e^{2\pi i\tau}$, $y_v = e^{2\pi i z_v}$ and the label a can be identified with elements of the set of $\mathrm{Spin}^c(M_3)/\mathbb{Z}_2$. The theta function reads

$$\Theta_a^M(\tau, \mathbf{z}) = \sum_{\mathbf{n}\in 2M\mathbb{Z}^{|V|}+\mathbf{a}} q^{-\frac{\mathbf{n}^T M^{-1}\mathbf{n}}{4}} e^{2\pi i \mathbf{z}^T\mathbf{n}}.$$

where $\mathbf{a} \in (2\mathbb{Z}^{|V|}+\delta)/(2M\mathbb{Z}^{|V|})$ and $\delta_v = \deg(v)$.

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$SU(2)$ Homological blocks

[Gukov, Pei, Putrov, Vafa (2017)]

The radial limit, taking $\tau \in \mathcal{H}$ to the boundary of the upper-half plane, relates the homological blocks to the WRT invariants

$$Z_{\mathrm{CS}}(M_3; k) = (i\sqrt{2k})^{-1} \sum_{a,b \in (\mathbb{Z}^{|V|}/M\mathbb{Z}^{|V|})/\mathbb{Z}_2} e^{2\pi i k \lambda(a,a)} X_{ab} \lim_{\tau \to \frac{1}{k}} \widehat{Z}_b(M_3; \tau),$$

where $\lambda(a,b)$ is the linking pairing on $H_1(M_3, \mathbb{Z})$ and the matrix X has as elements

$$X_{ab} = \frac{e^{2\pi i \lambda(a,b)} + e^{-2\pi i \lambda(a,b)}}{|\mathcal{W}_a| \sqrt{|\mathrm{Det} M|}}.$$

where $\mathcal{W}_a = \mathrm{Stab}_{\mathbb{Z}_2}(a)$.

Homological blocks - False thetas

[Cheng, Chun, Ferrari, Gukov, Harrison (2018)], [Bringmann, Mahlburg, Milas (2018)]

The quantum invariants $\widehat{Z}_a(M_3;\tau)$ are given by

$$q^{-c}\widehat{Z}_a(M_3;\tau) = \sum_{r\in S} \Psi_{m,r}(\tau) + p(\tau)$$

where $p(\tau)$ is a polynomial, $c \in \mathbb{Q}$, S is a subset of $\mathbb{Z}/2m\mathbb{Z}$.

The function $\Psi_{m,r}(\tau)$ is a **false theta function**

$$\Psi_{m,r}(\tau) := \widetilde{\vartheta^1_{m,r}}(\tau) = \sum_{\substack{\ell\in\mathbb{Z} \\ \ell = r \bmod 2m}} \operatorname{sgn}(\ell)\, q^{\ell^2/4m}$$

$\vartheta^1_{m,r}$ is the weight 3/2 **unary theta function**

$$\vartheta^1_{m,r}(\tau, z) = \frac{1}{2\pi i}\partial_z \vartheta_{m,r}(\tau,z)|_{z=0} = \sum_{\substack{\ell\in\mathbb{Z} \\ \ell\equiv r\ (2m)}} \ell q^{\ell^2/4m}$$

Holomorphic Eichler integral

[Lawrence, Zagier (1999)]

The **holomorphic Eichler integral** of a weight w cusp form $g(\tau)$ is defined as

$$\widetilde{g}(\tau) := \sum_{n\geq 1} c(n) n^{1-w} q^n \,,$$

where the coefficients $c(n)$ are the Fourier coefficients of $g(\tau)$ and $w \in \frac{1}{2}\mathbb{Z}$.

Holomorphic Eichler integrals were first constructed to describe the $(w-1)$-fold primitive of a weight $w \in \mathbb{Z}$ cusp form $g(\tau)$. For integral weights $\widetilde{g}(\tau)$ can be expressed as

$$\widetilde{g}(\tau) = \int_{\tau}^{i\infty} d\tau' \, \frac{g(\tau')}{(\tau' - \tau)^{2-w}} \,, \qquad \tau \in \mathcal{H}$$

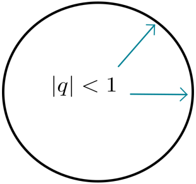
Radial Limit

[Lawrence, Zagier (1999)], [Cheng, Chun, Ferrari, Gukov, Harrison (2018)]

A false theta function does not transform nicely under the modular group, however its radial limit define a strong quantum modular form $Q(x)$

$$Q(x+it) = \lim_{t \to 0^+} \Psi_{m,r}(x+it)\,,$$

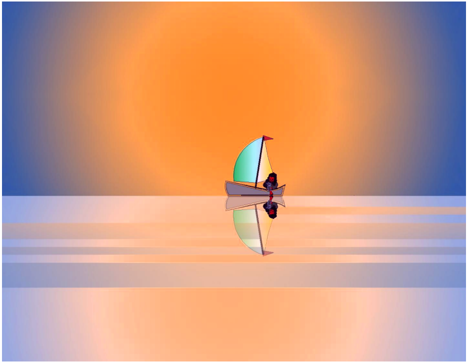
where $\tau = x + it$ with $x \in \mathbb{Q}$ and $t \in \mathbb{R}_+$.



The radial limit of false theta functions reproduces the WRT invariants $\longrightarrow$ **topological information** of M_3 can be easily extrapolated.

$M_3 \longrightarrow -M_3$

- **Q**: What happens in the lower half-plane?
- **Q**: What is $\widehat{Z}_a(-M_3;\tau)$ for a weakly positive definite 3-manifolds?



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∘ The Ohtsuki series obeys

$$Z_{\mathrm{CS}}(-M_3;k) = Z_{\mathrm{CS}}(M_3;-k), \qquad k \to \infty$$

∘ $\widehat{Z}_a(-M_3;\tau)$ is expected to be a holomorphic function on $\mathcal{H}$ with well-defined q-expansions and integral coefficients.

Mock modular forms

[Zwegers (2008)]

A **mock modular form** $f(\tau)$ of weight k is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$, whose completion

$$\hat{f}(\tau) = f(\tau) + g^*(\tau)$$

transforms like a modular form of weight k, $\hat{f}|_{k,\chi}\gamma(\tau) = \hat{f}(\tau)$.
The shadow, $g(\tau)$, is a holomorphic modular form of weight $2-k$ and

$$g^*(\tau) := \int_{-\bar{\tau}}^{i\infty} (\tau' + \tau)^{-k} \overline{g(-\bar{\tau}')} \, d\tau'$$

The non-holomorphic Eichler integral g^* transforms as

$$g^*(\tau) - g^*|_{k,\chi}\gamma(\tau) = \int_{-\gamma^{-1}(i\infty)}^{i\infty} g(\tau')(\tau' + \tau)^{-k} dw.$$

Non-holomorphic Eichler integral

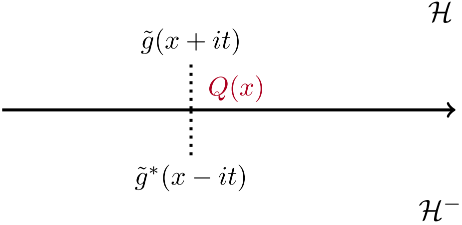
[Lawrence, Zagier (1999)], [Bringmann, Rolen (2015)]

Consider the non-holomorphic Eichler integral $\tilde{g}^* : \mathcal{H}^- \to \mathbb{C}$

$$\tilde{g}^*(\tau) := \int_{\bar{\tau}}^{i\infty} (\tau' - \tau)^{-k} g(\tau')\, d\tau'$$

The two Echler integrals $\tilde{g}^*$ and $\tilde{g}$ agree to infinite order at any $x \in \mathbb{Q}$, so that for $t > 0$

$$\tilde{g}(x+it) \sim \sum_{n\geq 0} \alpha_n t^n, \qquad \tilde{g}^*(x-it) \sim \sum_{n\geq 0} \alpha_n (-t)^n$$



Mock modular forms

[Griffin, Ono, Rolen (2013)], [Choi, Lim, Rhoades (2016)]

One of the distinctive features of mock theta functions is the infinite number of exponential singularities at roots of unity.

There is a collection of weakly holomorphic modular forms $\{G_j\}_{j=1}^n$ such that $(f - G_j)$ is bounded towards all cusps equivalent to x_j.
Given a choice of $\{G_j\}_{j=1}^n$, $f(\tau)$ defines a quantum modular form

$$Q(x) := \lim_{t\to 0^+} (f - G_x)(x + it),$$

Given a mock modular form $f(\tau)$ whose shadow is a cusp form $g(\tau)$ and $\tilde{g}(\tau)$ is its Eichler integral, then $f(\tau)$ and $\tilde{g}(\tau)$ have the "same" asymptotic series at $x \in \mathbb{Q}$

$$(f - G_x)(-x + it) \sim \sum_{n\geq 0} \alpha_x(n)(-t)^n, \quad \tilde{g}(x + it) \sim \sum_{n\geq 0} \alpha_x(n)t^n.$$

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Mock-False Conjecture

[Cheng, Chun, Ferrari, Gukov, Harrison (2018)]

Mock-False Conjecture

If the homological block $q^{-c}\widehat{Z}_a(M_3;\tau) = \sum_{r\in S}\Psi_{m,r}$ is a false theta function for some $c \in \mathbb{Q}$, then

$$q^{c}\widehat{Z}_a(-M_3;\tau) = \sum_{r\in S} f_{m,r}(\tau)$$

is a mock theta function, whose shadow is a linear combination of the weight $3/2$ unary theta series $\vartheta^1_{m,r}(\tau)$.

See also the more recent works

[Gukov, Manolescu (2019)], [Cheng, Ferrari, Sgroi (2019)], [Cheng, Sgroi (20xx)]

A little summary

$$
\begin{tikzcd}
\widehat{Z}_a(M_3;\tau) & & \Psi_{m,r}(\tau) \arrow[ll] \arrow[dr] & & \\
& \vartheta^1_{m,r}(\tau) \arrow[ur] \arrow[dr] & & Q(x+it) \arrow[r] & Z_{\mathrm{CS}}(M_3;k) \\
\widehat{Z}_a(-M_3;\tau) & & f_{m,r}(\tau) \arrow[ll] \arrow[ur, "-G_x(\tau)"'] & &
\end{tikzcd}
$$

$SU(2|1)$ Homological Blocks

[Ferrari, Putrov (20xx)]

The homological block associated to $M_3(\mathcal{G})$ with trivial $H_1(M_3,\mathbb{Z})$ and for gauge group $SU(2|1)$ is

$$\widehat{Z}^{\mathfrak{sl}(2|1)}(M_3;q) := (-1)^{\pi} q^{\frac{(3\sigma-\mathrm{Tr}M)}{2}(\rho,\rho)} \times \\ \times \oint \prod_{v\in \mathrm{V}} \frac{dx_v}{2\pi i x_v}\frac{dy_v}{2\pi i y_v}\left(\widetilde{\mathcal{D}}_{\mathfrak{sl}(2|1)}(\mathbf{x},\mathbf{y})\right)^{2-\deg(v)}\Theta_M(\tau,\mathbf{x},\mathbf{y})$$

$\widetilde{\mathcal{D}}_{\mathfrak{sl}(2|1)}$ is the super Weyl denominator and the theta function reads

$$\Theta_M(\tau,\mathbf{x},\mathbf{y}) = \sum_{\ell_1,\ell_2\in M\mathbb{Z}^{|V|}} q^{-(\ell_1,M^{-1}(\ell_1-\ell_2))}\prod_{v\in \mathrm{V}} x_v^{\ell_{1,v}} y_v^{\ell_{2,v}}$$

The invairant is defined when $-M^{-1}$ is weakly copositive.

Eisenstein series and strong QMFs

[Ferrari, Putrov (20xx)], [Bettin, Conrey (2013)]

The homological block associated to the 3-sphere turns out to be

$$\widehat{Z}(S^3;\tau) = -1 + 2\sum_{n\geq 1} d(n)q^n = -1 + 2q + 4q^2 + 4q^3 + 6q^4 + \dots$$

where $d(n)$ is the number of positive integer divisors. In terms of the weight 1 holomorphic Eisenstein series

$$\widehat{Z}(S^3;\tau) = -(1 + E_1(\tau))/2, \qquad E_1(\tau) := 1 - 4\sum_{n\geq 1} d(n)q^n$$

In the limit $\tau \to it$, $t \to 0^+$,

$$\left(E_1(\tau) - \frac{1}{\tau}E_1(-1/\tau)\right)|_{\tau\to 0^+} = -4\frac{\log(2\pi t) - \gamma}{2\pi i t} - \frac{4i}{\pi}\sum_{n=2}^{N}\frac{\zeta(n)B_n}{n}t^{n-1} + O(t^N)$$

the rhs defines a period function $p_S(t)$, which extends to an analytic function on $\mathbb{C}' = \mathbb{C} \setminus (-\infty, 0]$, the slit complex plane.

Some open questions

3-manifold invariants

- higher rank gauge groups
- more complicated 3-manifolds
- connection to logarithmic VOAs
- homological blocks for supergroups
- other works on Kashaev invariants

QMFs

- turn out to be useful to extract topological information
- hint at a hidden modular structure

Where else will we find QMFs ?

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